Demski
ACG 4352

Ralph's Shadow[1]

Ralph is anxious to learn about shadow prices, and is a devoted user of the Solver routine in Excel. To begin, consider the following constrained minimization:

$$\min_x f(x) = 25 + x^2 - 10x$$
$$\text{subject to } x \geq b$$

Use Excel to find the optimal x for two cases: $b = 0$ and $b = 7$. In the latter case, open the Sensitivity Report and notice the Lagrangian or shadow price is 4.00.

One way to see what is going on is to notice that if $x \geq 7$ the solution is $x = b = 7$ because $b = 7$ is larger than the value of x that minimizes $25+x^2-10x$. An easy way to see this is to graph $f(x) = 25+x^2-10x$. Anyway, for the $b = 7$ case notice the optimal solution is

$$25 + b^2 - 10b$$

Okay, now differentiate with respect to b

$$2b - 10$$

Now, if $b = 7$ this gradient is 2(7)-10=4. The shadow price is the rate at which the optimal objective function value changes with respect to b.

Try this for a few other values, say $b \in \{3, 5, 9\}$.

To dig deeper, suppose we want to solve the following

$$\min_x \omega(x)$$
$$\text{subject to } g(x) \geq b$$

where b is a constant. It now turns out that if x^* is a solution to this problem then there exists a shadow price, $\lambda \geq 0$, such that (1) $\omega'(x^*) - \lambda g'(x^*) = 0$, (2) $\lambda[g(x^*) - b] = 0$, and (3) $\frac{\partial \omega(x^*)}{\partial b} = \lambda$. Excel, thankfully, reports the optimal solution x^* and the shadow price λ.

Digging a little deeper, let's recast the problem in terms of the objective function less the constraint "priced" at its shadow price:

$$\Psi = \omega(x) - \lambda[[g(x) - b]$$

From here we have

$$\frac{\partial \Psi}{\partial x} = \omega'(x) - \lambda g'(x)$$

[1] Apologies to Lamont Cranston

and

$$\frac{\partial \Psi}{\partial x} = \lambda$$

From here we have two more Excel based exercises. First look at

$$\begin{aligned}&\min_x 10x\\&\text{subject to } x \geq b\end{aligned}$$

Solve this for $b \in \{1, 3, 5\}$. What is the intuition for the shadow price?

Second, look at a two variable problem

$$\begin{aligned}&\min_{x \geq 0, y \geq 0} 10x + 20y\\&\text{subject to } \sqrt{xy} \geq q \text{ and } -x \geq -b\end{aligned}$$

Notice the second constraint is really $x \leq b$. I have written it so each constraint is in $\geq$ format; be certain you do this in your Excel sheet. Solve this for three cases: (1) $q = 3, b = 7$, (2) $q = 5, b = 7$ and (3) $q = 15, b = 7$. Be certain to identify the shadow price on each constraint. What do you see?