The Term Structures of Co-Entropy in International Financial Markets

Fousseni Chabi-Yo Riccardo Colacito *

Abstract

We propose a new entropy-based correlation measure (co-entropy) to evaluate the performance of international asset pricing models. Co-entropy captures the codependence of two random variables beyond normality. We document that the co-entropy of international stochastic discount factors (SDFs) can be decomposed into a series of entropy-based correlations of permanent and transitory components of the SDFs. A large cross-section of countries is employed to obtain model-free estimates of all the components of co-entropy at various horizons. We confront several state-of-the-art international finance models with our empirical evidence, and find that they cannot account for the composition of codependence at all horizons.

JEL classification: F31; G12; G15.
This draft: November 11, 2013.

*Fousseni Chabi-Yo is affiliated with the Fisher College of Business, Ohio State University. Riccardo Colacito is affiliated with the University of North Carolina at Chapel Hill, Kenan Flagler School of Business. The authors acknowledge helpful discussions with Gurdip Bakshi, Ravi Bansal, Bruce Carlin, Mike Chernov, Eric Ghysels, Nikolai Roussanov, Georgios Skoulakis, and Rene Stulz. Any remaining errors are our responsibility alone. All computer codes are available from the authors.

1 Introduction

One of the major undertakings of the international macro-finance research in recent years has been the quest for a set of models capable of reproducing major stylized fact of international asset prices and quantities. While it is well understood that the central feature of all the models that have been proposed is a high degree of correlation of pricing kernels, less well known is the extent of co-movement of specific components of the stochastic discount factors.

In this paper we follow Alvarez and Jermann (2005) and decompose stochastic discount factors into a permanent and a transitory component.[1] This decomposition is relevant because the permanent component is the key ingredient that prices assets in the stock market, while the transitory component is the key ingredient that prices assets in bond markets, in particular, long-term discount bond. Understanding the extent of the co-movement of these components can thus shed light on the ability of economic models to accurately describe the dynamics of global financial markets.

Gavazzoni, Sambalaibat and Telmer (2013) argue that higher-order moments are critical for understanding currency dynamics. Motivated by their conclusion, we propose a novel entropy-based measure of correlation to better capture the extent of co-movement of stochastic discount factors and their components. We call this measure "co-entropy"

[1] More recently, Hansen and Scheinkman (2009), Hansen (2012), Koijen, Lustig and Van Nieuwerburgh (2010), and Bakshi and Chabi-Yo (2012) have also adopted this decomposition.

and we document that it can be seen as a model-free correlation that summarizes the extent of codependence of two variables beyond normality. We show that our measure is a weighted average of three international co-entropies: one that reflects the co-movement of transitory components of SDFs, one that has to do with the co-movement of permanent components, and one that is related to the cross-co-movement between permanent and transitory components.

We provide bounds and restrictions on all the co-entropies that account for the co-movement of stochastic discount factors. Similar to Alvarez and Jermann (2005), Hansen and Jagannathan (1991), and Brandt, Cochrane and Santa-Clara (2006), our bounds and restrictions are model-free and can be estimated given the time series of financial assets. By including exchange rates, we can extend our analysis to the study of international financial markets. In this respect, our analysis breaks new ground on the dynamics of international SDFs above and beyond what prescribed by the extensive literature on bounds on the dynamics of individual SDFs (see inter alia Hansen and Jagannathan (1991) and Bansal and Lehmann (1997)).

Using our new measure of entropy-based correlation, we first confirm the finding concerning the high degree of codependence of SDFs, which is needed to account for the degree of dispersion of exchange rate fluctuations. We also document several novel empirical regularities concerning the co-movement of the components of SDFs. First of all, we establish in a model-free setting that the correlation between permanent components across countries is high at all horizons. Second, we show that the term-structure of correlation of the transitory components is sharply upward sloping. Third, the cross-co-movement between transitory and permanent components is usually around zero, irrespective of the horizon. These findings appear to be consistent for a large cross-section of major industrialized countries, and subsequently put tighter restrictions on economic models.

Lustig, Stathopoulos and Verdelhan (2013) were the first ones to point out the relevance of studying the correlations of permanent and transitory components of the one period SDFs in international finance. Our paper provides the first systematic identification of all these correlations in a model free environment. We are able to achieve such identification using a novel measure, which is robust to non-normalities in the distributions of the SDFs. Furthermore, our results are not limited to one period SDFs, as we provide a very precise description of the entire term structure of correlations.

To evaluate international asset pricing models, we solve the eigenfunction problem of Alvarez and Jermann (2005), Hansen (2012), and Hansen and Scheinkman (2009). The set of restrictions on the co-entropy of permanent and transitory components obtained from observed asset prices can be used to evaluate whether the model's implied co-entropies are consistent with co-entropies estimated in a model-free manner from international asset prices. Our analysis reveals that a common feature of all these models lies in the difficulty of reproducing the *composition* of codependence of SDFs.

In particular, it seems to be the case that international finance models rely heavily on the correlation of transitory components. Our empirical evidence, however, suggests that these correlations are well below the levels impelled by these economic models. Perhaps even more interesting is the term structure of co-entropy of transitory components. Asset prices reveal that transitory components of SDFs display an increasing degree of co-movement through time, a robust finding for an overwhelming majority of developed countries in our sample. Existing international asset pricing models instead feature consistently flat term structures of co-entropy of transitory components of SDFs.

Taken together, these findings seem to suggest that more attention should be devoted to the maturity structure of the comovement of the components of the SDFs across countries. While existing models have typically relied on high contemporaneous degrees of correlation of international shocks to replicate key features of international financial markets, our analysis suggests that high degrees of correlation of international shocks across different dates may be even more relevant.

The rest of the paper is organized as follows. Section 2 provides the definition of entropy-based correlation, along with an example of its usefulness. Sections 3 and 4 focus on the derivation of model-free restrictions on the codependence of stochastic discount factors and their components. Section 5 shows the empirical evidence concerning the co-entropies in a large cross-section of countries. Section 6 presents several international asset pricing models with the co-entropy's restrictions impelled by the data. Section 7 provides concluding remarks.

2 Entropy-Based Correlation

Definition. In this section, we define the entropy-based correlation of two positive random variables. Entropy-based correlation summarizes in a single number codependence between two random variables beyond the standard Pearson correlation measure. Let X and Y be two positive real valued random variables. We define the entropy-based correlation as

$$\varrho_{X,Y} = 1 - \frac{L\left[X/Y\right]}{\left(L\left[X\right] + L\left[Y\right]\right)}, \tag{1}$$

where $L\left[u\right]$ represents the entropy of the positive random variable u, and is defined as

$$L\left[u\right] \equiv \log\left(E\left[u\right]\right) - E\left[\log\left(u\right)\right]. \tag{2}$$

The use of (1) as a codependence measure is motivated from the observation that when X and Y are jointly log-normally distributed with the same variance, the codependence measure in (1) boils down to the standard Pearson correlation between $\log(X)$ and $\log(Y)$:

$$\varrho_{X,Y} = \frac{cov\left(\log(X), \log(Y)\right)}{Var(\log(X))} \equiv \rho_{\log(X),\log(Y)}. \tag{3}$$

Discussion. We propose an example to document some of the differences between the proposed entropy-based correlation measure and the standard Pearson correlation measure. The example proceeds as follows. We shall consider two sets of bivariate distributions. In one case, we focus on joint-normal distributions with a varying degree of correlation; in the other case, instead, we focus on a mixture of two joint-normal distributions. The two sets of distributions have two things in common: all the moments of the marginal distributions are identical, and, most importantly, the Pearson correlations are exactly the same. We shall make the case, however, that the the high-order codependence terms are very different across the two cases, thus resulting in a different degree of entropy-correlation.

We shall now describe in greater detail the construction of the two sets of distributions. In one case, we draw a sample from a bivariate standard normal distribution, in which the correlation of the random variables is a non negative value that we denote w_1. The four plots in panel (a) of Figure 1 report the contours of the joint probability distribution associated to four cases for w_1. Note that in all four cases the entropy correlation coincides with the standard measure of correlation.

For the case of the multivariate mixture of normals, we draw a random sample (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) according to the probability distribution function

$$p(x, y|\mu, S, w) \quad = \quad (1 - w_1)\phi_1 (x, y|\mu_1, S_1) + w_1\phi_2 (x, y|\mu_2, S_2)$$

where $\phi_1(x, y|\mu_1, S_1)$ and $\phi_1(x, y|\mu_1, S_1)$ are normal probability distribution functions with means μ_1 and μ_2 and covariance matrices S_1 and S_2, respectively; w_1 and $1 - w_1$ are non-negative weights attached to them.

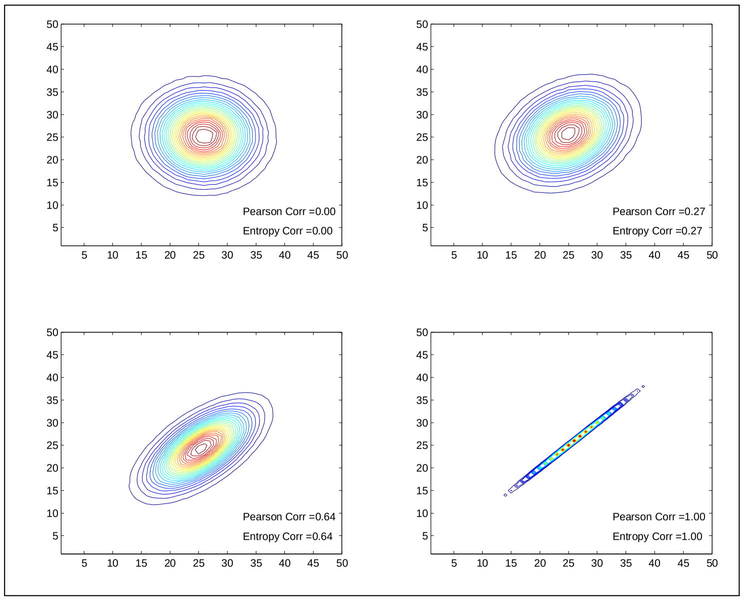
For simplicity, we shall assume that $\mu_1 = \mu_2 = [0, 0]'$ and that

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

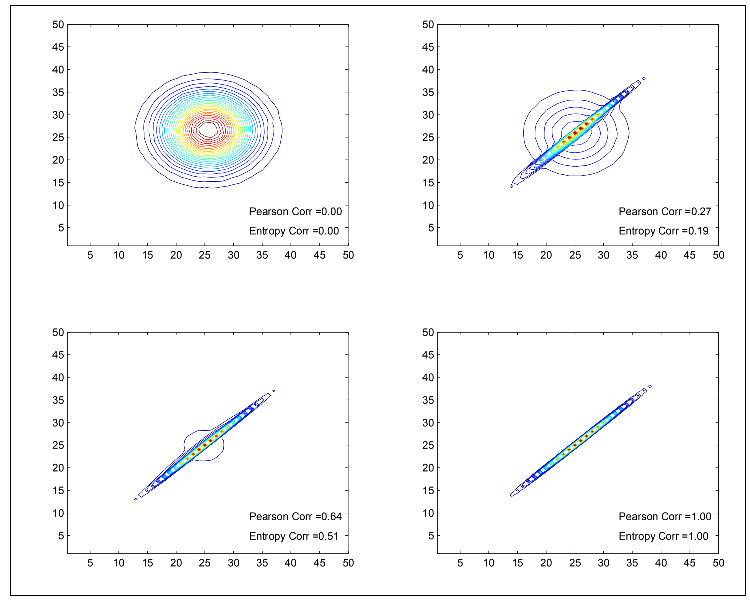
Similarly, we are going to draw a sample from a mixture of two normals: one in which x and y are uncorrelated and one in which x and y are perfectly correlated.

Panel (a) of Figure 1 reports the contours of the joint-normal probability distribution. Note that in all four plots, the Pearson correlation is exactly identical to the corresponding plots in panel (b). Judging by the moments of the marginal distributions and by the degree of Pearson correlations, we would not be able to distinguish the distributions in the two sets of plots.

In panel (b), however, there is a very different extent of non linear codependence between the two random variables. Specifically, it seems to be the case that the extent of codependence in the tails of the distribution is lower in the case of the mixture of normals. This difference is being captured by the entropy-based correlation, which is always below the standard measure of correlation. In the same manner, the entropy-based correlation captures high order codependence terms, whereas the Pearson correlation appears to be



(a) Contours of Normal Distributions



(b) Contours of Mixture of Normal Distributions

Figure 1: An example. Each panel shows the contours of the joint probability density functions of two random variables. The plots in panel (a) refer to the case in which the two random variables are distributed according to a joint normal. The plots in panel (b) refer to the case in which the two random variables are distributed according to a mixture of two normals. Each panel also shows the corresponding values of Pearson and Entropy correlations.

insensitive to it.

This observation is relevant for our analysis because it suggests that our entropy-based correlation is a more comprehensive measure of codependence, one that seems to be robust to non-normalities, a common feature of SDF's in the cross-section of countries.

3 Asset Pricing Restrictions on One-Period Co-Entropies

3.1 Stochastic Discount Factors

Preliminaries and Notation. We start by considering two countries, domestic and foreign, with SDFs M and M^*, respectively. To ease notations, we drop the time subscript from the SDF notation. Lowercase letters indicate logarithms of uppercase letters. We denote by $m = \log(M)$ and $m^* = \log(M^*)$. We then consider the sets of SDFs that correctly price the risk-free bond, the return on the long-term bond, and a generic set of risky assets

$$\begin{aligned} \mathbb{S} &\equiv \left\{M > 0 : E[M] = 1/R^f,\, E[MR_\infty] = 1 \text{ and } E[M\mathbf{R}] = 1\right\}, \\ \mathbb{S}^* &\equiv \left\{M^* > 0 : E[M^*] = 1/R^{*f},\, E[M^*R^*_\infty] = 1 \text{ and } E[M^*\mathbf{R}^*] = 1\right\}. \end{aligned}$$

R and $\mathbf{R}^*$ are the set of risky asset returns for both domestic and foreign countries. R^*_∞ and R_∞ are the gross returns on the long-term bond in both countries. If we further assume that markets are complete, then the growth rate between the currencies of the two countries is uniquely identified as the the ratio of the stochastic discount factors: $\exp(\Delta e) = M^*/M$.

The dispersion in the growth rate of the exchange rate can be decomposed into the sum of foreign and individual SDF entropies, as well as codependence between the foreign and domestic SDFs

$$L\left[\exp\left(\Delta e\right)\right] = \underbrace{L\left[\frac{M^*}{M}\right] - \left(L\left[M^*\right] + L\left[M\right]\right)}_{\text{co-entropy between } M^* \text{ and } M} + \underbrace{\left(L\left[M^*\right] + L\left[M\right]\right)}_{\text{sum of individual entropies}} . \tag{4}$$

Expression (4) clearly shows that there are two sources of dispersion in the growth rate of exchange rates. First, dispersion in the growth rate of exchange rates can be attributed to dispersion in individual entropies. Second, dispersion in the growth rate of exchange rates can be attributed to the entropy-based correlation of M^* and M. Dividing both the left and right hand sides of (4) by the sum of individual entropies and rearranging terms yields the entropy-based codependence between foreign and domestic SDFs:

$$\varrho_{M^*,M} = 1 - \frac{L\left[\exp\left(\Delta e\right)\right]}{\left(L\left[M^*\right] + L\left[M\right]\right)}, \tag{5}$$

which can be expressed in terms of the cumulants of the distribution of Δe as

$$\varrho_{M^*,M} = 1 - \frac{\sum_{j=1}^{\infty} \kappa_j^e / j!}{\sum_{j=1}^{\infty} \kappa_j^M / j! + \sum_{j=1}^{\infty} \kappa_j^{M^*} / j!}, \tag{6}$$

where κ_j^e is the jth cumulant of $\exp(\Delta e)$. κ_j^M is the cumulant of M, and $\kappa_j^{M^*}$ is the cumulant of M^*. For example, the first cumulant κ_1^e is the mean of $\exp(\Delta e)$. The second cumulant κ_2^e is the variance of $\exp(\Delta e)$. The third cumulant κ_3^e is the skewness of $\exp(\Delta e)$, and the fourth cumulant κ_4^e is the excess kurtosis of $\exp(\Delta e)$. Alvarez and Jermann (2005), Martin (2013), Backus, Chernov and Martin (2011) were among the first to use cumulants to explore asset pricing quantities.

A lower bound on co-entropy. We use the idea of Bansal and Lehmann (1997) and Backus, Chernov and Zin (2013) to obtain a bound on $L[M]$ in presence of a single risky asset[2]:

$$L[M] \geq E[\log(R)] - \log(R_f) = E[r_{ex}]. \tag{7}$$

A similar lower bound applies to $L[M^*]$. Then the following is a lower bound on the entropy correlation:

$$\varrho_{M^*,M} \geq \underline{\varrho}_{M^*,M} = 1 - \frac{L[\exp(\Delta e)]}{E[r_{ex}] + E[r_{ex}^*]}.$$

The bound is computable from the time-series of risky asset returns, risk-free rates, and

[2]The lower bound on the co-entropy of the SDFs can be easily extended to allow for multiple risky assets. To generalize it to multiple assets, one needs to use the multiple assets entropy bound on the SDF provided in Bakshi and Chabi-Yo (2013).

exchange rates.

3.2 Decomposing the Entropy-based Correlation

Motivated by recent works in Alvarez and Jermann (2005), Hansen, Heaton and Li (2008), Hansen and Scheinkman (2009), and Hansen (2012), we postulate that any SDF $M \in \mathbb{S}$, and $M^* \in \mathbb{S}^*$ can be decomposed into permanent and transitory components

$$\begin{aligned} M^* &= M^{*P} M^{*T} \quad \text{with} \quad E\left[M^{*P}\right] = 1 \quad \text{and} \quad M^{*T} = 1/R^*_\infty, \\ M &= M^P M^T \quad \text{with} \quad E\left[M^P\right] = 1 \quad \text{and} \quad M^T = 1/R_\infty. \end{aligned}$$

To examine the source of dispersion in $\exp(\Delta e)$, we show in Online Appendix I that the entropy-based correlation measure (6) can be decomposed as

$$\varrho_{M^*,M} = \left(1 - \frac{\Sigma_P}{\Sigma} - \frac{\Sigma_T}{\Sigma}\right) + \frac{\Sigma_P}{\Sigma} \cdot \varrho_{M^{*P},M^P} + \frac{\Sigma_T}{\Sigma} \cdot \varrho_{M^T,M^{*T}} + \frac{\Sigma_{P,T}}{\Sigma} \cdot \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}, \tag{8}$$

where ϱ_{M^{*P},M^P}, $\varrho_{M^T,M^{*T}}$, and $\varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}$ are entropy-based correlations of permanent and transitory components and Σ, Σ_P, Σ_T, and $\Sigma_{P,T}$ are defined as

$$\Sigma = L[M] + L[M^*], \quad \Sigma_P = L\left[M^P\right] + L\left[M^{*P}\right], \quad \Sigma_T = L\left[M^T\right] + L\left[M^{*T}\right], \quad \Sigma_{P,T} = L\left[\frac{M^{*P}}{M^P}\right] + L\left[\frac{M^T}{M^{*T}}\right].$$

Equation (8) shows that our entropy-based correlation can be decomposed into a weighted sum of the entropy correlations of permanent and transitory components of SDFs. This allows us to investigate not only the total amount of codependence between stochastic discount factors but also the source of such codependence.

More specifically, codependence between SDFs can be attributed to codependence in permanent components of SDFs, codependence in transitory components of SDFs, or codependence of the ratios of permanent components and the ratio of transitory components of SDFs. In what follows, we provide new restrictions on the additional codependence terms reported in equation (8).

Co-entropy of Transitory Components. We consider the entropy-based index of transitory components of SDFs across foreign and domestic countries,

$$\varrho_{M^T,M^{*T}} = 1 - \frac{L\left[\frac{M^T}{M^{*T}}\right]}{\left(L\left[M^T\right] + L\left[M^{*T}\right]\right)}, \tag{9}$$

with $M^T = \exp(-r_\infty)$ and $M^{*T} = \exp(-r^*_\infty)$. The entropy-based index (9) can be computed given a time-series of the long-term bond returns in both domestic and foreign countries. Equation (9) summarizes, beyond log normality, codependence between the transitory components in domestic and foreign countries.

Co-entropy of Permanent Components. We consider the entropy-based correlation of permanent components of SDFs across foreign and domestic countries as

$$\varrho_{M^{*P},M^P} = 1 - \frac{L\left[\frac{M^{*P}}{M^P}\right]}{\left(L\left[M^{*P}\right] + L\left[M^P\right]\right)}. \tag{10}$$

The permanent component of the SDF is a key ingredient for pricing assets in the stock market. Under the assumption of complete markets, this index can be expressed as

$$\varrho_{M^{*P},M^P} = 1 - \frac{L\left[\exp\left(\Delta e + r^*_\infty - r_\infty\right)\right]}{\left(L\left[M^{*P}\right] + L\left[M^P\right]\right)}. \tag{11}$$

Using the idea in Alvarez and Jermann (2005), we can obtain a bound on $L\left[M^P\right]$:

$$L[M^P] \geq E\left[\log(R)\right] - E\log(R_\infty) = E[r_{ex,\infty}]. \tag{12}$$

A similar lower bound applies to $L[M^{*P}]$. Then the following is a lower bound on the entropy-based index

$$\varrho_{M^{*P},M^P} \geq 1 - \frac{L\left[\exp\left(\Delta e + r^*_\infty - r_\infty\right)\right]}{E[r_{ex,\infty}] + E[r^*_{ex,\infty}]}. \tag{13}$$

This bound is computable from the time-series asset returns, risk-free rates, and long-term discount bonds across countries, as well as the exchange rates.

Co-Entropy of the Ratio of Permanent and Ratio of Transitory Components. The entropy-based correlation between the ratio of permanent components and the ratio of the transitory components of SDFs is

$$\varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}} = 1 - \frac{L\left[\frac{M^{*P}}{M^P} / \frac{M^T}{M^{*T}}\right]}{L\left[\frac{M^{*P}}{M^P}\right] + L\left[\frac{M^T}{M^{*T}}\right]}. \tag{14}$$

Using an argument similar to the earlier sections, it can be shown that

$$\varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}} = 1 - \frac{L\left[\exp\left(\Delta e\right)\right]}{L\left[\exp\left(\Delta e + r_\infty^* - r_\infty\right)\right] + L\left[\exp\left(r_\infty^* - r_\infty\right)\right]}. \tag{15}$$

This expression can be recovered from a time-series of long-term bond returns and exchange rates. If $\frac{M^{*P}}{M^P}$ and $\frac{M^T}{M^{*T}}$ are highly dependent, (14) will be close to 1.

4 Dynamics in Entropy-Based Correlation

Preliminaries and Notation. In this section, we propose measures to capture dynamics in the entropy-based index. Hansen (2013), Borovicka and Hansen (2013), and Backus et al. (2013) emphasize the importance of evaluating asset pricing models over alternative horizons. Specifically, we are interested in the impact of compounding SDFs, compounding permanent (transitory) components of SDFs over alternative investment horizons on entropy-based correlation.

We consider the sets of SDFs that correctly price the risk-free asset, the long-term discount bound, and a set of risky assets:

$$\mathbb{S}_n = \left\{M_{t,t+n} > 0 : E\left[M_{t,t+n}\right] = \overline{M}_n,\, E\left[M_{t,t+n}R_{\infty,t,t+n}\right] = 1 \text{ and } E\left[M_{t,t+n}\mathbf{R}_{t,t+n}\right] = \mathbf{1}\right\},$$

$$\mathbb{S}_n^* = \left\{M_{t,t+n}^* > 0 : E\left[M_{t,t+n}^*\right] = \overline{M}_n^*,\, E\left[M_{t,t+n}^*R_{\infty,t,t+n}^*\right] = 1 \text{ and } E\left[M_{t,t+n}^*\mathbf{R}_{t,t+n}^*\right] = \mathbf{1}\right\},$$

where $R_{\infty,t,t+n}$ and $R^*_{\infty,t,t+n}$ are returns on long-term discount bonds when bonds are held from t to $t+n$. $\mathbf{R}_{t,t+n}$ and $\mathbf{R}^*_{t,t+n}$ are the set of n-period risky asset returns. Under no arbitrage conditions, the growth rate between currencies of two countries, when the investment horizon is n, is given by

$$\exp\left(\Delta e_{t,t+n}\right) = \frac{M^*_{t,t+n}}{M_{t,t+n}},$$

and the SDFs can be decomposed as a product of one-period SDFs (compounding SDFs):

$$M^*_{t,t+n} = \prod_{i=1}^{n} M^*_{t+i-1,t+i} \text{ and } M_{t,t+n} = \prod_{i=1}^{n} M_{t+i-1,t+i},$$

where $M^*_{t+i-1,t+i}$ and $M_{t+i-1,t+i}$ represent the foreign and domestic stochastic discount factors over the time interval $[t+i-1, t+i]$. Similar to Section 3.2, we postulate that SDFs can be decomposed into permanent and transitory components:

$$\begin{aligned} M^*_{t+i-1,t+i} &= M^{*P}_{t+i-1,t+i} M^{*T}_{t+i-1,t+i}, \\ M_{t+i-1,t+i} &= M^{P}_{t+i-1,t+i} M^{T}_{t+i-1,t+i}, \end{aligned}$$

where

$$M^{*T}_{t+i-1,t+i} = 1/R^*_{\infty,t+i-1,t+i} \text{ and } M^{T}_{t+i-1,t+i} = 1/R_{\infty,t+i-1,t+i}.$$

$R^*_{\infty,t+i-1,t+i}$ and $R_{\infty,t+i-1,t+i}$ are the returns on the long-term bond in foreign and domestic countries. In sections to follow, the lowercase letters are a log of the uppercase letters.

For example, $r_{\infty,t+i-1,t+i} = \log\left(R_{\infty,t+i-1,t+i}\right)$.

Multiperiod Co-Entropy of SDFs. We apply the co-entropy operator to the multi-period stochastic discount factors defined in equation (4) and obtain:

$$\varrho_{M_{t,t+n},M^*_{t,t+n}} = 1 - \frac{L\left[\frac{M^*_{t,t+n}}{M_{t,t+n}}\right]}{\left(L\left[M_{t,t+n}\right] + L\left[M^*_{t,t+n}\right]\right)}.$$

We shall define the n-periods horizon codependence of the entropy-based correlation as the difference between the co-entropies of the n-periods and the one-period stochastic discount factors:

$$H\left[n\right] = \varrho_{M_{t,t+n},M^*_{t,t+n}} - \varrho_{M_{t,t+1},M^*_{t,t+1}}. \tag{16}$$

Co-Entropy of Permanent and Transitory Components. The decomposition of the entropy-based correlation reported in the equation in Section 3.2 also applies to the n-periods stochastic discount factors, after defining the n-periods multi-period entropy indices of permanent and transitory components as

$$\begin{aligned} \varrho_{M^i_{t,t+n},M^{i*}_{t,t+n}} &= 1 - \frac{L\left[\frac{M^{*i}_{t,t+n}}{M^i_{t,t+n}}\right]}{\left(L\left[M^i_{t,t+n}\right] + L\left[M^{*i}_{t,t+n}\right]\right)}, \quad \forall i \in \{P,T\} \\ \varrho_{\frac{M^{*P}_{t,t+n}}{M^P_{t,t+n}},\frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}}} &= 1 - \frac{L\left[\frac{M^{*P}_{t,t+n}}{M^P_{t,t+n}} / \frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}}\right]}{L\left[\frac{M^{*P}_{t,t+n}}{M^P_{t,t+n}}\right] + L\left[\frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}}\right]}. \end{aligned} \tag{17}$$

The definition of horizon codependence also applies to the entropy-based correlations in (17), and we denote them as $H^P[n]$, $H^T[n]$, and $H^{PT}[n]$:

$$H^P[n] = \varrho_{M^P_{t,t+n}, M^{*P}_{t,t+n}} - \varrho_{M^P_{t,t+1}, M^{*P}_{t,t+1}}, \tag{18}$$

$$H^T[n] = \varrho_{M^{*T}_{t,t+n}, M^T_{t,t+n}} - \varrho_{M^{*T}_{t,t+1}, M^T_{t,t+1}}, \tag{19}$$

$$H^{PT}[n] = \varrho_{\frac{M^{*P}_{t,t+n}}{M^P_{t,t+n}}, \frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}}} - \varrho_{\frac{M^{*P}_{t,t+1}}{M^P_{t,t+1}}, \frac{M^T_{t,t+1}}{M^{*T}_{t,t+1}}}. \tag{20}$$

The lower bound on the co-entropy index of the permanent components of the one-period stochastic discount factors in equation (13) generalizes to the permanent components of the n-periods stochastic discount factor. We omit the formula of the lower bound in the interest of space.

Additionally, we show in Online Appendix II that the following upper bound applies to the horizon codependence of the permanent components of SDFs:

$$H^P[n] \leq \frac{L\left[\exp\left(\Delta e_{t,t+1} + r^*_{t,t+1,\infty} - r_{t,t+1,\infty}\right)\right] - \frac{1}{n} L\left[\exp\left(\Delta e_{t,t+n} + r^*_{t,t+n,\infty} - r_{t,t+n,\infty}\right)\right]}{E\left[r_{t,t+1,ex,\infty}\right] + E\left[r^*_{t,t+1,ex,\infty}\right]}, \tag{21}$$

with $E[r_{t,t+1,ex,\infty}] = E\left[r_{t,t+1} - r_{\infty,t,t+1}\right]$ and $E[r^*_{t,t+1,ex,\infty}] = E\left[r^*_{t,t+1} - r^*_{\infty,t,t+1}\right]$. While one may expect the entropy-based correlation of permanent components of SDFs to exhibit dynamics when the investor horizon increases, (21) suggests that we should not expect "too much" dynamics in this co-entropy measure. The maximum amount of dynamics can be estimated from asset prices without making any modeling assumptions about the permanent and transitory components of SDFs.

5 Empirical Analysis

Description of the data. We collected monthly data on stock market returns, risk-free rates, ten-year bond yields, CPI inflation, and exchange rates versus US dollars for 16 countries, in addition to the United States. For Belgium, Canada, France, Germany, Italy, Japan, Spain, Switzerland, United Kingdom, and United States, the sample starts in January 1975 and ends in May 2013. For Austria, the sample starts in June 1989, for Denmark and Finland in January 1987, for Ireland in February 1995, for Denmark in January 1986, Norway in January 1979, and for Sweden in January 1982. Real variables are constructed by dividing nominal variables by realized CPI inflation.

Stock market returns for all countries are value-weighted returns in local currency collected from Kenneth French's website.[3] Risk-free rates are collected as the three-month interest rates on Government Bills. The data source is the International Monetary Fund (International Financial Statistics) for Canada, France, Germany, Italy, Japan, the Netherlands, Spain, the UK, and the US ; for the following countries data are instead collected from the OECD: Austria, Belgium, Denmark, Finland, Ireland, Norway, Sweden, and Switzerland.

Long-term rates are collected as the ten-year interest rates on government bonds. The data source is the International Monetary Fund (International Financial Statistics) for

[3]This dataset is publicly available at `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html#International`.

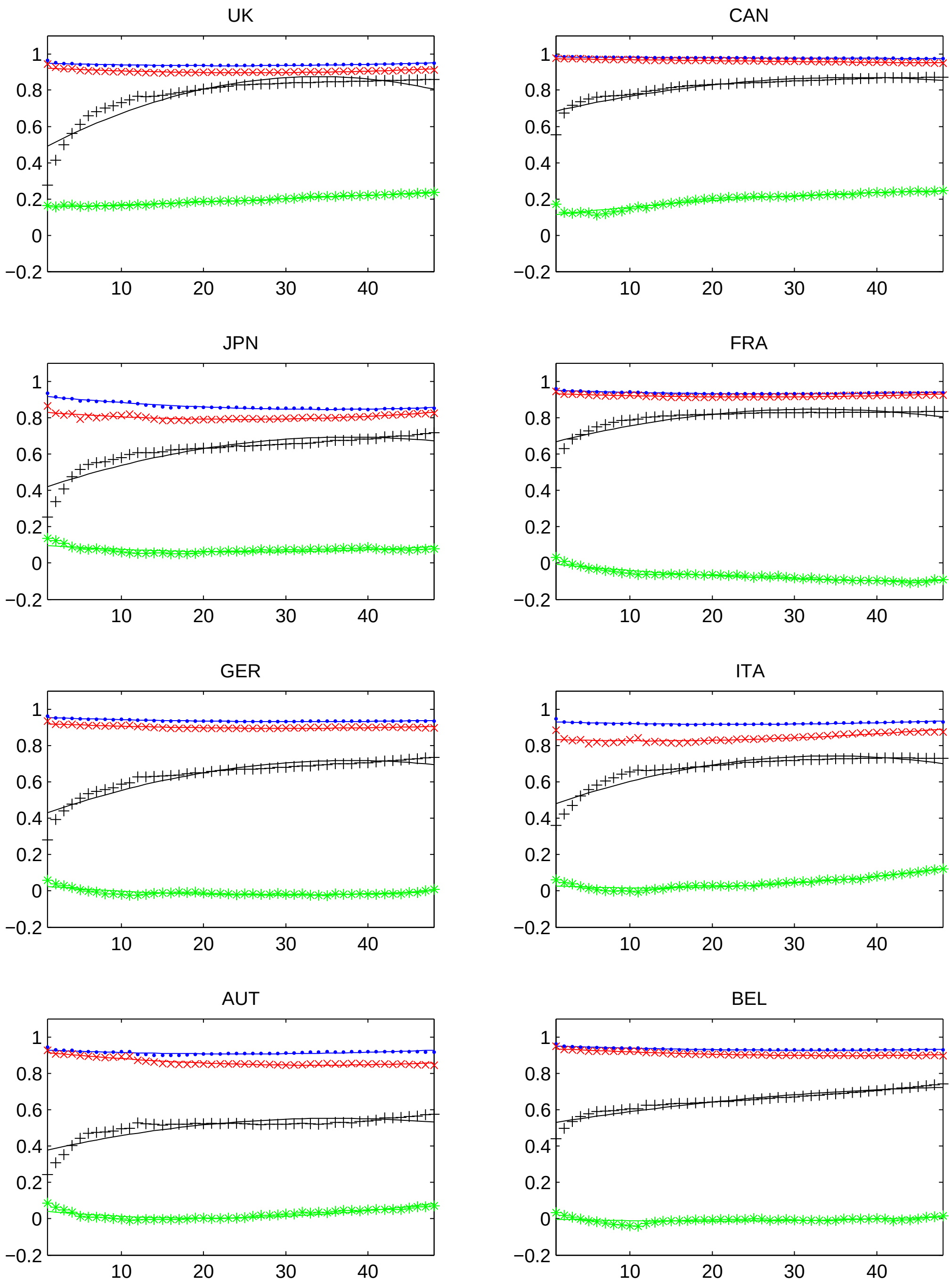
all countries. CPI inflation is computed as the growth rate of the "Total Items" index in two consecutive months. The data source is OECD for all countries. Exchange rates are collected in units of foreign currency per US dollar. The source is the Federal Reserve of St. Louis for all countries, with the exception of Canada, Denmark, Japan, Norway, Sweden, Switzerland, and the United Kingdom, for which the source is the Federal Reserve Board.

Co-entropies in the cross-section of countries. Figure 2 reports the estimated entropy correlations for the 16 countries in our sample. The United States is always assumed to be the home country. Specifically, for each country pair, we report the estimated entropy correlations for horizons ranging between 1 and 48 months. A few consistent findings appear to emerge from this analysis.

The lower bounds on the entropy correlations of the total SDFs (black dots in each panel) are always extremely large. This confirms the findings of Brandt et al. (2006) that SDFs ought to be very correlated across countries to explain the degree of dispersion of exchange rate fluctuations.

Our analysis allows us to dig deeper into the components responsible for such a high degree of codependence of the SDFs. The co-entropy of the permanent components (red crosses) are also very close to 1 across all panels, suggesting that an overwhelming majority of the comovement of SDFs is accounted for by these components.

The entropy-based correlations of the transitory components (black pluses) display the



Continued on next page ↪

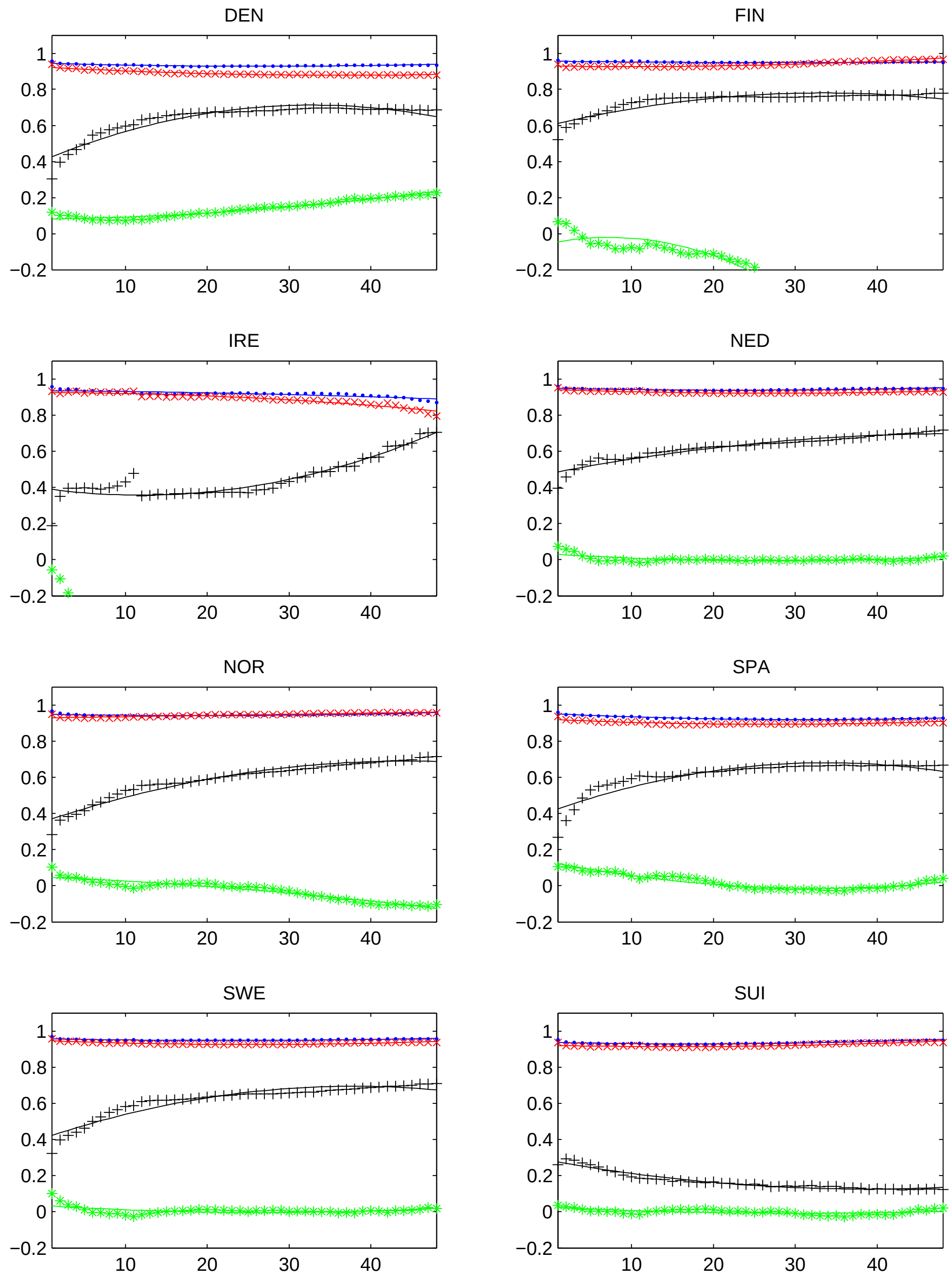


Figure 2: In each panel, the blue dots are the estimated entropy correlations of the SDFs, the red "x" marks are the estimated entropy correlations of the permanent components of the SDFs, the black "+" marks are the estimated entropy correlations of the transitory components of the SDFs, and the green "*" marks are the estimated entropy correlations of the transitory and permanent components. The horizontal axis reports the horizon in months. The lines are second-order polynomials fitted to the estimated correlations of the corresponding colors.

most interesting pattern.[4] They typically start off around 0.4 at relatively shorter horizons, and they steadily increase toward 1 at longer horizons. Similarly, the slope of the term structure (or the horizon codependence) of the co-entropy of the transitory components is consistently positive. This finding is very robust across all countries in our sample, with the only exception being that of Switzerland.

Figure 2 also reports the co-entropies between the ratios of the permanent components and the ratios of the transitory components (green asterisks). These correlations are always around zero and they appear to be quite insensitive to the horizon, except for the United Kingdom and Canada, where the implied co-entropy of the ratio of permanent components and the ratio of transitory components is around 0.2.

Tests. In this section we provide additional statistical evidence supporting the entropy-based correlations reported in Figure 2. Tables 1, 2, 3, and 4 report the estimated correlations along with the 95% bootstrapped confidence intervals. Additionally, the last column of each table reports the difference between the correlation at horizon 48 months and the correlation at 1-month horizon for each country. We interpret these numbers as the horizon codependence at 48 months, or equivalently as the slope of the term structure of correlation. For each of these slopes, we report the p-value associated with the null hypothesis that the slope is less than or equal to zero.

[4]These correlations reflect the extent of co-movement of long-term bonds, a topic recently studied by Jotikasthira, Le and Lundblad (2013). Our findings are also related to the analysis of van Binsbergen, Brandt and Koijen (2012) and van Binsbergen, Hueskes, Koijen and Vrugt (2013) who have documented a similar pattern for the term structure of correlation of equity yields.

The numbers in these tables reinforce the graphical analysis of Figure 2. The lower bounds on the entropy-based correlations of SDFs appear to be very precisely estimated, and we cannot reject the null hypothesis that these correlations are equal to 1, regardless of the country and of the specific horizon. A similar statement applies to the lower bounds on the permanent components of SDFs. Table 1 and Table 2 confirm that for both sets of correlations the term structure is virtually flat, with an upper bound on the horizon codependence of the permanent components very precisely estimated at about 0 (see Table 2).

The situation changes dramatically when it comes to the estimated entropy correlations of the transitory components. Table 3 shows that most of these correlations are statistically different from 1, and the slopes of their term structures are positive, with the exception of Switzerland. In all these cases, we reject the null hypothesis of non positive horizon codependence at conventional levels of confidence. When it comes to the correlations between the ratio of permanent and the ratio of transitory components, Table 4 documents that these correlations are typically undistinguishable from zero.

We further investigate the relationship between the three components of the entropy correlation of the stochastic discount factors. Specifically, in Table 5 we test whether we can reject the null hypothesis that the co-entropy of the transitory components is larger than the lower bound on the co-entropy of the permanent components. The numbers in the table are the differences of correlations (reported as transitory minus permanent) for each horizon and each country pair. The point estimates are all negative, significantly so for most countries at horizons less than 12 months. For horizons longer than one year, we can still reject the null for a majority of the cases, although the result appears to be less and less evident as the horizon increases.

The picture that seems to emerge from this analysis is one in which the correlation of the transitory components increases over time and is significantly smaller than the correlation of the permanent components at the very least for horizons up to one year. These findings are important because they impose very tight restrictions on international asset pricing models as far as the dynamics and the ranking of correlations are concerned.

We conclude this analysis by documenting that the co-entropy between the ratio of permanent and the ratio of the transitory components is always significantly smaller than the lower bound on the correlation of the permanent components. This set of tests is carried out in Table 6.

Table 1: Co-Entropy Indices: Total SDF

	1	**3**	**12**	**24**	**36**	**48**	**Slope**
UK	0.96	0.95	0.93	0.94	0.94	0.95	−0.01
	(0.88, 1.00)	(0.73, 1.00)	(0.77, 1.00)	(0.77, 1.00)	(0.78, 1.00)	(0.79, 1.00)	[0.657]
CAN	0.99	0.98	0.98	0.98	0.98	0.97	−0.01
	(0.93, 1.00)	(0.90, 1.00)	(0.89, 1.00)	(0.92, 1.00)	(0.91, 1.00)	(0.91, 1.00)	[0.747]
JPN	0.94	0.91	0.88	0.86	0.85	0.85	−0.03
	(0.68, 1.00)	(0.55, 1.00)	(0.31, 1.00)	(0.23, 1.00)	(0.23, 1.00)	(0.27, 1.00)	[0.610]
FRA	0.96	0.95	0.94	0.93	0.93	0.94	−0.02
	(0.83, 1.00)	(0.71, 1.00)	(0.71, 1.00)	(0.68, 1.00)	(0.69, 1.00)	(0.72, 1.00)	[0.671]
GER	0.96	0.95	0.94	0.93	0.93	0.93	−0.02
	(0.86, 1.00)	(0.76, 1.00)	(0.76, 1.00)	(0.71, 1.00)	(0.69, 1.00)	(0.70, 1.00)	[0.687]
ITA	0.95	0.93	0.92	0.92	0.92	0.93	−0.02
	(0.72, 1.00)	(0.61, 1.00)	(0.58, 1.00)	(0.59, 1.00)	(0.61, 1.00)	(0.61, 1.00)	[0.611]
AUT	0.94	0.93	0.90	0.91	0.92	0.92	−0.01
	(0.71, 1.00)	(0.67, 1.00)	(0.51, 1.00)	(0.53, 1.00)	(0.56, 1.00)	(0.55, 1.00)	[0.574]
BEL	0.96	0.95	0.94	0.93	0.93	0.93	−0.03
	(0.85, 1.00)	(0.72, 1.00)	(0.74, 1.00)	(0.70, 1.00)	(0.72, 1.00)	(0.73, 1.00)	[0.707]
DEN	0.96	0.94	0.93	0.93	0.93	0.93	−0.02
	(0.77, 1.00)	(0.69, 1.00)	(0.66, 1.00)	(0.63, 1.00)	(0.64, 1.00)	(0.67, 1.00)	[0.623]
FIN	0.96	0.95	0.95	0.95	0.95	0.95	−0.01
	(0.77, 1.00)	(0.74, 1.00)	(0.78, 1.00)	(0.73, 1.00)	(0.73, 1.00)	(0.69, 1.00)	[0.618]
IRE	0.96	0.94	0.92	0.92	0.92	0.87	0.00
	(0.76, 1.00)	(0.73, 1.00)	(0.61, 1.00)	(0.62, 1.00)	(0.60, 1.00)	(−1.00, 1.00)	[0.530]
NED	0.96	0.95	0.94	0.94	0.94	0.95	−0.01
	(0.81, 1.00)	(0.71, 1.00)	(0.72, 1.00)	(0.70, 1.00)	(0.73, 1.00)	(0.74, 1.00)	[0.592]
NOR	0.97	0.95	0.94	0.95	0.95	0.95	−0.01
	(0.84, 1.00)	(0.72, 1.00)	(0.65, 1.00)	(0.68, 1.00)	(0.71, 1.00)	(0.78, 1.00)	[0.615]
SPA	0.96	0.94	0.93	0.92	0.92	0.93	−0.03
	(0.83, 1.00)	(0.71, 1.00)	(0.71, 1.00)	(0.69, 1.00)	(0.65, 1.00)	(0.67, 1.00)	[0.713]
SWE	0.97	0.95	0.95	0.95	0.95	0.96	−0.01
	(0.87, 1.00)	(0.76, 1.00)	(0.79, 1.00)	(0.84, 1.00)	(0.85, 1.00)	(0.86, 1.00)	[0.643]
SUI	0.95	0.94	0.93	0.93	0.94	0.95	−0.00
	(0.82, 1.00)	(0.67, 1.00)	(0.70, 1.00)	(0.72, 1.00)	(0.76, 1.00)	(0.79, 1.00)	[0.504]

Notes - This table reports the estimated co-entropy indices of the total Stochastic Discount Factors. The numbers 1 through 48 on the first row denote the number of months to which each corresponding co-entropy index applies. The label "Slope" denotes the difference between the co-entropy index at the 48-month horizon and the corresponding co-entropy index at the 1-month horizon. The numbers in parentheses denote the bootstrapped 95 percent confidence intervals associated with each co-entropy index. The numbers in brackets denote the bootstrapped p-values associated with the null hypothesis that the slope is less than or equal to zero, one star, two stars, and three stars denote p-values less than 0.10, 0.05, and 0.01, respectively.

Table 2: Co-Entropy Indices: Permanent Component of the SDF

	1	**3**	**12**	**24**	**36**	**48**	**Slope**	**Slope U.B.**
UK	0.94	0.92	0.90	0.90	0.90	0.91	−0.03	−0.0003
	(0.75, 1.00)	(0.55, 1.00)	(0.50, 1.00)	(0.52, 1.00)	(0.53, 1.00)	(0.57, 1.00)	[0.657]	(−0.004, 0.003)
CAN	0.98	0.97	0.96	0.96	0.96	0.95	−0.02	−0.0003
	(0.87, 1.00)	(0.87, 1.00)	(0.81, 1.00)	(0.80, 1.00)	(0.76, 1.00)	(0.73, 1.00)	[0.738]	(−0.005, 0.002)
JPN	0.86	0.82	0.81	0.80	0.80	0.82	−0.00	−0.0003
	(0.33, 1.00)	(0.33, 1.00)	(0.10, 1.00)	(0.08, 1.00)	(0.17, 1.00)	(0.27, 1.00)	[0.557]	(−0.009, 0.007)
FRA	0.94	0.93	0.92	0.91	0.92	0.92	−0.02	−0.0002
	(0.71, 1.00)	(0.65, 1.00)	(0.57, 1.00)	(0.55, 1.00)	(0.58, 1.00)	(0.60, 1.00)	[0.613]	(−0.005, 0.002)
GER	0.93	0.91	0.90	0.89	0.90	0.90	−0.02	−0.0003
	(0.66, 1.00)	(0.58, 1.00)	(0.50, 1.00)	(0.42, 1.00)	(0.46, 1.00)	(0.47, 1.00)	[0.621]	(−0.005, 0.006)
ITA	0.88	0.83	0.82	0.84	0.86	0.87	−0.02	−0.0002
	(0.58, 1.00)	(0.49, 1.00)	(0.30, 1.00)	(0.35, 1.00)	(0.40, 1.00)	(0.41, 1.00)	[0.617]	(−0.005, 0.007)
AUT	0.93	0.91	0.87	0.85	0.85	0.84	−0.02	−0.0002
	(0.62, 1.00)	(0.65, 1.00)	(0.43, 1.00)	(0.38, 1.00)	(0.39, 1.00)	(0.36, 1.00)	[0.599]	(−0.006, 0.008)
BEL	0.95	0.93	0.92	0.90	0.90	0.90	−0.03	−0.0005
	(0.73, 1.00)	(0.58, 1.00)	(0.56, 1.00)	(0.51, 1.00)	(0.46, 1.00)	(0.45, 1.00)	[0.681]	(−0.009, 0.004)
DEN	0.94	0.92	0.90	0.88	0.88	0.88	−0.03	−0.0004
	(0.69, 1.00)	(0.61, 1.00)	(0.47, 1.00)	(0.40, 1.00)	(0.23, 1.00)	(0.22, 1.00)	[0.656]	(−0.008, 0.011)
FIN	0.94	0.92	0.92	0.93	0.95	0.97	0.01	0.0000
	(0.72, 1.00)	(0.70, 1.00)	(0.63, 1.00)	(0.64, 1.00)	(0.67, 1.00)	(0.78, 1.00)	[0.463]	(−0.003, 0.003)
IRE	0.93	0.93	0.90	0.90	0.88	0.80	0.02	0.0002
	(0.72, 1.00)	(0.72, 1.00)	(0.61, 1.00)	(0.62, 1.00)	(0.63, 1.00)	(0.53, 1.00)	[0.446]	(−0.004, 0.008)
NED	0.95	0.93	0.93	0.92	0.93	0.93	−0.01	−0.0002
	(0.76, 1.00)	(0.68, 1.00)	(0.64, 1.00)	(0.58, 1.00)	(0.60, 1.00)	(0.61, 1.00)	[0.601]	(−0.003, 0.003)
NOR	0.95	0.93	0.93	0.95	0.95	0.96	0.00	0.0001
	(0.72, 1.00)	(0.66, 1.00)	(0.63, 1.00)	(0.71, 1.00)	(0.74, 1.00)	(0.81, 1.00)	[0.494]	(−0.002, 0.001)
SPA	0.94	0.92	0.89	0.90	0.90	0.90	−0.03	−0.0004
	(0.69, 1.00)	(0.58, 1.00)	(0.49, 1.00)	(0.48, 1.00)	(0.49, 1.00)	(0.50, 1.00)	[0.650]	(−0.005, 0.005)
SWE	0.96	0.94	0.93	0.93	0.93	0.94	−0.02	−0.0002
	(0.80, 1.00)	(0.69, 1.00)	(0.65, 1.00)	(0.67, 1.00)	(0.70, 1.00)	(0.77, 1.00)	[0.663]	(−0.002, 0.001)
SUI	0.94	0.92	0.91	0.92	0.93	0.94	0.00	0.0000
	(0.70, 1.00)	(0.56, 1.00)	(0.58, 1.00)	(0.62, 1.00)	(0.67, 1.00)	(0.70, 1.00)	[0.489]	(−0.002, 0.003)

Notes - This table reports the estimated co-entropy indices of the permanent components. The numbers 1 through 48 on the first row denote the number of months to which each corresponding co-entropy index applies. The label "Slope" denotes the difference between the co-entropy index at the 48-month horizon and the corresponding co-entropy index at the 1-month horizon.The label Slope U.B. denotes the on the 48-month horizon co-dependence. The numbers in parentheses denote the bootstrapped 95 percent confidence intervals associated with each co-entropy index. The numbers in brackets denote the bootstrapped p-values associated with the null hypothesis that the slope is less than or equal to zero. One star, two stars, and three stars denote p-values less than 0.10, 0.05, and 0.01, respectively.

Table 3: Co-Entropy Indices: Transitory Component of the SDF

	1	**3**	**12**	**24**	**36**	**48**	**Slope**
UK	0.28	0.50	0.77	0.83	0.85	0.86	0.58^{***}
	(0.20, 0.36)	(0.29, 0.67)	(0.59, 0.88)	(0.66, 0.92)	(0.70, 0.94)	(0.70, 0.98)	[0.000]
CAN	0.55	0.72	0.79	0.84	0.86	0.87	0.31^{**}
	(0.47, 0.64)	(0.57, 0.82)	(0.68, 0.88)	(0.70, 0.94)	(0.69, 0.96)	(0.70, 0.99)	[0.013]
JPN	0.25	0.41	0.61	0.64	0.67	0.72	0.45^{**}
	(0.18, 0.34)	(0.23, 0.57)	(0.39, 0.80)	(0.35, 0.88)	(0.25, 0.92)	(0.22, 0.93)	[0.048]
FRA	0.52	0.68	0.80	0.82	0.83	0.84	0.32^{**}
	(0.44, 0.60)	(0.53, 0.80)	(0.68, 0.91)	(0.67, 0.95)	(0.62, 0.97)	(0.60, 1.00)	[0.028]
GER	0.28	0.44	0.63	0.67	0.70	0.74	0.44^{***}
	(0.20, 0.37)	(0.29, 0.58)	(0.49, 0.77)	(0.49, 0.86)	(0.49, 0.94)	(0.51, 0.98)	[0.002]
ITA	0.36	0.47	0.66	0.71	0.73	0.73	0.37^{**}
	(0.26, 0.47)	(0.29, 0.65)	(0.49, 0.81)	(0.52, 0.85)	(0.48, 0.91)	(0.41, 0.95)	[0.032]
AUT	0.24	0.35	0.53	0.52	0.53	0.57	0.32^{**}
	(0.15, 0.33)	(0.16, 0.55)	(0.29, 0.76)	(0.23, 0.87)	(0.22, 0.97)	(0.27, 1.00)	[0.043]
BEL	0.44	0.54	0.62	0.65	0.69	0.74	0.30^{*}
	(0.33, 0.55)	(0.35, 0.70)	(0.42, 0.81)	(0.40, 0.90)	(0.38, 0.93)	(0.43, 0.96)	[0.051]
DEN	0.30	0.44	0.63	0.68	0.70	0.69	0.38^{*}
	(0.16, 0.45)	(0.19, 0.67)	(0.36, 0.83)	(0.33, 0.91)	(0.27, 0.98)	(0.23, 0.99)	[0.080]
FIN	0.52	0.61	0.74	0.76	0.76	0.78	0.27^{**}
	(0.44, 0.60)	(0.41, 0.79)	(0.62, 0.85)	(0.60, 0.90)	(0.60, 0.98)	(0.60, 1.00)	[0.018]
IRE	0.19	0.40	0.35	0.37	0.52	0.70	0.45
	(0.07, 0.30)	(0.08, 0.69)	(−0.02, 0.77)	(−0.08, 0.94)	(−0.12, 1.00)	(−0.14, 1.00)	[0.193]
NED	0.39	0.50	0.59	0.63	0.67	0.72	0.32^{**}
	(0.29, 0.48)	(0.34, 0.65)	(0.40, 0.76)	(0.40, 0.86)	(0.37, 0.91)	(0.39, 0.94)	[0.044]
NOR	0.28	0.38	0.55	0.61	0.67	0.72	0.43^{**}
	(0.20, 0.38)	(0.16, 0.58)	(0.35, 0.73)	(0.36, 0.83)	(0.34, 0.90)	(0.30, 0.95)	[0.035]
SPA	0.27	0.42	0.60	0.65	0.67	0.67	0.40^{**}
	(0.19, 0.34)	(0.21, 0.61)	(0.39, 0.79)	(0.40, 0.84)	(0.41, 0.88)	(0.37, 0.92)	[0.018]
SWE	0.32	0.42	0.61	0.65	0.67	0.71	0.39^{**}
	(0.26, 0.39)	(0.24, 0.60)	(0.45, 0.76)	(0.44, 0.83)	(0.39, 0.92)	(0.31, 0.96)	[0.045]
SUI	0.26	0.29	0.18	0.15	0.13	0.12	−0.10
	(0.18, 0.34)	(0.07, 0.50)	(−0.12, 0.50)	(−0.27, 0.56)	(−0.34, 0.64)	(−0.37, 0.67)	[0.644]

Notes - This table reports the estimated co-entropy indices of the transitory components. The numbers 1 through 48 on the first row denote the number of months to which each corresponding co-entropy index applies. The label “Slope” denotes the difference between the co-entropy index at the 48-month horizon and the corresponding co-entropy index at the 1-month horizon. The numbers in parenthesis denote the bootstrapped 95 percent confidence intervals associated with each co-entropy index. The numbers in brackets denote the bootstrapped p-values associated to the null hypothesis that the slope is less than or equal to zero, one star, two stars, and three stars denote p-values less than 0.10, 0.05, and 0.01, respectively.

Table 4: Co-Entropy Indices: Permanent and Transitory Components of the SDF

	1	**3**	**12**	**24**	**36**	**48**	**Slope**
UK	0.16	0.17	0.17	0.19	0.21	0.24	0.06
	(0.12, 0.22)	(0.07, 0.26)	(0.08, 0.25)	(0.09, 0.28)	(0.07, 0.34)	(0.01, 0.39)	[0.218]
CAN	0.17	0.12	0.15	0.21	0.22	0.25	0.08
	(0.10, 0.24)	(−0.02, 0.26)	(−0.01, 0.31)	(−0.04, 0.42)	(−0.11, 0.50)	(−0.14, 0.61)	[0.308]
JPN	0.14	0.11	0.05	0.07	0.08	0.08	−0.06
	(0.10, 0.17)	(0.04, 0.18)	(−0.07, 0.18)	(−0.12, 0.26)	(−0.18, 0.32)	(−0.33, 0.36)	[0.653]
FRA	0.03	−0.01	−0.06	−0.07	−0.09	−0.09	−0.12
	(0.01, 0.05)	(−0.06, 0.05)	(−0.15, 0.01)	(−0.23, 0.08)	(−0.35, 0.17)	(−0.46, 0.25)	[0.733]
GER	0.06	0.03	−0.02	−0.02	−0.02	0.01	−0.05
	(0.03, 0.09)	(−0.03, 0.09)	(−0.12, 0.07)	(−0.21, 0.16)	(−0.32, 0.24)	(−0.33, 0.35)	[0.623]
ITA	0.06	0.04	0.00	0.03	0.06	0.12	0.06
	(0.03, 0.09)	(−0.05, 0.12)	(−0.14, 0.14)	(−0.20, 0.24)	(−0.29, 0.35)	(−0.30, 0.48)	[0.396]
AUT	0.09	0.05	−0.01	0.00	0.04	0.07	−0.01
	(0.06, 0.11)	(−0.02, 0.12)	(−0.10, 0.08)	(−0.19, 0.19)	(−0.27, 0.34)	(−0.33, 0.46)	[0.525]
BEL	0.03	0.01	−0.03	−0.00	−0.00	0.02	−0.03
	(0.01, 0.06)	(−0.06, 0.08)	(−0.13, 0.08)	(−0.21, 0.21)	(−0.32, 0.29)	(−0.35, 0.38)	[0.563]
DEN	0.12	0.10	0.08	0.13	0.18	0.23	0.10
	(0.06, 0.18)	(0.01, 0.20)	(−0.04, 0.19)	(−0.06, 0.32)	(−0.13, 0.44)	(−0.17, 0.53)	[0.311]
FIN	0.07	0.02	−0.06	−0.16	−0.52	−1.00	−0.93
	(−0.08, 0.21)	(−0.26, 0.30)	(−0.62, 0.32)	(−1.00, 0.31)	(−1.00, 0.34)	(−1.00, 0.26)	[0.882]
IRE	−0.06	−0.18	−0.32	−0.31	−0.47	−0.77	−0.43
	(−0.28, 0.14)	(−0.77, 0.19)	(−1.00, 0.15)	(−1.00, 0.35)	(−1.00, 0.56)	(−1.00, 0.72)	[0.611]
NED	0.07	0.05	−0.01	−0.00	−0.00	0.02	−0.08
	(0.04, 0.10)	(−0.02, 0.11)	(−0.12, 0.09)	(−0.24, 0.19)	(−0.33, 0.31)	(−0.39, 0.41)	[0.651]
NOR	0.10	0.05	−0.01	−0.01	−0.08	−0.11	−0.21
	(0.04, 0.16)	(−0.08, 0.17)	(−0.14, 0.14)	(−0.25, 0.24)	(−0.46, 0.29)	(−0.60, 0.35)	[0.804]
SPA	0.11	0.10	0.05	−0.01	−0.03	0.04	−0.09
	(0.02, 0.18)	(−0.04, 0.21)	(−0.13, 0.20)	(−0.34, 0.25)	(−0.48, 0.36)	(−0.41, 0.48)	[0.651]
SWE	0.10	0.04	−0.02	0.01	−0.01	0.02	−0.08
	(0.05, 0.15)	(−0.06, 0.14)	(−0.11, 0.09)	(−0.19, 0.20)	(−0.32, 0.30)	(−0.37, 0.42)	[0.623]
SUI	0.04	0.02	−0.00	0.00	−0.03	0.02	0.02
	(0.01, 0.06)	(−0.05, 0.09)	(−0.13, 0.12)	(−0.25, 0.21)	(−0.43, 0.31)	(−0.51, 0.55)	[0.470]

Notes - This table reports the estimated co-entropy indices between the ratio of the permanent components and the ratio of the transitory components. The numbers 1 through 48 on the first row denote the number of months to which each corresponding co-entropy index applies. The label Slope denotes the difference between the co-entropy index at the 48-month horizon and the corresponding co-entropy index at the 1-month horizon. The numbers in parenthesis denote the bootstrapped 95 percent confidence intervals associated with each co-entropy index. The numbers in brackets denote the bootstrapped p-values associated with the null hypothesis that the slope is less than or equal to zero, one star, two stars, and three stars denote p-values less than 0.10, 0.05, and 0.01, respectively.

Table 5: Difference of Co-entropy Index of Transitory Components and Co-Entropy Index of Permanent Components

	1	**3**	**12**	**24**	**36**	**48**
UK	-0.66^{**}	-0.43^{**}	-0.14	-0.07	-0.05	-0.04
	[0.016]	[0.044]	[0.137]	[0.216]	[0.277]	[0.304]
CAN	-0.42^{**}	-0.26^{**}	-0.17^{**}	-0.12^{*}	-0.09^{*}	-0.07
	[0.013]	[0.022]	[0.047]	[0.063]	[0.096]	[0.155]
JPN	-0.66^{**}	-0.51^{*}	-0.27	-0.23	-0.20	-0.17
	[0.045]	[0.059]	[0.126]	[0.154]	[0.165]	[0.170]
FRA	-0.42^{**}	-0.26^{*}	-0.12	-0.09	-0.08	-0.08
	[0.026]	[0.056]	[0.136]	[0.186]	[0.208]	[0.221]
GER	-0.66^{**}	-0.50^{**}	-0.28^{*}	-0.23^{*}	-0.20	-0.17
	[0.022]	[0.036]	[0.075]	[0.100]	[0.129]	[0.183]
ITA	-0.58^{**}	-0.47^{**}	-0.25	-0.20	-0.18	-0.18
	[0.033]	[0.049]	[0.114]	[0.119]	[0.132]	[0.140]
AUT	-0.70^{**}	-0.59^{**}	-0.39^{*}	-0.38^{*}	-0.37	-0.33
	[0.025]	[0.028]	[0.066]	[0.084]	[0.105]	[0.141]
BEL	-0.51^{**}	-0.40^{**}	-0.29^{*}	-0.24^{*}	-0.19	-0.14
	[0.023]	[0.044]	[0.067]	[0.094]	[0.131]	[0.179]
DEN	-0.64^{**}	-0.49^{**}	-0.28^{*}	-0.22	-0.19	-0.19
	[0.022]	[0.033]	[0.084]	[0.126]	[0.172]	[0.184]
FIN	-0.44^{**}	-0.34^{**}	-0.20^{*}	-0.18^{*}	-0.18^{*}	-0.17^{*}
	[0.028]	[0.037]	[0.079]	[0.091]	[0.092]	[0.085]
IRE	-0.77^{**}	-0.55^{**}	-0.58^{**}	-0.54^{**}	-0.41^{**}	-0.27^{*}
	[0.017]	[0.025]	[0.032]	[0.037]	[0.047]	[0.071]
NED	-0.55^{**}	-0.44^{**}	-0.33^{**}	-0.27^{*}	-0.24^{*}	-0.20^{*}
	[0.018]	[0.034]	[0.044]	[0.068]	[0.082]	[0.098]
NOR	-0.66^{**}	-0.56^{**}	-0.38^{**}	-0.32^{**}	-0.26^{**}	-0.22^{**}
	[0.020]	[0.027]	[0.042]	[0.035]	[0.042]	[0.049]
SPA	-0.67^{**}	-0.51^{**}	-0.29^{*}	-0.24^{*}	-0.22^{*}	-0.22
	[0.019]	[0.040]	[0.071]	[0.085]	[0.090]	[0.102]
SWE	-0.63^{**}	-0.52^{**}	-0.31^{**}	-0.27^{*}	-0.24^{*}	-0.21^{*}
	[0.014]	[0.026]	[0.047]	[0.056]	[0.074]	[0.095]
SUI	-0.67^{**}	-0.63^{**}	-0.71^{**}	-0.73^{**}	-0.76^{**}	-0.76^{**}
	[0.018]	[0.029]	[0.025]	[0.018]	[0.019]	[0.019]

Notes - This table reports the estimated differences between the co-entropy indices of the transitory components and the co-entropy indices of the permanent components. The first row reports the number of months. The numbers in brackets denote the p-values associated to the null that the difference is positive: one star, two stars, and three stars denote p-values less than 0.10, 0.05, and 0.01, respectively.

6 International Asset Pricing Models

This section describes state-of-the art international asset pricing models. In each model, the economy consists of two countries (*home* and *foreign*). We index variables in the foreign country with a superscript, "*". The setup is identical in foreign and domestic

Table 6: Difference of Co-Entropy Index of Cross Components and Co-Entropy Index of Permanent Components

	1	**3**	**12**	**24**	**36**	**48**
UK	-0.78^{**}	-0.76^{**}	-0.73^{**}	-0.70^{**}	-0.68^{**}	-0.66^{**}
	[0.014]	[0.027]	[0.027]	[0.025]	[0.026]	[0.023]
CAN	-0.80^{***}	-0.85^{***}	-0.81^{**}	-0.74^{**}	-0.71^{**}	-0.68^{**}
	[0.006]	[0.006]	[0.011]	[0.011]	[0.015]	[0.017]
JPN	-0.78^{**}	-0.82^{**}	-0.84^{**}	-0.81^{**}	-0.80^{**}	-0.81^{**}
	[0.038]	[0.039]	[0.049]	[0.049]	[0.046]	[0.041]
FRA	-0.92^{**}	-0.95^{**}	-0.98^{**}	-0.98^{**}	-0.99^{**}	-0.99^{**}
	[0.011]	[0.018]	[0.017]	[0.020]	[0.018]	[0.016]
GER	-0.89^{**}	-0.91^{**}	-0.94^{**}	-0.92^{**}	-0.92^{**}	-0.88^{**}
	[0.016]	[0.021]	[0.023]	[0.027]	[0.026]	[0.025]
ITA	-0.89^{**}	-0.91^{**}	-0.91^{**}	-0.88^{**}	-0.84^{**}	-0.78^{**}
	[0.023]	[0.025]	[0.036]	[0.031]	[0.031]	[0.032]
AUT	-0.86^{**}	-0.90^{**}	-0.94^{**}	-0.91^{**}	-0.87^{**}	-0.82^{**}
	[0.021]	[0.018]	[0.027]	[0.031]	[0.029]	[0.037]
BEL	-0.92^{**}	-0.93^{**}	-0.94^{**}	-0.91^{**}	-0.90^{**}	-0.88^{**}
	[0.013]	[0.019]	[0.019]	[0.021]	[0.027]	[0.028]
DEN	-0.83^{**}	-0.84^{**}	-0.84^{**}	-0.77^{**}	-0.72^{**}	-0.66^{*}
	[0.016]	[0.020]	[0.027]	[0.035]	[0.047]	[0.054]
FIN	-0.89^{**}	-0.91^{**}	-0.97^{**}	-1.06^{**}	-1.36^{**}	-1.83^{***}
	[0.014]	[0.016]	[0.018]	[0.017]	[0.016]	[0.008]
IRE	-1.01^{**}	-1.13^{**}	-1.25^{**}	-1.21^{**}	-1.27^{**}	-1.39^{**}
	[0.013]	[0.013]	[0.015]	[0.017]	[0.023]	[0.039]
NED	-0.88^{**}	-0.90^{**}	-0.93^{**}	-0.92^{**}	-0.92^{**}	-0.92^{**}
	[0.011]	[0.017]	[0.014]	[0.020]	[0.020]	[0.018]
NOR	-0.85^{**}	-0.89^{**}	-0.93^{**}	-0.94^{**}	-1.00^{**}	-1.05^{***}
	[0.015]	[0.016]	[0.016]	[0.011]	[0.010]	[0.008]
SPA	-0.83^{**}	-0.83^{**}	-0.84^{**}	-0.89^{**}	-0.91^{**}	-0.86^{**}
	[0.016]	[0.024]	[0.023]	[0.024]	[0.023]	[0.024]
SWE	-0.85^{***}	-0.90^{**}	-0.93^{**}	-0.91^{**}	-0.92^{**}	-0.89^{***}
	[0.009]	[0.014]	[0.015]	[0.014]	[0.014]	[0.009]
SUI	-0.90^{**}	-0.91^{**}	-0.91^{**}	-0.89^{**}	-0.90^{**}	-0.85^{**}
	[0.013]	[0.021]	[0.019]	[0.012]	[0.013]	[0.014]

Notes - This table reports the estimated differences between the co-entropy indices of the cross components and the co-entropy indices of the permanent components. The first row reports the number of months. The numbers in brackets denote the p-values associated with the null hypothesis that the difference is positive: one star, two stars, and three stars denote p-values less than 0.10, 0.05, and 0.01, respectively.

countries. Due to space considerations, we describe only the domestic side of the economy.

We present the SDF in each international asset pricing models and use the eigenfunction problem of Alvarez and Jermann (2005) and Hansen (2012) to solve for the permanent and transitory components of SDFs. For more details, we refer readers to each model. We keep the same notations as in the original models to facilitate comparisons. The parameter values used are specific to each model and have no analogous meaning to other models. The choice of the models is guided by their success in explaining several features of international asset pricing quantities and puzzles.

The main finding is the same for all models. While they are all successful in accounting for the degree of codependence of the stochastic discount factors (which is what most of these models were designed to do), they all seem to struggle in reproducing the right mix of entropy-based correlations for the permanent and transitory components of SDFs. In general, it seems to be the case that all the models overshoot, in terms of the contribution of the correlation of the transitory components at horizons less than one year, and cannot reproduce the upward sloping term structure of co-entropy of these components.

6.1 A Model with Long-Run Risks

Setup of the economy. This model follows the setup described by Colacito and Croce (2011) and Colacito (2012).[5] Agents order consumption profiles according to the following utility function:

$$U_t = (1-\delta)\log C_t + \delta\theta \log E_t \exp\left\{\frac{U_{t+1}}{\theta}\right\}, \tag{22}$$

[5]More recently, Lewis and Liu (2012), Bansal and Shaliastovich (forthcoming), and Zviadadze (2013) have also built on this international version of the Bansal and Yaron (2004) long-run risks model.

where $\theta = 1/(1-\gamma)$, γ is the coefficient of risk aversion, and δ is the subjective discount factor. The logarithm of the growth rate of consumption follows this process:

$$\begin{aligned} \log \frac{C_{t+1}}{C_t} = \Delta c_{t+1} &= \mu_c + x_t + \sigma_c \varepsilon_{c,t+1}, \\ x_t &= \rho x_{t-1} + \sigma_x \varepsilon_{x,t}, \end{aligned} \tag{23}$$

where $\varepsilon_{c,t+1}$ and $\varepsilon_{x,t+1}$ are jointly normally distributed as independent standard normals. While shocks are orthogonal within each country, we allow them to be correlated across countries. Specifically, we let ρ_{x,x^*} and ρ_{c,c^*} denote the cross-country correlations of the shocks to x and Δc, respectively, and ρ_{c,x^*} denote the correlation between shocks to x in one country and shocks to Δc in the other country.

Stochastic discount factor and its components. We solve the eigenfunction problem of Alvarez and Jermann (2005) and Hansen (2012) and show (see Online Appendix I) that the logarithms of SDF, and its permanent and transitory components can be expressed as follows:

$$\begin{aligned} m_{t+1} &= \log \delta - \mu_c - \frac{B^2 \sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2} - x_t + \frac{B \sigma_x}{\theta} \varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right) \sigma_c \varepsilon_{c,t+1}, \\ m_{t+1}^T &= \log(\zeta) - x_t - \xi \sigma_x \varepsilon_{x,t+1}, \\ m_{t+1}^P &= \left(\frac{B}{\theta} + \xi\right) \sigma_x \varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right) \sigma_c \varepsilon_{c,t+1} - \frac{1}{2}\left(\frac{B}{\theta} + \xi\right)^2 \sigma_x^2 - \frac{1}{2}\left(\frac{1}{\theta} - 1\right)^2 \sigma_c^2, \end{aligned} \tag{24}$$

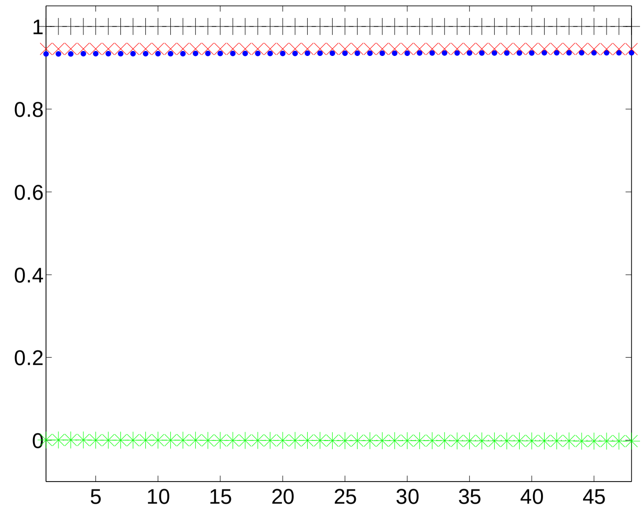
with $B = \delta/(1-\delta\rho)$, $\xi = 1/(\rho-1)$ and

$$\log(\zeta) = \log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2} + \frac{1}{2}\left(\frac{B\sigma_x}{\theta} + \frac{1}{\rho-1}\sigma_x\right)^2 + \frac{1}{2}\left(\frac{1}{\theta}-1\right)^2\sigma_c^2.$$

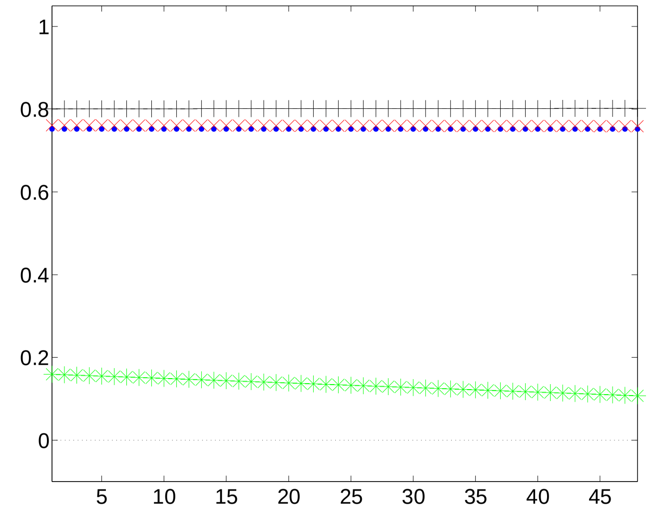
Discussion. Figure 3 reports the results of some alternative calibrations of the international correlations of the shocks. Risk aversion γ is set to 10, while the elasticity of intertemporal substitution, ψ, is equal to 1. All other parameters are set to the same numbers of the original paper. Panel (a) refers to the baseline calibration, according to which the predictive components of consumption growth rates are perfectly correlated across countries. This clearly results in an overstatement of the role of the codependence of the transitory components, while all other entropy-based correlations appear to be in line with the data.

The issue of the excessive codependence of the transitory components in this model can be mitigated by lowering the correlation of long-run risks, while offsetting such reduction with an increase in the correlation between the long-run shocks in one country and the short-run shocks in the other (see panel (c)).[6] This choice of parameters appears to enable the model to replicate the average degrees of correlations across all horizons, but it still falls short of accounting for the upward sloping term structure of co-entropy of the transitory components that we observe in the data.

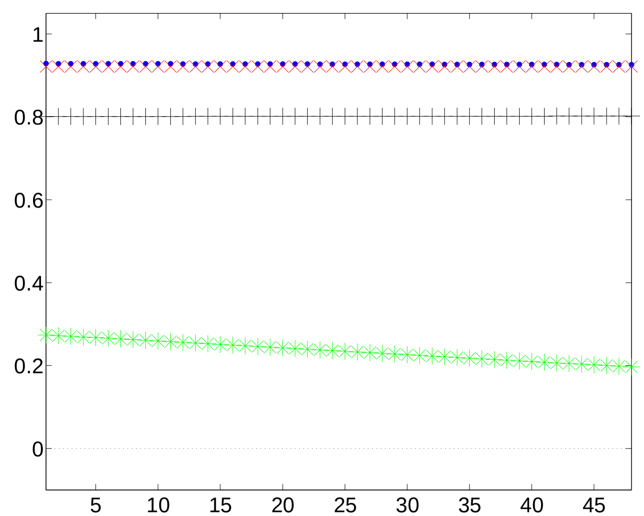
[6]Colacito (2012) also argues in favor of such calibration to better account for the imperfect degree of correlation of interest rates across major industrialized countries.



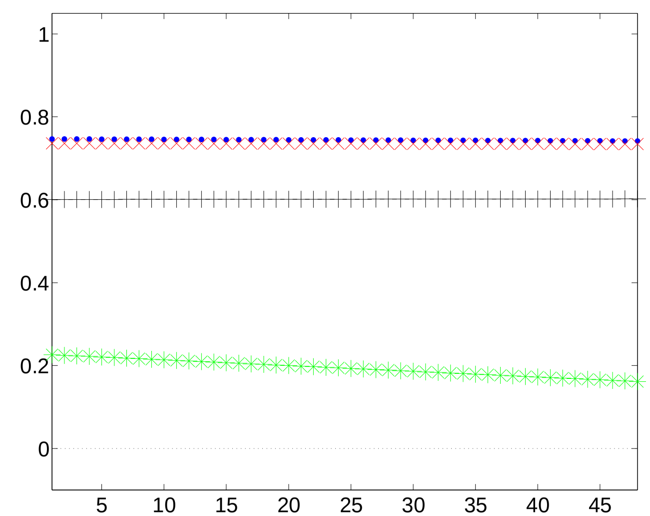
(a) $\rho_{x,x^*} = 1$, $\rho_{c,c^*} = 0.3$, $\rho_{c,x^*} = 0$



(b) $\rho_{x,x^*} = 0.8$, $\rho_{c,c^*} = 0.3$, $\rho_{c,x^*} = 0$



(c) $\rho_{x,x^*} = 0.8$, $\rho_{c,c^*} = 0.3$, $\rho_{c,x^*} = 0.3$



(d) $\rho_{x,x^*} = 0.6$, $\rho_{c,c^*} = 0.30$, $\rho_{c,x^*} = 0.30$

Figure 3: The long-run risks model. In each panel, the blue dots are the entropy correlations of the SDFs, the red “x” marks are the entropy correlations of the permanent components of the SDFs, the black “+” marks are the entropy correlations of the transitory components of the SDFs, and the green “*” marks are the entropy correlations of the transitory and permanent components. The calibrations of the correlations are reported below each panel.

6.2 A Model with Habits

Setup of the Economy. In this section, we consider the two-country model with external habits of Verdelhan (2010). Agents order consumption streams and maximize their expected utility (see also Campbell and Cochrane (1999))

$$E\sum_{t=0}^{\infty}\beta^t\frac{(C_t-H_t)^{1-\gamma}-1}{1-\gamma}, \tag{25}$$

where γ is the risk-aversion coefficient, H_t is the external habit level, and C_t is the agent's consumption. Define $S_t = \frac{C_t-H_t}{C_t}$ as the surplus consumption ratio, and $s_t = \log(S_t)$. The consumption growth is log-normally distributed $\Delta c_{t+1} = g + u_{t+1}$ with $u_{t+1} \backsim N(0, \sigma^2)$. We denote by ρ_{u,u^*} the cross-country correlation of the endowment shocks. The sensitivity function $\lambda(s_t)$ governs the dynamics of the surplus consumption ratio

$$\lambda(s_t) \quad = \quad \tfrac{1}{\overline{S}}\sqrt{1-2(s_t-\overline{s})}-1 \quad \text{, when} \quad s_t \le s_{\max} \quad \text{and} \quad 0 \quad \text{elsewhere} \tag{26}$$

and the dynamic of the surplus consumption ratio is given by

$$s_{t+1} \equiv (1-\phi)\overline{s} + \phi s_t + \lambda(s_t)u_{t+1},$$

with $s_{\max} = \overline{s} + \frac{1}{2}\left(1-\overline{S}^2\right)$.

Stochastic discount factor and its components. In Online Appendix II, we solve the eigenfunction problem and derive the permanent and transitory components of SDFs. The logarithms of the SDF and its permanent and transitory components can be expressed as follows:

$$\begin{aligned} m_{t+1} &= \log(\beta) - \gamma\left[g + (\phi - 1)(s_t - \overline{s}) + (1 + \lambda(s_t))\, u_{t+1}\right], \qquad (27) \\ m_{t+1}^P &= -\frac{1}{2}\gamma^2 - \gamma u_{t+1}, \\ m_{t+1}^T &= \log(\beta) - \gamma g + \frac{1}{2}\gamma^2 + \gamma(1 - \phi)(s_t - \overline{s}) - \gamma\lambda(s_t)\, u_{t+1}. \end{aligned}$$

Discussion. Given that there is only one shock in each country, the whole structure of cross-country correlations is driven by the correlations of consumption shocks. It is apparent that if we calibrate such correlation to produce a large enough codependence of the stochastic discount factors, the model would not be able to replicate the composition of correlation of its components.

6.3 A Model with Rare Disasters

Setup of the Economy. This section describes a two-country version of the model in Barro (2006). Agents order consumption streams using time-additive CRRA preferences. In this model, the representative consumer maximizes a time-additive utility function with isoelastic utility:

$$E_0 \sum_{t=0}^{\infty} \left[e^{-\rho t} \cdot \frac{C_t^{1-\theta} - 1}{1-\theta} \right],$$

where ρ is the rate of time preference and θ is the coefficient of relative risk aversion. The logarithm of consumption evolves as a random walk:

$$\log\left(C_{t+1}\right) = \log\left(C_t\right) + \gamma + u_{t+1} + \upsilon_{t+1}.$$

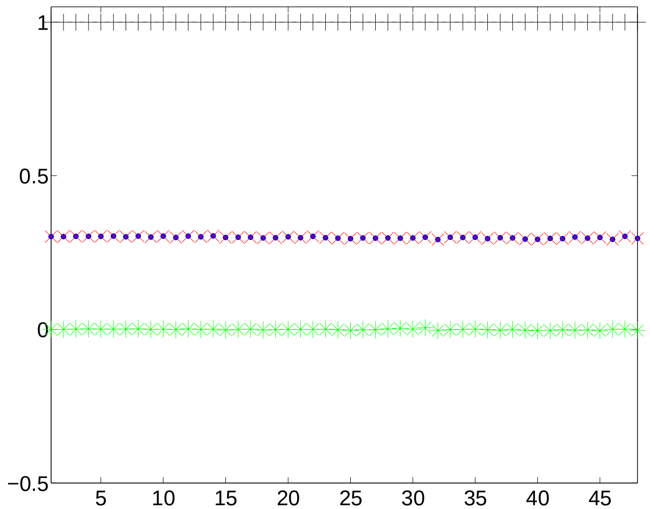
The random variable u_{t+1} is i.i.d normal with mean 0 and variance σ^2. The random variable υ_{t+1} picks up low-probability disasters (see Rietz (1988)). The random variable υ_{t+1} takes values 0 with probability $\exp\left(-p\right)$ and $\log\left(1-b\right)$ otherwise, where b is the size of the downward jump in output.

Stochastic discount factor and its components. We document in Online Appendix II that the logarithms of the SDF and its permanent and transitory components can be expressed as follows:

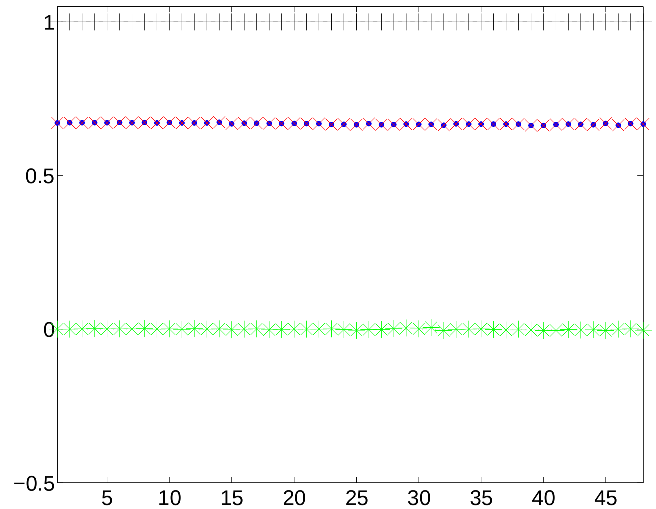
$$\begin{aligned} m_{t+1} &= -\rho - \theta\gamma - \theta u_{t+1} - \theta \upsilon_{t+1}, \qquad (28) \\ m_{t+1}^{T} &= -\theta \log\left(1-b\right) - \rho - \theta\gamma + \frac{1}{2}\theta^2\sigma^2, \\ m_{t+1}^{P} &= -\theta u_{t+1} - \theta \upsilon_{t+1} + \theta \log\left(1-b\right) - \frac{1}{2}\theta^2\sigma^2. \end{aligned}$$

Discussion. Figure 4 reports the entropy correlations for this model. We calibrate all the parameters according to the original paper. Furthermore, we set the international correlation of the non disaster shocks to consumption to 0.3 and the correlation of the disaster shocks to 1. We document the performance of the model for three alternative values of the disaster's intensity.

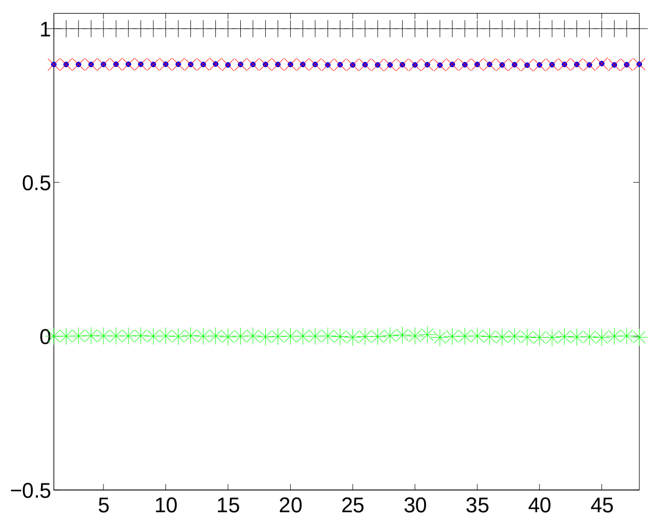
The results are quite intuitive. Since disasters are perfectly correlated across countries, as the consumption loss in case of disaster increases, so does the correlation of the stochastic discount factors. This upward shift is accompanied one for one by an increase in the correlation of the permanent components. Given that growth rates are $i.i.d.$ and homoskedastic, interest rates are constant in the model. This implies that the entropy correlation of the transitory component is always equal to 1 and the entropy correlation of the ratio of permanent and the ratio of transitory components is always equal to 0, regardless of the calibration. In sum, this model cannot overcome the shortcomings already discussed in the context of the earlier models.



(a) $\log(1-b)=0$



(b) $\log(1-b)=0.4$



(c) $\log(1-b)=0.8$

Figure 4: The Rare Events model. In each panel, the blue dots are the entropy correlations of the SDFs, the red "x" marks are the entropy correlations of the permanent components of the SDFs, the black "+" marks are the entropy correlations of the transitory components of the SDFs, and the green "*" marks are the entropy correlations of the transitory and permanent components. The calibrations of the loss in case of disaster are reported below each panel.

6.4 A Model of the Cross-Section of Currency Risk Premia

Setup of the Economy. This section describes the model by Lustig, Roussanov, and Verdelhan (2013).[7] The log SDF in each country is defined as

$$\begin{aligned} -m_{t+1} &= \alpha + \chi z_t + \sqrt{\gamma z_t} u_{t+1} + \tau z_t^{\omega} + \sqrt{\delta z_t^{\omega}} u_{t+1}^{\omega} + \sqrt{\kappa z_t} u_{t+1}^{g} \qquad (29) \\ -m_{t+1}^{*} &= \alpha + \chi z_t^{*} + \sqrt{\gamma z_t^{*}} u_{t+1}^{*} + \tau z_t^{\omega} + \sqrt{\delta^{*} z_t^{\omega}} u_{t+1}^{\omega} + \sqrt{\kappa z_t^{*}} u_{t+1}^{g} \end{aligned}$$

where the three innovations u_{t+1}, u_{t+1}^{ω}, and u_{t+1}^{g} are independent of one another. Moreover, u_{t+1} and u_{t+1}^{*} are independent. The state variables dynamics are given by

$$\begin{aligned} z_{t+1} &= (1-\phi)\,\theta + \phi z_t - \sigma \sqrt{z_t} u_{t+1}, \qquad (30) \\ z_{t+1}^{*} &= (1-\phi)\,\theta + \phi z_t^{*} - \sigma \sqrt{z_t^{*}} u_{t+1}^{*}, \\ z_{t+1}^{\omega} &= (1-\phi^{\omega})\,\theta^{\omega} + \phi^{\omega} z_t^{\omega} - \sigma^{\omega} \sqrt{z_t^{\omega}} u_{t+1}^{\omega}. \end{aligned}$$

Stochastic discount factor and its components. We solve the eigenfunction problem and derive, in Online Appendix II, the permanent and transitory components of the SDF. The log SDF and the log of the permanent and transitory components of the SDF can be expressed as follows:

$$\begin{aligned} m_{t+1} &= -\left(\alpha + \chi z_t + \sqrt{\gamma z_t} u_{t+1} + \tau z_t^{\omega} + \sqrt{\delta z_t^{\omega}} u_{t+1}^{\omega} + \sqrt{\kappa z_t} u_{t+1}^{g}\right), \qquad (31) \\ m_{t+1}^{T} &= \log(\zeta) + \xi_0 \left(z_t - z_{t+1}\right) + \xi_1 \left(z_t^{\omega} - z_{t+1}^{\omega}\right), \\ m_{t+1}^{P} &= m_{t+1} - m_{t+1}^{T}, \end{aligned}$$

with $\log(\zeta)$, ξ_0, and ξ_1 defined in the Appendix.

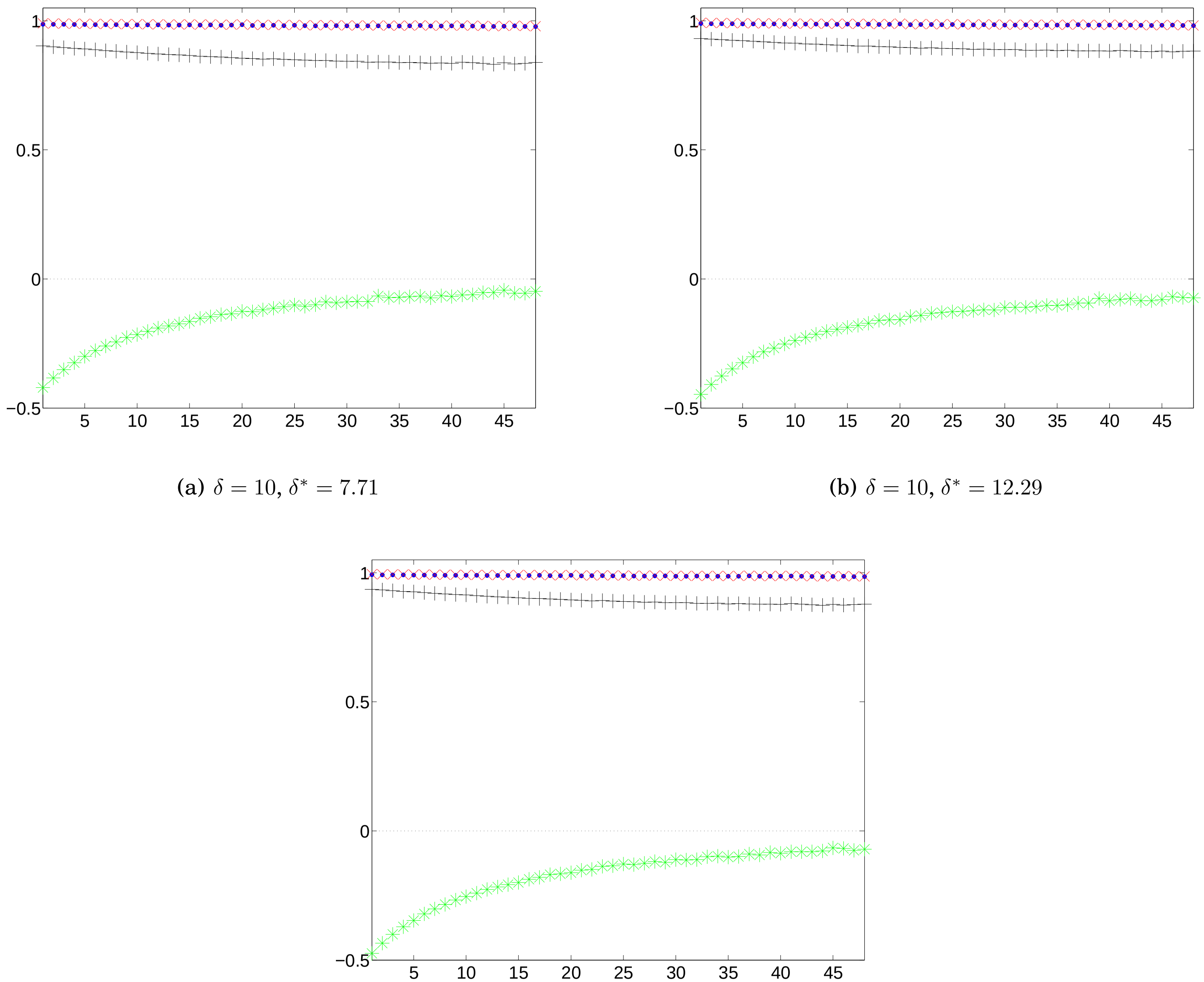
[7] More recently, Ready, Roussanov and Ward (2013) have proposed an additional explanation for the cross-section of currency risk premia.

Discussion. Figure 5 reports the results for this model. All parameters are calibrated according to the original paper. We report three panels in which the coefficient δ^* takes on the bottom, top, and average of the range of values studied by Lustig, Roussanov, and Verdelhan (2013). The results appear to be almost insensitive to the specific choice of this parameter.

We note that this model does an excellent job at matching the degrees of correlation of the stochastic discount factors and their permanent components. While the co-entropy of the transitory components is smaller than the previously mentioned correlations, it is still too large compared to the cross-section of countries that we considered in our empirical analysis.

Additionally, its term structure appears to be slightly downward sloping, contrary to the positive horizon dependence that we typically find in the data. The most striking departure from our empirical evidence is perhaps the strongly upward sloping term structure of entropy correlations of the ratio of permanent and the ratio of transitory components. The very negative correlations at the short end of this term structure are difficult to reconcile with the numbers in Table 4.

This example documents that even a reduced form model that is capable of accounting for several features of the cross-section of currency risk premia and for the degree of volatility of exchange rates' fluctuations has a hard time replicating the decomposition of codependence of permanent and transitory components of the SDFs.



(a) $\delta = 10$, $\delta^* = 7.71$

(b) $\delta = 10$, $\delta^* = 12.29$

(c) $\delta = \delta^* = 10$

Figure 5: The LRV model. In each panel, the blue dots are the entropy correlations of the SDFs, the red "x" marks are the entropy correlations of the permanent components of the SDFs, the black "+" marks are the entropy correlations of the transitory components of the SDFs, and the green "*" marks are the entropy correlations of the transitory and permanent components. The calibrations of the exposures to the world factor are reported below each panel.

6.5 A Potential Resolution: Intertemporal Correlation of the Shocks

The common assumption of the models discussed up to this point is that shocks are $i.i.d.$ across dates. One possible resolution for the shortcomings of the models discussed in this section consists in letting shocks be correlated across time. In this subsection we propose a small amendment of the model with the long-run risks discussed in section 6.1, with the understanding that this generalization could be extended to the other models that we have investigated so far.

Following the notation of section 6.1, we allow for three additional sources of correlation:

$$\begin{aligned}
\rho_{x,x^*}^{lag} &= corr\left(\varepsilon_{x,t}, \varepsilon_{x,t-1}^*\right) = corr\left(\varepsilon_{x,t-1}, \varepsilon_{x,t}^*\right), \\
\rho_{c,c^*}^{lag} &= corr\left(\varepsilon_{c,t}, \varepsilon_{c,t-1}^*\right) = corr\left(\varepsilon_{c,t-1}, \varepsilon_{c,t}^*\right), \\
\rho_{c,x^*}^{lag} &= corr\left(\varepsilon_{c,t}, \varepsilon_{x,t-1}^*\right) = corr\left(\varepsilon_{c,t-1}, \varepsilon_{x,t}^*\right).
\end{aligned}$$

The introduction of these intertemporal correlations does not affect the decomposition of the stochastic discount factors into permanent and transitory components in (24). This is because the additional correlations only involve the cross-country correlations of shocks, while leaving the dynamics within each country unaffected.

Figure 6 presents the results associated with this model. The choice of parameters' values is reported in the note to the figure. The correlations of the shocks are chosen in order to minimize the squared differences between the term structures of co-entropies in the model and in the data for the US and the UK. Note that when the model is allowed to choose the term structure of correlations that best fits the data, it will decrease the contemporaneous correlation of long-run risks and compensate this with an increase in the intertemporal correlation of these shocks across countries. Additionally, note that the correlation between the long-run shocks in one country and the short-run shocks in the other country are quite large.

The resulting term structure of correlations of the transitory components is now sharply upward sloping, in line with what we found in the data. The model now also features positive horizon codependence of the stochastic discount factors and their permanent components. While these appear to be at odds with the empirical evidence reported in section 5, we note from Table 1 and Table 2 that these co-entropies are usually within the 95% confidence intervals, with the only possible exception of very short horizons.

Overall, while this model represents a significant step forward, relative to the other models discussed in this section, there still seems to be a tension between the ability of generating a positive horizon codependence of transitory components and matching the empirical evidence on the virtually flat term structures of all other co-entropies.

We take the simple exercise of this section as suggesting the relevance of modeling intertemporal correlations. From a technical point of view, this appears to be a necessary step toward a better replication of the term structure of co-entropies in international financial markets. In terms of economic modeling, this evidence suggests that more attention should be focused on understanding the mechanism through which shocks

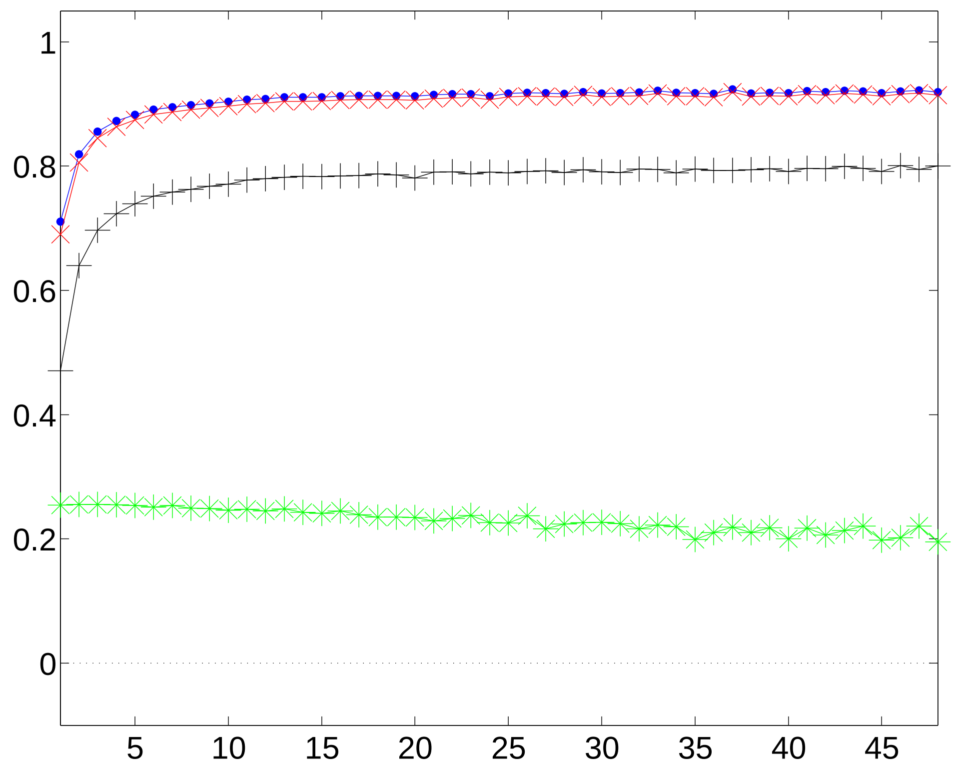


Figure 6: An amended model with Long Run Risks. The blue dots are the entropy correlations of the SDFs, the red "x" marks are the entropy correlations of the permanent components of the SDFs, the black "+" marks are the entropy correlations of the transitory components of the SDFs, and the green "*" marks are the entropy correlations of the transitory and permanent components. The correlations are set to $\rho_{c,c^*} = 0.372$, $\rho_{x,x^*} = 0.473$, $\rho_{c,x^*} = 0.421$, $\rho^{lag}_{x,x^*} = 0.170$, $\rho^{lag}_{c,c^*} = -0.045$, and $\rho^{lag}_{c,x^*} = -0.069$. They are chosen to minimize the squared differences between the term structures of co-entropies in the model and in the data for the US and the UK.

propagate across countries through time. We consider this an important challenge for future research in this field.

7 Concluding Remarks

The common feature across the models that we analyzed appears to be the general ability of accounting for the large degree of co-entropy (codependence) of the stochastic discount factors, whether coming from a high correlation of long-run risks, a common disaster state, exposure to a world risk factor, etc. Our analysis reveals that, while this major task can be accomplished through a variety of models, reproducing the *composition* of co-entropy is still very difficult for these state-of-the-art models.

In particular, it seems to be the case that economic models rely heavily on the correlation of transitory components. If, on the one hand, this hypothesis cannot be rejected at longer horizons, on the other hand, it is hard to reconcile with our empirical evidence from a large cross-section of major industrialized countries.

Perhaps even more interesting are the level and the term structure of co-entropy of transitory components across countries. The data reveal that transitory shocks display a significant and increasing degree of codependence through time, a robust finding for an overwhelming majority of developed countries in our sample. Existing international asset pricing models feature a very high level of co-entropy and a flat term structure of co-entropies of transitory components across countries. In addition, some models are not able to reproduce level and slope of the term structure of the co-entropy between the ratio of permanent components across countries and the ratio of transitory components.

Taken together, these findings seem to suggest that future research should devote more attention to the temporal distribution of correlations. Our analysis reveals the need of paying more attention to the intertemporal distribution of co-movement between various components of the SDFs across countries, as opposed to the contemporaneous co-movement of shocks. While the former has typically been neglected in existing models, the latter has been carrying all the weight to deliver a high degree of correlation co-movement of the stochastic discount factors.

This is an important direction for future research, as it requires a deeper understanding about the way in which shocks of different nature are shared internationally across time and about the timing with which technological, preference, and endowment shocks spread across countries. General equilibrium models of the likes of Pavlova and Rigobon (2007) are now faced with a tighter set of constraints, which will perhaps allow us to gain a deeper understanding of the sources of risk in international macro-finance models.

References

Alvarez, Fernando and Urban Jermann, "Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth," *Econometrica*, 2005, *73*, 1977–2016.

Backus, D, M Chernov, and I Martin, "Disasters implied by equity index options," *Journal of Finance*, 2011, *66*, 1969–2012.

Backus, David, Mikhail Chernov, and Stanley Zin, "Sources of entropy in representative agent models," *Journal of Finance (forthcoming)*, 2013.

Bakshi, Gurdip and Fousseni Chabi-Yo, "Variance bounds on the permanent and transitory components of stochastic discount factors," *Journal of Financial Economics*, 2012, *105*, 191–208.

____ **and** ____ , "New Entropy Restrictions and the Quest for Better Specified Asset Pricing Models," Working paper, Ohio State University 2013.

Bansal, R. and Ivan Shaliastovich, "A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets," *Review of Financial Studies*, forthcoming.

Bansal, Ravi and Amir Yaron, "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 2004, *59*, 1481–1509.

____ **and Bruce Lehmann**, "Growth-Optimal Portfolio Restrictions on Asset Pricing Models," *Macroeconomic Dynamics*, 1997, *1*, 333–354.

Barro, Robert, "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 2006, *121* (9), 823–866.

Borovicka, Jaroslav and Lars Hansen, "Examining Macroeconomic Models through the Lens of Asset Pricing?," *Journal of Econometrics (forthcoming)*, 2013.

Brandt, Michael, John Cochrane, and Pedro Santa-Clara, "International Risk Sharing is Better Than You Think, or Exchange Rates are Too Smooth," *Journal of Monetary Economics*, 2006, *53*, 671–698.

Campbell, J. and J. Cochrane, "Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 1999, *107*, 205–251.

Colacito, Riccardo, "Six anomalies looking for a model. A consumption based explanation of international finance puzzles.," *Working Paper, Department of Finance, University of North Carolina, Chapel Hill NC*, 2012.

____ **and Mariano M. Croce**, "Risks for the long run and the real exchange rate," *Journal of Political Economy*, 2011, *19(1)*, 223–250.

Gavazzoni, F., B. Sambalaibat, and C. Telmer, "Currency Risk and Pricing Kernel Volatility," *Carnegie Mellon University and INSEAD Working Paper*, 2013.

Hansen, L and J Scheinkman, "Long-Term Risk: An Operator Approach," *Econometrica*, 2009, *77*, 177–234.

____ **, J Heaton, and N Li**, "Consumption Strikes Back? Measuring Long-Run Risk," *Journal of Political Economy*, 2008, *116*, 260–302.

Hansen, Lars, "Dynamic value decomposition in stochastic economies," *Econometrica*, 2012, *80*, 911–967.

——— , *Handbook of the Economics of Finance*, New York, NY: North-Holland, 2013.

——— **and Ravi Jagannathan**, "Implications of Security Market Data for Dynamic Economies," *Journal of Political Economy*, 1991, *99*, 225–261.

Jotikasthira, Pab, Anh Le, and Chris Lundblad, "Why Do Term Structures in Different Currencies Comove?," *Working Paper UNC Chapel Hill*, 2013.

Koijen, Ralph, Hanno Lustig, and Stijn Van Nieuwerburgh, "The Cross-Section and Time-Series of Stock and Bond Returns," Unpublished Working Paper, NBER # 15688 2010.

Lewis, Karen and Edith Liu, "International Consumption Risk is Shared After All: An Asset Return View," *Working Paper, University of Pennsylvania and Cornell University*, 2012.

Lustig, Hanno, Andreas Stathopoulos, and Adrien Verdelhan, "The Term Structure of Currency Carry Trade Risk Premia," *MIT, UCLA, and USC Working Paper*, 2013.

——— **, Nikolai Roussanov, and Adrien Verdelhan**, "Countercyclical Currency Risk Premia," *Journal of Financial Economics (forthcoming)*, 2013.

Martin, Ian, "Consumption-Based Asset Pricing with Higher Cumulants," *Review of Economic Studies*, 2013, *80*, 745–773.

Pavlova, Anna and Roberto Rigobon, "Asset Prices and Exchange Rates," *Review of Financial Studies*, 2007, *20(4)*, 1139–1181.

Ready, Robert, Nick Roussanov, and Colin Ward, "Commodity Trade and the Carry Trade: a Tale of Two Countries," *Working Paper, University of Pennsylvania and Rochester University*, 2013.

Rietz, T, "The equity risk premium: A solution," *Journal of Monetary Economics*, 1988, *22*, 117–131.

van Binsbergen, Jules H., Michael W. Brandt, and Ralph Koijen, "On the Timing and Pricing of Dividends," *American Economic Review*, 2012, *102(4)*, 1596–1618.

——— **, Wouter Hueskes, Ralph Koijen, and Evert B. Vrugt**, "Equity Yields," *Journal of Financial Economics*, 2013, p. forthcoming.

Verdelhan, Adrien, "A Habit-Based Explanation of the Exchange Rate Risk Premium," *Journal of Finance, Vol. 65, No 1, pp 123-145*, 2010.

Zviadadze, Irina, "Term-Structure of Consumption Risk Premia in the Cross-Section of Currency Returns," *Working Paper Stockholm School of Economics*, 2013.

Online Appendix I: Proofs and Decomposition

Decomposition of the dispersion in the growth rate of exchange rate as the weighted sum of co-entropies. We first observe that the growth rate of the exchange rate is

$$\exp(\Delta e) = \frac{M^{*P}}{M^P} / \frac{M^T}{M^{*T}}. \qquad \text{(A1)}$$

Second, we note that the entropy of $L\left[\exp\left(\Delta e\right)\right]$ can be decomposed as

$$\begin{aligned} L\left[\exp\left(\Delta e\right)\right] &= L\left[\frac{M^{*P}}{M^P} / \frac{M^T}{M^{*T}}\right] - \left(L\left[\frac{M^{*P}}{M^P}\right] + L\left[\frac{M^T}{M^{*T}}\right]\right) \\ &+ \left(L\left[\frac{M^{*P}}{M^P}\right] - \left(L\left[M^{*P}\right] + L\left[M^P\right]\right)\right) \\ &+ \left(L\left[\frac{M^T}{M^{*T}}\right] - \left(L\left[M^T\right] + L\left[M^{*T}\right]\right)\right) \\ &+ \left(L\left[M^{*P}\right] + L\left[M^P\right]\right) + \left(L\left[M^{*T}\right] + L\left[M^T\right]\right). \end{aligned} \qquad \text{(A2)}$$

After defining

$$\Sigma = L\left[M\right] + L\left[M^*\right], \quad \Sigma_P = L\left[M^P\right] + L\left[M^{*P}\right], \quad \Sigma_T = L\left[M^T\right] + L\left[M^{*T}\right]$$

and

$$\Sigma_{P,T} = L\left[\frac{M^{*P}}{M^P}\right] + L\left[\frac{M^T}{M^{*T}}\right],$$

it follows that

$$\begin{aligned} \varrho_{M^*,M} &= 1 - \frac{L\left[\frac{M^*}{M}\right]}{\left(L\left[M^*\right] + L\left[M\right]\right)} \\ &= 1 - \frac{1}{\Sigma}\left(\Sigma_P + \Sigma_T - \Sigma_P \cdot \varrho_{M^{*P},M^P} - \Sigma_T \cdot \varrho_{M^T,M^{*T}} - \Sigma_{P,T} \cdot \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}}\right) \\ &= 1 - \frac{\Sigma_P + \Sigma_T}{\Sigma} + \frac{\Sigma_P}{\Sigma} \cdot \varrho_{M^{*P},M^P} + \frac{\Sigma_T}{\Sigma} \cdot \varrho_{M^T,M^{*T}} + \frac{\Sigma_{P,T}}{\Sigma} \cdot \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}} \\ &= \left(1 - \frac{\Sigma_P}{\Sigma} - \frac{\Sigma_T}{\Sigma}\right) + \frac{\Sigma_P}{\Sigma} \cdot \varrho_{M^{*P},M^P} + \frac{\Sigma_T}{\Sigma} \cdot \varrho_{M^T,M^{*T}} + \frac{\Sigma_{P,T}}{\Sigma} \cdot \varrho_{\frac{M^{*P}}{M^P},\frac{M^T}{M^{*T}}} \end{aligned} \qquad \text{(A3)}$$

Bound on the Entropy-Based Correlation of SDFs. Notice that

$$\begin{aligned}
\varrho_{M^*,M} &= 1-\frac{L\left[\frac{M^*}{M}\right]}{L\left[M^*\right]+L\left[M\right]} \\
&= 1-\frac{L\left[\exp\left(\Delta e\right)\right]}{L\left[M^*\right]+L\left[M\right]} \text{ (since } \exp\left(\Delta e\right)=\frac{M^*}{M}\text{)} \\
&\geq 1-\frac{L\left[\exp\left(\Delta e\right)\right]}{\left(E\left[r_{ex}\right]+E\left[r_{ex}^*\right]\right)} \text{ (since } L\left[M\right]\geq E\left[r_{ex}\right] \text{ and } L\left[M^*\right]\geq E\left[r_{ex}^*\right]\text{)},
\end{aligned} \tag{A4}$$

where

$$\begin{aligned}
E\left[r_{ex}\right] &= E\left[\log\left(R\right)\right]-\log\left(R_f\right), \\
E\left[r_{ex}^*\right] &= E\left[\log\left(R^*\right)\right]-\log\left(R_f^*\right).
\end{aligned} \tag{A5}$$

Bound on the Entropy-Based Correlation of the Permanent Component SDFs. Notice that

$$\begin{aligned}
\varrho_{M^{*P},M^P} &= 1-\frac{L\left[\frac{M^{*P}}{M^P}\right]}{L\left[M^{*P}\right]+L\left[M^P\right]} \\
&= 1-\frac{L\left[\frac{M^*}{M}/\frac{M^{*T}}{M^T}\right]}{L\left[M^{*P}\right]+L\left[M^P\right]} \text{ (since } \Delta e=\frac{M^*}{M}=\frac{M^{*P}}{M^P}\frac{M^{*T}}{M^T}\text{)} \\
&= 1-\frac{L\left[\exp\left(\Delta e-r_\infty+r_\infty^*\right)\right]}{L\left[M^{*P}\right]+L\left[M^P\right]} \text{(since } \frac{M^{*T}}{M^T}=\exp\left(r_\infty-r_\infty^*\right)\text{)} \\
&\geq 1-\frac{L\left[\exp\left(\Delta e-r_\infty+r_\infty^*\right)\right]}{\left(E\left[r_{ex,\infty}\right]+E\left[r_{ex,\infty}^*\right]\right)},
\end{aligned} \tag{A6}$$

since $L\left[M^P\right]\geq E\left[r_{ex,\infty}\right]$, $L\left[M^{*P}\right]\geq E\left[r_{ex,\infty}^*\right]$.

Entropy-Based Correlation of the Transitory Component SDFs. Notice that

$$\begin{aligned}
\varrho_{M^T,M^{T*}} &= 1-\frac{L\left[\frac{M^T}{M^{*T}}\right]}{L\left[M^{*T}\right]+L\left[M^T\right]} \\
&= 1-\frac{L\left[\exp\left(r_\infty^*-r_\infty\right)\right]}{L\left[\exp\left(-r_\infty^*\right)\right]+L\left[\exp\left(-r_\infty\right)\right]},
\end{aligned} \tag{A7}$$

since $M^{*T}=\exp\left(-r_\infty^*\right)$, $M^T=\exp\left(-r_\infty\right)$.

Entropy-Based Correlation of the Ratio of Permanent and the Ratio of the Transitory Components SDFs. The entropy-based correlation between the ratio of permanent components and the ratio of the transitory components of SDFs is

$$\varrho_{\frac{M^{*P}}{M^{P}},\frac{M^{T}}{M^{*T}}} = 1 - \frac{L\left[\frac{M^{*P}}{M^{P}}/\frac{M^{T}}{M^{*T}}\right]}{L\left[\frac{M^{*P}}{M^{P}}\right] + L\left[\frac{M^{T}}{M^{*T}}\right]}, \tag{A8}$$

$$\varrho_{\frac{M^{*P}}{M^{P}},\frac{M^{T}}{M^{*T}}} = 1 - \frac{L\left[\frac{M^{*P}}{M^{P}}/\frac{M^{T}}{M^{*T}}\right]}{L\left[\frac{M^{*P}}{M^{P}}\right] + L\left[\frac{M^{T}}{M^{*T}}\right]}.$$

We observe that

$$\frac{M^{*P}}{M^{P}}/\frac{M^{T}}{M^{*T}} = \exp(\Delta e) \quad \textbf{and} \quad \frac{M^{*P}}{M^{P}} = \exp(\Delta e)/\frac{M^{*T}}{M^{T}} \quad \textbf{and} \quad \frac{M^{T}}{M^{*T}} = \exp(r^{*}_{\infty} - r_{\infty}), \tag{A9}$$

and hence

$$\varrho_{\frac{M^{*P}}{M^{P}},\frac{M^{T}}{M^{*T}}} = 1 - \frac{L[\exp(\Delta e)]}{L[\exp(\Delta e + r^{*}_{\infty} - r_{\infty})] + L[\exp(r^{*}_{\infty} - r_{\infty})]}. \tag{A10}$$

This ends the proof.

Proof of the Upper Bound on Horizon Codependence in Entropy Correlation of Permanent Components of SDFs. We first assume a stationary environment:

$$E\left[\log\left(M^{P}_{t+i-1,t+i}\right)\right] = E\left[\log\left(M^{P}_{t+i,t+i+1}\right)\right] \ \forall i = 1, ..., n-1. \tag{A11}$$

Using the definition of the n-period entropy-based index of permanent components, we have:

$$\begin{aligned}
\varrho_{M^{P*}_{t,t+n},M^{P}_{t,t+n}} &= \frac{\left(L\left[M^{P}_{t,t+n}\right] + L\left[M^{*P}_{t,t+n}\right]\right) - L\left[\frac{M^{*P}_{t,t+n}}{M^{P}_{t,t+n}}\right]}{\left(L\left[M^{P}_{t,t+n}\right] + L\left[M^{*P}_{t,t+n}\right]\right)} \\
&= \frac{-nE\left[\log\left(M^{P}_{t,t+1}\right)\right] - nE\left[\log\left(M^{*P}_{t,t+1}\right)\right] - L\left[\frac{M^{*P}_{t,t+n}}{M^{P}_{t,t+n}}\right]}{-nE\left[\log\left(M^{P}_{t,t+1}\right)\right] - nE\left[\log\left(M^{*P}_{t,t+1}\right)\right]} \\
&= \frac{E\left[\log\left(M^{P}_{t,t+1}\right)\right] + E\left[\log\left(M^{*P}_{t,t+1}\right)\right] + \frac{1}{n}L\left[\frac{M^{*P}_{t,t+n}}{M^{P}_{t,t+n}}\right]}{E\left[\log\left(M^{P}_{t,t+1}\right)\right] + E\left[\log\left(M^{*P}_{t,t+1}\right)\right]},
\end{aligned} \tag{A12}$$

since $L\left[M^P_{t,t+1}\right] = -E\left[\log\left(M^P_{t,t+1}\right)\right]$ and $L\left[M^{*P}_{t,t+1}\right] = -E\left[\log\left(M^{*P}_{t,t+1}\right)\right]$. The one-period entropy-based correlation of permanent components of SDFs is

$$\varrho_{M^{P*}_{t,t+1},M^P_{t,t+1}} = \frac{E\left[\log\left(M^P_{t,t+1}\right)\right] + E\left[\log\left(M^{*P}_{t,t+1}\right)\right] + L\left[\frac{M^{*P}_{t,t+1}}{M^P_{t,t+1}}\right]}{E\left[\log\left(M^P_{t,t+1}\right)\right] + E\left[\log\left(M^{*P}_{t,t+1}\right)\right]}. \tag{A13}$$

The horizon codependence in the entropy-based correlation of permanent components is the difference between n-period and one-period entropy-based correlation of permanent components of SDFs.

$$\begin{aligned} H^P[n] &= \varrho_{M^{P*}_{t,t+n},M^P_{t,t+n}} - \varrho_{M^{P*}_{t,t+1},M^P_{t,t+1}} \\ &= \frac{\frac{1}{n}L\left[\frac{M^{*P}_{t,t+n}}{M^P_{t,t+n}}\right] - L\left[\frac{M^{*P}_{t,t+1}}{M^P_{t,t+1}}\right]}{E\left[\log\left(M^P_{t,t+1}\right)\right] + E\left[\log\left(M^{*P}_{t,t+1}\right)\right]} \\ &= \frac{L\left[e_{t,t+1}\frac{M^T_{t,t+1}}{M^{*T}_{t,t+1}}\right] - \frac{1}{n}L\left[e_{t,t+n}\frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}}\right]}{L\left[M^P_{t,t+1}\right] + L\left[M^{*P}_{t,t+1}\right]}. \end{aligned} \tag{A14}$$

Denote by

$$E\left[r_{t,t+1,ex,\infty}\right] = E\left[r_{t,t+1} - r_{\infty,t,t+1}\right] \text{ and } E\left[r^*_{t,t+1,ex,\infty}\right] = E\left[r^*_{t,t+1} - r^*_{\infty,t,t+1}\right]. \tag{A15}$$

Using Alvarez and Jermann (2005), it can be shown that $L\left[M^P_{t,t+1}\right] \geq E\left[r_{t,t+1,ex,\infty}\right]$ and $L\left[M^{*P}_{t,t+1}\right] \geq E\left[r^*_{t,t+1,ex,\infty}\right]$, allowing us to derive an upper bound on the horizon codependence of permanent components of SDFs.

$$H^P[n] \leq \frac{L\left[\exp\left(\Delta e_{t,t+1}\right)\frac{M^T_{t,t+1}}{M^{*T}_{t,t+1}}\right] - \frac{1}{n}L\left[\exp\left(\Delta e_{t,t+n}\right)\frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}}\right]}{E\left[r_{t,t+1,ex,\infty}\right] + E\left[r^*_{t,t+1,ex,\infty}\right]}. \tag{A16}$$

Observe that

$$\frac{M^T_{t,t+n}}{M^{*T}_{t,t+n}} = \exp\left(r^*_{t,t+n,\infty} - r_{t,t+n,\infty}\right) \tag{A17}$$

Replacing this ratio in the upper bound yields the final result:

$$H^P[n] \leq \frac{L\left[\exp\left(\Delta e_{t,t+1} + r^*_{t,t+1,\infty} - r_{t,t+1,\infty}\right)\right] - \frac{1}{n}L\left[\exp\left(\Delta e_{t,t+n} + r^*_{t,t+n,\infty} - r_{t,t+n,\infty}\right)\right]}{E\left[r_{t,t+1,ex,\infty}\right] + E\left[r^*_{t,t+1,ex,\infty}\right]}. \tag{A18}$$

This ends the proof.

Proof of the Bounds on Horizon Codependence in Entropy-Based Correlation of SDFs. We assume a stationary environment:

$$\begin{aligned} E\left[M_{t+i-1,t+i}\right] &= E\left[M_{t,t+1}\right], \\ E\left[M^*_{t+i-1,t+i}\right] &= E\left[M^*_{t,t+1}\right], \end{aligned} \tag{A19}$$

and define the n-period entropy-based correlation as

$$\varrho_{M^*_{t,t+n},M_{t,t+n}} = \frac{\left(L\left[M_{t,t+n}\right] + L\left[M^*_{t,t+n}\right]\right) - L\left[\frac{M^*_{t,t+n}}{M_{t,t+n}}\right]}{\left(L\left[M_{t,t+n}\right] + L\left[M^*_{t,t+n}\right]\right)}. \tag{A20}$$

The horizon codependence in the entropy-based correlation of SDFs is

$$H^M[n] = \varrho_{M_{t,t+n},M^*_{t,t+n}} - \varrho_{M_{t,t+1},M^*_{t,t+1}}. \tag{A21}$$

For characterization to follow, define

$$H[n] = \frac{1}{n}L\left[M_{t,t+n}\right] - L\left[M_{t,t+1}\right], \tag{A22}$$

$$H^*[n] = \frac{1}{n}L\left[M^*_{t,t+n}\right] - L\left[M^*_{t,t+1}\right], \tag{A23}$$

$$H^e[n] = \frac{1}{n}L\left[\frac{M^*_{t,t+n}}{M_{t,t+n}}\right] - L\left[\frac{M^*_{t,t+1}}{M_{t,t+1}}\right]. \tag{A24}$$

We then simplify expressions (A23), (A24), and (A24) below:

$$\begin{aligned} H[n] &= \frac{1}{n}L\left[M_{t,t+n}\right] - L\left[M_{t,t+1}\right] \\ &= \frac{1}{n}\log\left(E\left[M_{t,t+n}\right]\right) - \log\left(E\left[M_{t,t+1}\right]\right) \\ &= \frac{1}{n}\log\left(E\left[E_t\left[M_{t,t+n}\right]\right]\right) - \log\left(E\left[E_t\left[M_{t,t+1}\right]\right]\right) \\ &= \frac{1}{n}\log\left(E\left[\exp\left(-ny_t^n\right)\right]\right) - \log\left(E\left[\exp\left(-y_t\right)\right]\right). \end{aligned} \tag{A25}$$

Similarly,

$$H^*[n] = \frac{1}{n}\log\left(E\left[\exp\left(-ny_t^{*n}\right)\right]\right) - \log\left(E\left[\exp\left(-y_t^*\right)\right]\right) \tag{A26}$$

and

$$H^e[n] = \frac{1}{n} L[e_{t,t+n}] - L[e_{t,t+1}]. \tag{A27}$$

We exploit (A25), (A26), (A27), and express the n-period entropy-based correlation as

$$\varrho_{M^*_{t,t+n}, M_{t,t+n}} = \frac{nH[n] + nL[M_{t,t+1}] + nH^*[n] + nL\left[M^*_{t,t+1}\right] - nH^e[n] - nL\left[\frac{M^*_{t,t+1}}{M_{t,t+1}}\right]}{nH[n] + nL[M_{t,t+1}] + nH^*[n] + nL\left[M^*_{t,t+1}\right]}. \tag{A28}$$

The above expression simplifies to

$$\varrho_{M^*_{t,t+n}, M_{t,t+n}} = \frac{A + L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}{B + L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}, \tag{A29}$$

where

$$\begin{aligned} A &= B - \frac{1}{n} L[\Delta e_{t,t+n}], \\ B &= H[n] + H^*[n], \\ C &= -L[\Delta e_{t,t+n}]. \end{aligned} \tag{A30}$$

For $n = 1$, we have $B = 0$ and $A = C = -L[\Delta e_{t,t+1}]$:

$$\varrho_{M^*_{t,t+1}, M_{t,t+1}} = \frac{C + L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}{L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}. \tag{A31}$$

Therefore, the difference between n-period and one-period entropy-based correlation of SDFs is

$$\varrho_{M^*_{t,t+n}, M_{t,t+n}} - \varrho_{M^*_{t,t+1}, M_{t,t+1}} = \frac{A + L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}{B + L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]} - \frac{C + L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}{L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right]}. \tag{A32}$$

Now, we denote by

$$l = L[M_{t,t+1}] + L\left[M^*_{t,t+1}\right] \tag{A33}$$

and express the horizon dependence as

$$H^M[n] = \frac{(A - B - C)\, l - BC}{l\,(l + B)}. \tag{A34}$$

Now, define the function

$$g[x] = \frac{(A - B - C)\, x - BC}{x\,(x + B)}. \tag{A35}$$

Next, our goal is to derive bounds on $g[.]$. To proceed, we calculate the first derivative of $g[x]$:

$$g^{'}[x] = \frac{(A - B - C)\, x\,(x + B) - (2x + B)\,((A - B - C)\, x - BC)}{x^2\,(x + B)^2} \tag{A36}$$

and

$$g^{'}[x] = \frac{(B + C - A)\, x^2 + 2xBC + BBC}{x^2\,(x + B)^2}. \tag{A37}$$

Since $A - B - C < 0$, we have

$$B - A > -C \tag{A38}$$

and

$$(B - A)\,(-C) > (-C)^2\,. \tag{A39}$$

Because inequality (A39) holds, there are two solutions:

$$x_1 = \frac{-BC - \sqrt{(A - B)\, CB^2}}{(B + C - A)} \text{ and } x_2 = \frac{-BC + \sqrt{(A - B)\, CB^2}}{(B + C - A)}. \tag{A40}$$

(A39) can be used to show that $x_1 < 0$ and $x_2 > 0$

$$x_1 < \frac{-BC - \sqrt{(-C)^2\, B^2}}{(B + C - A)} = \frac{-BC - (-C)\, B}{(B + C - A)} = 0. \tag{A41}$$

and $x_2 > 0$. As a result,

$$g^{'}[x] \leq 0 \text{ when } x \in [0, x_2] \text{ and } g^{'}[x] \geq 0 \text{ when } x > x_2. \tag{A42}$$

(A42) is therefore used to derive the bounds on the horizon codependence:

$$\begin{cases} g[x_m] \leq H^M[n] \leq f[\underline{l}] & \text{if } \underline{l} \leq l \leq x_m \\ g[x_m] \leq H^M[n] & \text{if } \underline{l} \leq x_m \leq l \\ g[\underline{l}] \leq H^M[n] & \text{if } x_m \leq \underline{l} \leq l \end{cases} \tag{A43}$$

where

$$\underline{l} = E\left[r_{t,t+1,ex}\right] + E\left[r^*_{t,t+1,ex}\right]. \tag{A44}$$

Online Appendix II: Derivations of Permanent and Transitory Components in International Asset Pricing Models

In Online Appendix II, we present the SDF used in state-of-the art asset pricing models, and use the eigenfunction problem of Hansen (2012) to solve for the permanent and transitory components of SDFs. For more details on each model, we refer the reader to asset pricing models that are used.

II.1 A Model with Long-Run Risks

This model follows the setup described by Colacito and Croce (2011) and Colacito (2012).

The stochastic discount factor. We obtain the stochastic discount factor as the intertemporal marginal rate of substitution:

$$\begin{aligned} m_{t+1} &= \log \frac{\partial U_t / \partial C_{t+1}}{\partial U_t / \partial C_t} \\ &= \log \left(\delta\theta \frac{1}{E_t \exp\left\{\frac{U_{t+1}}{\theta}\right\}} \cdot \exp\left\{\frac{U_{t+1}}{\theta}\right\} \cdot \frac{1-\delta}{\theta} \cdot \frac{1}{C_{t+1}} \cdot \frac{C_t}{1-\delta} \right) \\ &= \log \left(\delta \frac{\exp\left\{\frac{U_{t+1}}{\theta}\right\}}{E_t \exp\left\{\frac{U_{t+1}}{\theta}\right\}} \cdot \frac{C_t}{C_{t+1}} \right). \end{aligned} \quad \text{(B1)}$$

In order to calculate the properties of the stochastic discount factor, we need to solve for the utility function first. We are going to solve for the function $v_t = U_t - \log C_t$. Following the same approach as in Colacito (2012), we obtain a recursive representation of v_t:

$$v_t = \delta\theta \log E_t \exp\left\{\frac{v_{t+1} + \Delta c_{t+1}}{\theta}\right\} \quad \text{(B2)}$$

Now we guess that v_t is a linear function of x_t,

$$v_t = A + B \cdot x_t, \quad \text{(B3)}$$

and find the coefficients A and B to verify this guess:

$$\begin{aligned} A + Bx_t &= \delta\theta \log E_t \exp\left\{\frac{A + Bx_{t+1} + \mu_c + x_t + \sigma_c\varepsilon_{c,t+1}}{\theta}\right\} \\ &= \delta\theta \log E_t \exp\left\{\frac{A + B\rho x_t + B\sigma_x\varepsilon_{x,t+1} + \mu_c + x_t + \sigma_c\varepsilon_{c,t+1}}{\theta}\right\} \\ &= \delta\left(A + \mu_c + \frac{B^2\sigma_x^2 + \sigma_c^2}{2\theta}\right) + \delta(B\rho + 1)\cdot x_t, \end{aligned} \tag{B4}$$

where the last equality follows or the properties of the log-normal distribution. Clearly, our guess is verified by setting:

$$\begin{aligned} A &= \frac{\delta}{1-\delta}\left(\mu_c + \frac{B^2\sigma_x^2 + \sigma_c^2}{2\theta}\right), \\ B &= \frac{\delta}{1-\delta\rho}. \end{aligned} \tag{B5}$$

It is now possible to rewrite the stochastic discount factor as a function of the shocks and of the state variable x_t. Focus on the certainty equivalent term in (B1):

$$\begin{aligned} \log E_t \exp\left\{\frac{v_{t+1} + \Delta c_{t+1}}{\theta}\right\} &= \log E_t \exp\left\{\frac{A + B\cdot(\rho x_t + \sigma_x\varepsilon_{x,t+1}) + \mu_c + x_t + \sigma_c\varepsilon_{c,t+1}}{\theta}\right\} \\ &= \frac{A+\mu_c}{\theta} + \frac{B^2\sigma_x^2}{2\theta^2} + \frac{\sigma_c^2}{2\theta^2} + \left(\frac{B\rho+1}{\theta}\right)x_t. \end{aligned} \tag{B6}$$

Hence,

$$\begin{aligned} m_{t+1} &= \log\delta - (\mu_c + x_t + \sigma_c\varepsilon_{c,t+1}) + \frac{A+\mu_c}{\theta} + \frac{(B\rho+1)}{\theta}x_t \\ &\quad + \frac{B\sigma_x}{\theta}\varepsilon_{x,t+1} + \frac{\sigma_c}{\theta}\varepsilon_{c,t+1} - \frac{A+\mu_c}{\theta} \\ &\quad - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2} - \left(\frac{B\rho+1}{\theta}\right)x_t \\ &= \left(\log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2}\right) - x_t + \frac{B\sigma_x}{\theta}\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1}. \end{aligned} \tag{B7}$$

Decomposition into Permanent and Transitory components. We will now use the eigenfunction problem of Alvarez and Jermann (2005) and Hansen (2012). Let $f_{t+1}^{(e)}$ be the eigenfunction, therefore we must have

$$E_t\left[M_{t+1}f_{t+1}^{(e)}\right]=\zeta f_t^{(e)}, \tag{B8}$$

where ζ is the eigenvalue. Accordingly, the permanent and transitory components of the SDF are

$$M_{t+1}^P=M_{t+1}\frac{f_{t+1}^{(e)}}{\zeta f_t^{(e)}},\text{ and } M_{t+1}^T=\frac{\zeta f_t^{(e)}}{f_{t+1}^{(e)}}. \tag{B9}$$

We conjecture that the eigenfunction takes the form

$$f_{t+1}^{(e)}=\exp\left(\xi x_{t+1}\right). \tag{B10}$$

Combining the SDF with (B10), it can be shown that

$$\begin{aligned}\log\left(M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right) &= \log\left(M_{t+1}\right)+\log\left(f_{t+1}^{(e)}\right)-\log\left(f_t^{(e)}\right)\\
&= m_{t+1}+\xi x_{t+1}-\xi x_t\\
&= \left(\log\delta-\mu_c-\frac{B^2\sigma_x^2}{2\theta^2}-\frac{\sigma_c^2}{2\theta^2}\right)-x_t+\frac{B\sigma_x}{\theta}\varepsilon_{x,t+1}+\left(\frac{1}{\theta}-1\right)\sigma_c\varepsilon_{c,t+1}\\
&\quad +\xi x_{t+1}-\xi x_t\\
&= \left(\log\delta-\mu_c-\frac{B^2\sigma_x^2}{2\theta^2}-\frac{\sigma_c^2}{2\theta^2}\right)+\left(\frac{B\sigma_x}{\theta}+\xi\sigma_x\right)\varepsilon_{x,t+1}+\left(\frac{1}{\theta}-1\right)\sigma_c\varepsilon_{c,t+1}\\
&\quad +\left(\xi\left(\rho-1\right)-1\right)x_t\end{aligned} \tag{B11}$$

and

$$\begin{aligned}E_t\left[M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right] &= \exp\left(\log\delta-\mu_c-\frac{B^2\sigma_x^2}{2\theta^2}-\frac{\sigma_c^2}{2\theta^2}+\frac{1}{2}\left(\frac{B\sigma_x}{\theta}+\xi\sigma_x\right)^2\right.\\
&\quad \left.+\frac{1}{2}\left(\frac{1}{\theta}-1\right)^2\sigma_c^2+\left(\xi\left(\rho-1\right)-1\right)x_t\right).\end{aligned} \tag{B12}$$

From (B12), it follows immediately that $\xi\left(\rho-1\right)-1=0$, or equivalently,

$$\xi=\frac{1}{\rho-1}, \tag{B13}$$

and

$$\log(\zeta) = \log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2} + \frac{1}{2}\left(\frac{B\sigma_x}{\theta} + \xi\sigma_x\right)^2 + \frac{1}{2}\left(\frac{1}{\theta} - 1\right)^2 \sigma_c^2 \quad \textbf{(B14)}$$

$$= \log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2} + \frac{1}{2}\left(\frac{B\sigma_x}{\theta} + \frac{1}{\rho - 1}\sigma_x\right)^2 + \frac{1}{2}\left(\frac{1}{\theta} - 1\right)^2 \sigma_c^2.$$

It follows that the transitory component is

$$\begin{aligned}
m_{t+1}^T &= \log\left(M_{t+1}^T\right) \\
&= \log(\zeta) + \log\left(f_t^{(e)}\right) - \log\left(f_{t+1}^{(e)}\right) \\
&= \log(\zeta) + \xi\left(x_t - x_{t+1}\right) \\
&= \log(\zeta) + \xi\left(x_t - \rho x_t - \sigma_x \varepsilon_{x,t+1}\right) \\
&= \log(\zeta) + \xi x_t (1 - \rho) - \xi\sigma_x\varepsilon_{x,t+1} \\
&= \log(\zeta) - x_t - \xi\sigma_x\varepsilon_{x,t+1}, \qquad \textbf{(B15)}
\end{aligned}$$

and the permanent component is

$$\begin{aligned}
m_{t+1}^P &= \log\left(M_{t+1}\right) - \log\left(M_{t+1}^T\right) \\
&= \left(\log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2}\right) - x_t + \frac{B\sigma_x}{\theta}\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1} \\
&\quad - \log(\zeta) + x_t + \xi\sigma_x\varepsilon_{x,t+1} \\
&= \left(\log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2}\right) + \frac{B\sigma_x}{\theta}\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1} - \log(\zeta) + \xi\sigma_x\varepsilon_{x,t+1} \\
&= \left(\log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2}\right) + \left(\frac{B}{\theta} + \xi\right)\sigma_x\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1} - \log(\zeta) \\
&= \left(\frac{B}{\theta} + \xi\right)\sigma_x\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1} - \frac{1}{2}\left(\frac{B}{\theta} + \xi\right)^2\sigma_x^2 - \frac{1}{2}\left(\frac{1}{\theta} - 1\right)^2\sigma_c^2. \qquad \textbf{(B16)}
\end{aligned}$$

To summarize,

$$\begin{aligned}
m_{t+1}^T &= \log(\zeta) - x_t - \xi\sigma_x\varepsilon_{x,t+1}, \qquad \textbf{(B17)} \\
m_{t+1}^P &= -\frac{1}{2}\left(\frac{B}{\theta} + \xi\right)^2\sigma_x^2 - \frac{1}{2}\left(\frac{1}{\theta} - 1\right)^2\sigma_c^2 + \left(\frac{B}{\theta} + \xi\right)\sigma_x\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1}, \\
m_{t+1} &= \left(\log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2}\right) - x_t + \frac{B\sigma_x}{\theta}\varepsilon_{x,t+1} + \left(\frac{1}{\theta} - 1\right)\sigma_c\varepsilon_{c,t+1},
\end{aligned}$$

with

$$\log(\zeta) = \log\delta - \mu_c - \frac{B^2\sigma_x^2}{2\theta^2} - \frac{\sigma_c^2}{2\theta^2} + \frac{1}{2}\left(\frac{B\sigma_x}{\theta} + \frac{1}{\rho-1}\sigma_x\right)^2 + \frac{1}{2}\left(\frac{1}{\theta} - 1\right)^2\sigma_c^2. \tag{B18}$$

A similar expression applies to the foreign country.

II.2 A Model with Habits

We shall derive the permanent and transitory components. We conjecture that the eigenfunction takes the form

$$f_{t+1}^{(e)} = \exp\left(\xi s_{t+1}\right), \tag{B19}$$

and express

$$\begin{aligned}
\log\left(M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right) &= \log\left(M_{t+1}\right) + \log\left(f_{t+1}^{(e)}\right) - \log\left(f_t^{(e)}\right) \\
&= m_{t+1} + \xi s_{t+1} - \xi s_t \\
&= \log(\beta) - \gamma g - \gamma(\phi - 1)(s_t - \overline{s}) - \gamma(1 + \lambda(s_t))\,u_{t+1} + \xi s_{t+1} - \xi s_t \\
&= \log(\beta) - \gamma g - \gamma(\phi - 1)(s_t - \overline{s}) + \xi(1 - \phi)\,\overline{s} + \xi\phi s_t - \xi s_t \\
&\quad + \left[-\xi + (\xi - \gamma)(1 + \lambda(s_t))\right]u_{t+1}
\end{aligned}$$

and

$$E_t\left[M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right] = \exp\left(\log(\beta) - \gamma g + (\gamma - \xi)(1 - \phi)(s_t - \overline{s}) + \frac{1}{2}\left[-\xi + (\xi - \gamma)(1 + \lambda(s_t))\right]^2\right). \tag{B20}$$

Setting $\xi = \gamma$ we show that

$$E_t\left[M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right] = \zeta = \exp\left(\log(\beta) - \gamma g + \frac{1}{2}\gamma^2\right). \tag{B21}$$

Finally, the permanent component is

$$\begin{aligned} m_{t+1}^{T} &= \log\left(M_{t+1}^{T}\right) \\ &= \log(\zeta) + \log\left(f_t^{(e)}\right) - \log\left(f_{t+1}^{(e)}\right) \\ &= \log(\beta) - \gamma g + \frac{1}{2}\gamma^2 + \gamma\left(s_t - s_{t+1}\right) \\ &= \log(\beta) - \gamma g + \frac{1}{2}\gamma^2 + \gamma s_t - \gamma s_{t+1} \\ &= \log(\beta) - \gamma g + \frac{1}{2}\gamma^2 + \gamma s_t - \gamma\left((1-\phi)\,\overline{s} + \phi s_t + \lambda\left(s_t\right) u_{t+1}\right) \\ &= \log(\beta) - \gamma g + \frac{1}{2}\gamma^2 + \gamma(1-\phi)\left(s_t - \overline{s}\right) - \gamma\lambda\left(s_t\right) u_{t+1}, \end{aligned}$$

and the permanent component is

$$\begin{aligned} m_{t+1}^{P} &= \log\left(M_{t+1}\right) - \log\left(M_{t+1}^{T}\right) \\ &= m_{t+1} - m_{t+1}^{T} \\ &= -\gamma\left[(\phi-1)\left(s_t - \overline{s}\right) + \left(1 + \lambda\left(s_t\right)\right) u_{t+1}\right] - \frac{1}{2}\gamma^2 - \gamma\left(s_t - s_{t+1}\right) \\ &= -\frac{1}{2}\gamma^2 - \gamma u_{t+1}. \end{aligned}$$

To summarize,

$$\begin{aligned} m_{t+1} &= \log(\beta) - \gamma\left[g + (\phi-1)\left(s_t - \overline{s}\right) + \left(1 + \lambda\left(s_t\right)\right) u_{t+1}\right], \qquad \text{(B22)} \\ m_{t+1}^{P} &= -\frac{1}{2}\gamma^2 - \gamma u_{t+1}, \\ m_{t+1}^{T} &= \log(\beta) - \gamma g + \frac{1}{2}\gamma^2 + \gamma(1-\phi)\left(s_t - \overline{s}\right) - \gamma\lambda\left(s_t\right) u_{t+1}. \end{aligned}$$

A similar expression applies to the foreign country.

II.3 A Model with Rare Disasters

This section describes the model of Barro (2006). The SDF in the domestic country is given by

$$m_{t+1} = e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\theta}, \qquad \text{(B23)}$$

where ρ is the rate of time preference and θ is the relative risk aversion coefficient. Setting

$$C_{t+1} = A_{t+1}, \tag{B24}$$

where A_{t+1} is an equity claim on period $t+1's$ output. The log of output evolves as a random walk,

$$\log(A_{t+1}) = \log(A_t) + \gamma + u_{t+1} + \upsilon_{t+1},$$

where $\gamma \geq 0$. The random variable u_{t+1} is i.i.d normal with mean 0 and variance σ^2. The random variable υ_{t+1} picks up low-probability disasters (see Rietz (1988)). The distribution of υ_{t+1} is given by

$$\begin{cases} \exp(-p) & \text{if} \quad \upsilon_{t+1} = 0, \\ 1 - \exp(-p) & \text{if} \quad \upsilon_{t+1} = \log(1-b), \end{cases} \tag{B25}$$

where b is the size of the downward jump in output. The log SDF can be written as

$$\log(m_{t+1}) = -\rho - \theta\gamma - \theta u_{t+1} - \theta\upsilon_{t+1}. \tag{B26}$$

We follow the convention of correlationing foreign variables with a superscript "*". Within this set up, the transitory component is a constant:

$$\begin{aligned} M_{t+1}^T &= E_t\left[\exp\left(-\rho - \theta\gamma - \theta u_{t+1} - \theta\upsilon_{t+1}\right)\right] \\ &= \left(\exp\left(-\rho - \theta\gamma\right)\right)\exp\left(\frac{1}{2}\theta^2\sigma^2\right)E\left[\exp\left(-\theta\upsilon_{t+1}\right)\right] \\ &= \left[\exp(-p) + (1-\exp(-p))\exp\left(-\theta\log(1-b)\right)\right]\exp\left(-\rho - \theta\gamma + \frac{1}{2}\theta^2\sigma^2\right). \end{aligned} \tag{B27}$$

The permanent component is

$$M_{t+1}^P = \frac{\exp\left(-\theta u_{t+1} - \frac{1}{2}\theta^2\sigma^2\right)\exp\left(-\theta\upsilon_{t+1}\right)}{\exp(-p) + (1-\exp(-p))\exp\left(-\theta\log(1-b)\right)}. \tag{B28}$$

To summarize,

$$M_{t+1}^{T} = \left(\exp\left(-p\right) + \left(1-\exp\left(-p\right)\right)\exp\left(-\theta\log\left(1-b\right)\right)\right)\exp\left(-\rho-\theta\gamma+\frac{1}{2}\theta^{2}\sigma^{2}\right) \quad \text{(B29)}$$

$$M_{t+1}^{P} = \frac{\exp\left(-\theta u_{t+1}-\frac{1}{2}\theta^{2}\sigma^{2}\right)\exp\left(-\theta\upsilon_{t+1}\right)}{\exp\left(-p\right)+\left(1-\exp\left(-p\right)\right)\exp\left(-\theta\log\left(1-b\right)\right)}$$

$$M_{t+1} = \exp\left(-\rho-\theta\gamma-\theta u_{t+1}-\theta\upsilon_{t+1}\right).$$

II.4 A Model of the Cross-Section of Currency Risk Premia

This section describes the model by Lustig, Roussanov, and Verdelhan (2013) . In their model (see equation (3) of Section 3.2), the log SDF in each country is defined as

$$-m_{t+1} = \alpha+\chi z_t+\sqrt{\gamma z_t}u_{t+1}+\tau z_t^{\omega}+\sqrt{\delta z_t^{\omega}}u_{t+1}^{\omega}+\sqrt{\kappa z_t}u_{t+1}^{g}, \quad \text{(B30)}$$

$$-m_{t+1}^{*} = \alpha+\chi z_t^{*}+\sqrt{\gamma z_t^{*}}u_{t+1}^{*}+\tau z_t^{\omega}+\sqrt{\delta^{*} z_t^{\omega}}u_{t+1}^{\omega}+\sqrt{\kappa z_t^{*}}u_{t+1}^{g}, \quad \text{(B31)}$$

where the three innovations u_{t+1}, u_{t+1}^{ω}, and u_{t+1}^{g} are independent of one another. Moreover, u_{t+1} and u_{t+1}^{*} are independent. The state variables dynamic is given by

$$z_{t+1} = \left(1-\phi\right)\theta+\phi z_t-\sigma\sqrt{z_t}u_{t+1}, \quad \text{(B32)}$$

$$z_{t+1}^{*} = \left(1-\phi\right)\theta+\phi z_t^{*}-\sigma\sqrt{z_t^{*}}u_{t+1}^{*}, \quad \text{(B33)}$$

$$z_{t+1}^{\omega} = \left(1-\phi^{\omega}\right)\theta^{\omega}+\phi^{\omega} z_t^{\omega}-\sigma^{\omega}\sqrt{z_t^{\omega}}u_{t+1}^{\omega}. \quad \text{(B34)}$$

We conjecture that the eigenfunction takes the form

$$f_{t+1}^{(e)} = \exp\left(\xi_0 z_{t+1}+\xi_1 z_{t+1}^{\omega}\right). \quad \text{(B35)}$$

We have

$$
\begin{aligned}
\log\left(M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right) &= \log\left(M_{t+1}\right)+\log\left(f_{t+1}^{(e)}\right)-\log\left(f_t^{(e)}\right) \\
&= m_{t+1}+\xi_0\left(z_{t+1}-z_t\right)+\xi_1\left(z_{t+1}^{\omega}-z_t^{\omega}\right) \\
&= -\alpha-\chi z_t-\sqrt{\gamma z_t}u_{t+1}-\tau z_t^{\omega}-\sqrt{\delta z_t^{\omega}}u_{t+1}^{\omega}-\sqrt{\kappa z_t}u_{t+1}^{g} \\
&\quad +\xi_0 z_{t+1}-\xi_0 z_t+\xi_1 z_{t+1}^{\omega}-\xi_1 z_t^{\omega} \\
&= -\alpha-\chi z_t-\sqrt{\gamma z_t}u_{t+1}-\tau z_t^{\omega}-\sqrt{\delta z_t^{\omega}}u_{t+1}^{\omega}-\sqrt{\kappa z_t}u_{t+1}^{g} \\
&\quad +\xi_0\left(\left(1-\phi\right)\theta+\phi z_t-\sigma\sqrt{z_t}u_{t+1}\right) \\
&\quad -\xi_0 z_t+\xi_1\left(\left(1-\phi^{\omega}\right)\theta^{\omega}+\phi^{\omega}z_t^{\omega}-\sigma^{\omega}\sqrt{z_t^{\omega}}u_{t+1}^{\omega}\right)-\xi_1 z_t^{\omega}
\end{aligned} \tag{B36}
$$

which simplifies to

$$
\begin{aligned}
\log\left(M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right) &= -\alpha+\xi_0\left(1-\phi\right)\theta+\xi_1\left(1-\phi^{\omega}\right)\theta^{\omega}-\chi z_t+\xi_0\phi z_t-\xi_1 z_t^{\omega} \\
&\quad -\xi_0 z_t+\xi_1\phi^{\omega}z_t^{\omega}+\left(-\sqrt{\gamma}-\xi_0\sigma\right)\sqrt{z_t}u_{t+1}-\tau z_t^{\omega} \\
&\quad +\left(-\sqrt{\delta}-\xi_1\sigma^{\omega}\right)\sqrt{z_t^{\omega}}u_{t+1}^{\omega}-\sqrt{\kappa}\sqrt{z_t}u_{t+1}^{g}
\end{aligned} \tag{B37}
$$

and

$$
\begin{aligned}
\log\left(E_t\left[M_{t+1}\frac{f_{t+1}^{(e)}}{f_t^{(e)}}\right]\right) &= -\alpha+\xi_0\left(1-\phi\right)\theta+\xi_1\left(1-\phi^{\omega}\right)\theta^{\omega} \\
&\quad +\left(-\chi+\xi_0\phi-\xi_0+\frac{1}{2}\left(-\sqrt{\gamma}-\xi_0\sigma\right)^2+\frac{1}{2}\kappa\right)z_t \\
&\quad +\left(\left(-\xi_1+\xi_1\phi^{\omega}-\tau\right)+\frac{1}{2}\left(-\sqrt{\delta}-\xi_1\sigma^{\omega}\right)^2\right)z_t^{\omega}.
\end{aligned} \tag{B38}
$$

We set

$$
-\chi+\xi_0\left(\phi-1\right)+\frac{1}{2}\left(-\sqrt{\gamma}-\xi_0\sigma\right)^2+\frac{1}{2}\kappa=0 \tag{B39}
$$

and

$$
\xi_1\left(\phi^{\omega}-1\right)-\tau+\frac{1}{2}\left(-\sqrt{\delta}-\xi_1\sigma^{\omega}\right)=0 \tag{B40}
$$

and show:

$$
-\alpha+\xi_0\left(1-\phi\right)\theta+\xi_1\left(1-\phi^{\omega}\right)\theta^{\omega}=\log\left(\zeta\right). \tag{B41}
$$

Therefore,

$$E_t\left[M_{t+1}f_{t+1}^{(e)}\right]=\zeta f_t^{(e)}. \tag{B42}$$

We solve (B39)

$$\frac{1}{2}\xi_0^2\sigma^2+\left(\sqrt{\gamma}\sigma+(\phi-1)\right)\xi_0+\frac{1}{2}\kappa-\chi+\frac{1}{2}\gamma=0 \tag{B43}$$

for ξ_0. Following Hansen (2012), we choose the solution with a negative root:

$$\xi_0=\frac{-\left(\sqrt{\gamma}\sigma+(\phi-1)\right)-\sqrt{\left(\sqrt{\gamma}\sigma+(\phi-1)\right)^2-2\sigma^2\left(\frac{1}{2}\kappa-\chi+\frac{1}{2}\gamma\right)}}{\sigma^2} \tag{B44}$$

We also solve (B40)

$$\left(\frac{1}{2}\delta-\tau\right)+\xi_1\left((\phi^\omega-1)+\sqrt{\delta}\sigma^\omega\right)+\frac{1}{2}\xi_1^2\left(\sigma^\omega\right)^2=0 \tag{B45}$$

for ξ_1. Following Hansen (2012), we choose the solution with a negative root

$$\xi_1=\frac{-\left((\phi^\omega-1)+\sqrt{\delta}\sigma^\omega\right)-\sqrt{\left((\phi^\omega-1)+\sqrt{\delta}\sigma^\omega\right)^2-2\left(\frac{1}{2}\delta-\tau\right)\left(\sigma^\omega\right)^2}}{\left(\sigma^\omega\right)^2}. \tag{B46}$$

To summarize, the log transitory component, log permanent component and log SDF are

$$\begin{aligned}
m_{t+1}^T &= \log(\zeta)+\xi_0\left(z_t-z_{t+1}\right)+\xi_1\left(z_t^\omega-z_{t+1}^\omega\right), && \text{(B47)}\\
m_{t+1}^P &= m_{t+1}-m_{t+1}^T, && \text{(B48)}\\
-m_{t+1} &= \alpha+\chi z_t+\sqrt{\gamma z_t}u_{t+1}+\tau z_t^\omega+\sqrt{\delta z_t^\omega}u_{t+1}^\omega+\sqrt{\kappa z_t}u_{t+1}^g, && \text{(B49)}
\end{aligned}$$

with

$$\begin{aligned}
\log(\zeta) &= -\alpha+\xi_0(1-\phi)\theta+\xi_1(1-\phi^\omega)\theta^\omega, && \text{(B50)}\\
\xi_0 &= \frac{-\left(\sqrt{\gamma}\sigma+(\phi-1)\right)-\sqrt{\left(\sqrt{\gamma}\sigma+(\phi-1)\right)^2-2\sigma^2\left(\frac{1}{2}\kappa-\chi+\frac{1}{2}\gamma\right)}}{\sigma^2}, && \text{(B51)}\\
\xi_1 &= \frac{-\left((\phi^\omega-1)+\sqrt{\delta}\sigma^\omega\right)-\sqrt{\left((\phi^\omega-1)+\sqrt{\delta}\sigma^\omega\right)^2-2\left(\frac{1}{2}\delta-\tau\right)\left(\sigma^\omega\right)^2}}{\left(\sigma^\omega\right)^2}. && \text{(B52)}
\end{aligned}$$

The foreign SDF has similar formulas with superscript "$*$".