Complex Times: Asset Pricing and Conditional Moments under Non-Affine Diffusions*

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Abstract

Many applications in continuous-time financial economics require conditional moments or contingent claims prices, but such expressions are known in closed-form for only a few specific models. Power series (in the time variable) for these quantities are easily derived, but often fail to converge, even for very short time horizons. We characterize a large class of continuous-time non-affine conditional moment and contingent claim pricing problems with solutions that are analytic in the time variable, and that therefore can be represented by convergent power series. The ability to approximate solutions accurately and in closed-form simplifies the estimation of latent variable models, since the state vector must be extracted from observed quantities for many different parameter vectors during a typical estimation procedure.

1 Introduction

Many applications in financial economics require solutions to second order parabolic partial differential equations with a final condition. Continuous-time processes are often expressed as solutions to stochastic differential equations; estimation of the model parameters is frequently by maximum likelihood or method of moments. Likelihood functions solve the Chapman-Kolmogorov forward and backward equations, whereas conditional moments solve the backward equation. Prices of derivative securities with European-style exercise are solutions to the Feynman-Kac equation with a final condition. In term structure models, bonds are often treated as derivatives written on the interest rate, and are therefore also solutions to the Feynman-Kac equation. In estimation of latent variable models, both equations are typically encountered; the values of the state variables must be inferred from observed quantities by inverting the Feynman-Kac solution, and the fit must be evaluated by calculating the likelihood function or conditional moments, using the Chapman-Kolmogorov equations.

Very few continuous-time models have closed-form conditional moments, likelihood functions, or derivative prices. For the geometric Brownian motion model used by Black and Scholes (1973) and Merton (1973),

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likelihoods, conditional moments, and prices of standard derivative securities are known in closed-form. In the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), likelihoods, conditional moments, and bond prices are all known in closed-form.[1] However, more complicated models almost always lose some of the tractability of these early models; for example, Heston (1993) uses Fourier transforms to find option prices in the stochastic volatility model of Hull and White (1987). In the affine yield models of Duffie and Kan (1996), conditional moments are known in closed-form, but in general, bond prices and likelihoods can only be found by numeric or other approximation methods.[2] Nonetheless, much research on the term structure of interest rates has focused on affine yield models, since, the numeric procedure to calculate bond prices is very fast.[3] Estimation for this class of models has been by simulated method of moments, from Dai and Singleton (2000), by quasi-maximum likelihood, from Duffee (2002), and by approximate maximum likelihood, as in Thompson (2004), Mosburger and Schneider (2005), and Cheridito, Filipović, and Kimmel (2007).

Non-linear models are much less common in the literature, despite evidence from, for example, Aït-Sahalia (1996) and Stanton (1997) of non-linear evolution of the short interest rate process. For some such models, such as Beaglehole and Tenney (1992), Constantinides (1992), Ahn and Gao (1999), and Ahn, Dittmar, and Gallant (2002), bond prices are known in closed-form. Chan, Karolyi, Longstaff, and Sanders (1992) estimate a strongly non-linear model of the interest rate, but do not derive bond prices. In general, the numeric analysis required of many non-linear models makes their use difficult or impossible for many applications. Aït-Sahalia (2002) and Aït-Sahalia (2008) construct closed-form approximations to the likelihood function of non-affine diffusions, and Cheridito, Filipović, and Kimmel (2007), Thompson (2004), Mosburger and Schneider (2005), and Aït-Sahalia and Kimmel (2008) use this method to estimate affine yield models. In principle, this method of approximation could be extended to general solutions to the Feynman-Kac equation (such as bond prices in a non-affine term structure model) by integrating over a fundamental solution. But apart from the integration, there are problems with this approach; power series may not converge at all for longer maturities, and even if they do, the rate of convergence may be very slow. Thus, despite recent advances in the estimation of non-linear models, significant challenges remain in estimation when the values of latent state variables must be extracted from observed prices.

We therefore develop a technique for the construction of convergent series solutions to the Chapman-Kolmogorov backward and Feynman-Kac equations, which, as discussed, are conditional moments and contingent claim prices. Specifically, we use power series in the time variable. The coefficients of the power series can

[1]Our usage of "closed-form" includes such expressions as the cumulative Gaussian distribution function and modified Bessel functions of the first kind. With a narrower definition of "closed-form," the class of models for which such closed-form solutions exist is even more limited.

[2]Although partial differential equation techniques have been used to price bonds and other derivative securities under affine term structure models for some time, theoretical justification of this practice for all affine models was not provided until recently; see Levendorskii (2004a) and Levendorskii (2004b). Affine models have been widely studied; see, for example, Dai and Singleton (2000) and Gouriéroux and Sufana (2006) for classification schemes, Dai and Singleton (2002) for examination of expectations puzzles, and Duffee (2002), Duarte (2004), and Cheridito, Filipović, and Kimmel (2007) for extended market price of risk specifications.

[3]In a general multiple factor term structure model, numeric solution of the partial differential equation that bond prices satisfy is typically very slow. However, for affine yield models, this equation is equivalent to a system of Ricatti-type ODEs, which can be solved numerically very quickly.

be found by a simple recursive relation. Applying this approach to a large class of scalar diffusion processes (and also multiple diffusions, provided the state variables evolve independently), we construct a large class of final conditions such that the corresponding moments are analytic in the time variable, and also a large class of non-affine term structure models for which bond prices are analytic in maturity. Analyticity in the time variable is both a necessary and sufficient condition for convergence of the power series representation of the solution. Furthermore, the method of time transformations, as described in Kimmel (2008), can often greatly increase the range of maturities for which the series converge (and also the speed of convergence). Our technique is then suitable for bond pricing applications, in which we must often consider time horizons of many years; see Kimmel (2008) and Jarrow, Li, Liu, and Wu (2006) for applications to the pricing of non-callable and callable bonds, respectively.

Although a conditional moment or bond pricing function has meaning only for positive time horizons, the behavior of such functions for all complex values of the time variable determines the region of convergence of a power series. Singularities at complex times (which are not meaningful for applications) can still prevent convergence of a power series for positive real times (which are relevant for applications). Throughout, we therefore take the perspective that conditional moments and bond prices are complex functions of a complex time argument, even though these quantities really only deserve to be called "moments" or "prices" for real positive values of the time argument.

The rest of this paper is organized as follows. In Section 2, we discuss the general problem of constructing series representations to solutions of conditional moments or contingent claim pricing problems, and illustrate some of the problems with this approach. In Section 3, we show that, for an arbitrary scalar diffusion process and interest rate specification, there exists an infinite-dimensional family of conditional moment and diffusion problems with solutions that are analytic, and which therefore have convergent power series representations. In Section 4, we explicitly characterize two large families of contingent claim and conditional moment problems with analytic solutions, and determine the range of convergence of the power series representations of the solutions. Section 5 illustrates these methods with examples motivated by bond pricing problems. In some examples, bond prices are known in closed-form, allowing assessment of the accuracy of the approximations; in others, bond prices are not known in closed-form, but can be approximated by our technique. Finally, Section 6 concludes. Proofs of all theorems and corollaries are found in the appendix, which also includes some auxiliary lemmas not shown in the main text.

2 Series Solutions

We consider an N-dimensional diffusion process

$$X_{t+\Delta} = X_t + \int_t^{t+\Delta} \mu\left(X_u\right) du + \int_t^{t+\Delta} \sigma\left(X_u\right) dW_u$$

with initial condition $X_t = x$, where W_t is an N-dimensional standard Brownian motion, X_t is an N-vector of state variables, $\mu\left(X_t\right)$ is an $N \times 1$ vector-valued function, and $\sigma\left(X_t\right)$ is an $N \times N$ matrix-valued function. We assume that $\mu\left(X_t\right)$ and $\sigma\left(X_t\right)$ are chosen so that a unique strong solution X_t exists. See, for example, Karatzas and Shreve (1991), Stroock and Varadhan (1979), or Liptser and Shiryaev (2001) for existence and uniqueness criteria. We are interested in finding expectations, conditional on knowledge of the state vector at an earlier time, which are given by

$$f(\Delta, x) = E\left[e^{-\int_t^{t+\Delta} r(X_u)du} g(X_{t+\Delta}) \,\middle|\, X_t = x\right] \tag{2.1}$$

for some scalar-valued functions $r(x)$ and $g(x)$, and a time horizon $\Delta \geq 0$.[4] For conditional moment problems, the expectation in (2.1) is usually taken under true probabilities, with $r(x) = 0$, whereas for asset pricing problems, it is taken under risk-neutral or risk-forward probabilities. Under technical regularity conditions,[5] the solution $f(\Delta, x)$ to the probabilistic problem is also a solution to the partial differential equation

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \sum_{i=1}^{N} \mu_i(x) \frac{\partial f}{\partial x_i}(\Delta, x) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(\Delta, x) - r(x) f(\Delta, x) \tag{2.2}$$

with the final condition $f(0, x) = g(x)$, where $\mu_i(x)$ denotes the ith element of the vector $\mu(x)$, and $\sigma_{ij}^2(x)$ denotes the element in the ith row and jth column (or, by symmetry, the jth row and ith column) of the matrix $\sigma(x)\sigma^T(x)$. A solution of (2.1) and (2.2) is the price of a derivative instrument with final payoff $g(X_t)$ at maturity. The Chapman-Kolmogorov backward equation is obtained by setting $r(x) = 0$; in this case, solutions to the partial differential equation are conditional expectations (also subject to technical regularity conditions).

Solutions to (2.2) are known in closed-form only for a few special cases. Approximations to conditional likelihood functions have been developed by Aït-Sahalia (2002) for scalar diffusions; see Aït-Sahalia (1999) for examples. This technique was extended to the case of multiple diffusions by Aït-Sahalia (2008). Since a conditional moment is the integral of the final condition over the likelihood function, it might seem this approach could be used to approximate solutions to (2.2) as well, at least in the case $r(x) = 0$. However, this approach is problematic. Consider a convergent series of approximations to a likelihood function

$$\rho_n(\Delta, x, y) \Longrightarrow \rho(\Delta, x, y) \tag{2.3}$$

where x is the backward state variable, y is the forward state variable, and Δ is the time between the backward and forward observations. Convergence of a series of approximations to a conditional moment

$$\int_{-\infty}^{+\infty} \rho_n(\Delta, x, y) g(y) \, dy \Longrightarrow \int_{-\infty}^{+\infty} \rho(\Delta, x, y) g(y) \, dy \tag{2.4}$$

does not necessarily follow. Note that we have not specified the type of convergence in (2.3), e. g., pointwise or uniform. However, even if this convergence is uniform in y, there is no guarantee of any meaningful kind of convergence in (2.4), nor even the existence of the integrals on the left-hand side. Furthermore, even in cases

[4] This approach is a very typical one for pricing of contingent claims, such as options, or bonds in a term structure model. For a very different approach to modeling the term structure of interest rates, see Heath, Jarrow, and Morton (1992).

[5] See Levendorskii (2004a) and Levendorskii (2004b) for a recent discussion of this issue for affine models. General conditions for the equivalence of the probabilistic and partial differential equation problems that are necessary, sufficient, and simple to apply remain elusive.

where the conditional moment approximations do converge, it may be difficult or impossible to calculate the integrals explicitly. Finally, even if these difficulties can be overcome, the approximation method still must be extended to take the $r(x)$ coefficient in (2.2) into account, for approximation of contingent claims prices.

We therefore take a different approach, which is to construct a power series representation to the conditional moment or contingent claims price directly, without going through the intermediate step of constructing a likelihood representation or fundamental solution. The form of the partial differential equation (2.2) suggests that solutions can be written as a power series in Δ, centered at zero,

$$f(\Delta, x) = a_0(x) + \sum_{n=1}^{\infty} a_n(x) \frac{\Delta^n}{n!}. \tag{2.5}$$

A power series converges in $|\Delta| < r$ for some $r \geq 0$. The final condition requires

$$a_0(x) = g(x). \tag{2.6}$$

Substituting the proposed solution into (2.2), and gathering terms of like order in Δ, the functions $a_n(x)$ for $n \geq 1$ must satisfy a recursive relation

$$a_n(x) = \sum_{i=1}^{N} \mu_i(x) \frac{\partial a_{n-1}}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2(x) \frac{\partial^2 a_{n-1}}{\partial x_i \partial x_j}(x) - r(x) a_{n-1}(x). \tag{2.7}$$

Provided $g(x)$, $\mu(x)$, $\sigma(x)\sigma^T(x)$, and $r(x)$ are all infinitely differentiable in a neighborhood of x, the coefficients $a_n(x)$ exist.[6] The series described in (2.5), (2.6), and (2.7) can be interpreted as the deterministic part of the stochastic Itô-Taylor expansions as discussed in, for example, Kloeden and Platen (1999).

Given the requisite smoothness conditions of the three coefficients of (2.2) and the final condition, derivation of a power series representation of a solution is straightforward. Much less straightforward is determining whether and where the series converges. Any power series converges trivially at the point where the series is centered, since all terms but the first are zero. However, for large Δ (and possibly for any $\Delta \neq 0$), the power series may not converge; worse still, it may converge to the wrong function. Although the probabilistic problem (2.1) is meaningful only for non-negative real values of the time horizon, $\Delta \in [0, +\infty)$, it is nonetheless advantageous to consider the partial differential equation problem (2.2) (with final condition) in the more general setting of complex values of time, since the behavior of the solution for complex values of Δ affects the region of the convergence of the power series, even when the coefficients of the partial differential equation and final condition satisfy strong smoothness conditions for positive time horizons.

To illustrate some of the problems that can occur, we consider the very simple special case of finding conditional moments of a function of the terminal value of a Brownian motion. We seek the conditional

[6] Infinite differentiability of these functions is sufficient, but not necessary, for the existence of the coefficients $a_n(x)$. For example, if $g(x)$ and $\mu(x)$ are both affine in x, then the coefficients $a_n(x)$ can be found even if $\sigma(x)\sigma^T(x)$ is not differentiable. The coefficients $a_n(x)$ can even be found in some cases in which the coefficients of the partial differential equation do not specify a valid diffusion process in the analogous probabilistic problem. However, existence of the power series coefficients does not by itself guarantee convergence of the series anywhere but the origin.

moment

$$f(\Delta, x) = E\left[\exp\left(\frac{cX_{t+\Delta}^2}{2}\right)|X_t = x\right]$$

where X_t is a Brownian motion. In this case, the coefficients of (2.2) are $\mu(x) = 0$, $\sigma(x) = 1$, and $r(x) = 0$, and the final condition is $g(x) = \exp(cx^2/2)$. The solution is

$$f(\Delta, x) = \frac{\exp\left(\frac{cx^2}{2(1-\Delta c)}\right)}{\sqrt{1-\Delta c}}.$$

This function has a singularity at $\Delta = 1/c$, and a power series around $\Delta = 0$ therefore converges only for $|\Delta| < 1/|c|$ (and possibly also for some $|\Delta| = 1/|c|$). If $c > 0$, the conditional expectation is not defined for $\Delta \geq 1/c$, so non-convergence is appropriate for these values; the tails of the final condition grow too quickly as a function of x, so the conditional expectation becomes undefined for excessively large Δ. However, for $c < 0$, the conditional moment is defined and satisfies strong smoothness conditions for all $\Delta \geq 0$. Nonetheless, the singularity at $\Delta = 1/c$ still prevents convergence of the series for $\Delta > 1/|c|$, even though the conditional moment function is well-behaved in this range. In the $c > 0$ case, the series fails to converge for $\Delta > 1/c$ because the tails of the final condition grow too quickly; in the $c < 0$ case, it fails to converge for $\Delta > 1/|c|$ because the tails go to zero too quickly.

In addition to excessively thick or thin tails in the final condition, excessive oscillation in the tails can also be a problem. Consider the conditional moment

$$f(\Delta, x) = E\left[\cos\left(\frac{cX_{t+\Delta}^2}{2}\right)|X_t = x\right]$$

for any real $c \neq 0$, where X_t is (as before) a Brownian motion. The cosine function is even, so we take $c > 0$ without loss of generality. The solution is

$$f(\Delta, x) = \frac{\exp\left(-\frac{x^2\Delta c^2}{2(1+\Delta^2 c^2)}\right)}{\sqrt[4]{1+c^2\Delta^2}} \cos\left[\frac{cx^2}{2(1+c^2\Delta^2)} + \frac{\arctan(c\Delta)}{2}\right].$$

For real values of Δ we take the fourth root to be the positive branch, and extend it by analytic continuation. This solution is then well-behaved for all real Δ; however, there are singularities in $f(\Delta, x)$ for imaginary values, at $\Delta = \pm\imath/c$. As in the previous example, this power series converges for all $|\Delta| < 1/c$ (and possibly also for some $|\Delta| = 1/c$), but diverges elsewhere; the singularities at imaginary Δ prevent convergence of the power series for large positive Δ.

Some power series fail to converge for any values except $\Delta = 0$. Consider the contingent claim price

$$f(\Delta, x) = E\left[e^{-\int_t^{t+\Delta} r(X_u)du} \max(X_{t+\Delta} - K, 0)\right]$$

where $\mu(x) = \mu x$, $\sigma(x) = \sigma x$, $r(x) = r$, and $g(x) = \max(x - K, 0)$. The solution is the well-known option pricing formula of Black and Scholes (1973) and Merton (1973),

$$f(\Delta, x) = X_t N\left(\frac{\ln\frac{X_t}{K} + \left(r + \frac{\sigma^2}{2}\right)\Delta}{\sigma\sqrt{\Delta}}\right) - Ke^{-r\Delta} N\left(\frac{\ln\frac{X_t}{K} + \left(r - \frac{\sigma^2}{2}\right)\Delta}{\sigma\sqrt{\Delta}}\right), \tag{2.8}$$

where $N(\bullet)$ is the cumulative normal distribution function. This solution is analytic in a neighborhood of any value of Δ except $\Delta = 0$,[7] so a power series constructed as in (2.6) and (2.7) converges to the solution only for this value, which is trivial, since the solution at $\Delta = 0$ is assumed as part of the problem statement.

Other than for their use as illustrative examples, there is little point in finding power series representations of functions that are already known in closed-form. However, the problems encountered in the examples above can also occur in those cases for which the solutions are not known in closed-form. Even if a conditional moment or asset price function is well-behaved for positive real values of Δ (i. e., those values of interest in typical applications), a power series fails to converge if the final (or payoff) condition has tails that, for example, are too thin, or oscillate too quickly. In these cases, singularities for negative or complex values of Δ prevent convergence of the series for positive real values of Δ. The next section considers the problem of determining when solutions to conditional moment or contingent claims pricing problems have analytic (in the time variable) solutions.

3 Existence of Analytic Solutions

For some choices of $\mu(x)$, $\sigma(x)$, and $r(x)$, it may not be obvious that there are any final conditions $g(x)$ at all such that the solution $f(\Delta, x)$ of (2.2) is analytic in the time variable in some neighborhood of the origin. Analyticity of solutions for Δ with positive real part follows from well-known results in the literature; for example, analyticity of the fundamental solution to a general problem on a bounded domain follows from the construction of Friedman (1964). Colton (1979) shows that, on a bounded domain, there exists, for any final condition, an approximate solution to the general scalar PDE problem, which is analytic in the time variable in a neighborhood of the origin.[8] However, these results are not useful for our purposes. The construction of Friedman (1964) does not establish analyticity of the solution at $\Delta = 0$, necessary for convergence of a power series around that point, and the results apply to bounded domains, not the unbounded domains typical in economic and financial applications. Furthermore, although Colton (1979) demonstrates the existence of an approximate solution, no practical method to find it is given. We therefore analyze the scalar case, and show that there is an infinite-dimensional family of $g(x)$ that give rise to analytic solutions in a neighborhood of the origin, on domains that are not necessarily bounded. We first consider a special case

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{\sigma^2(y)}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y). \tag{3.1}$$

[7] The cumulative normal function may be extended to complex values of the argument by analytic continuation, using its power series representation.

[8] Colton (1979) focuses only on negative values of the time variable; although the solutions constructed are analytic, this property is not emphasized, or even noted.

Continuity of $\sigma(y)$, and positivity on the interior of the state space, suffice for the existence of an infinite-dimensional set of final conditions, such that the solution to (3.1) (with final condition) is everywhere analytic. This follows by a simple construction. Consider an interval $y \in [c, d]$ on which the function $\sigma^2(y)$ is continuous and non-zero, and let $\sigma_{\min}$ and $\sigma_{\max}$ denote the minimum and maximum values of $\sigma(y)$ on this interval; without loss of generality, we take $\sigma(y) > 0$. Then $a_0(y) = 1$ and $a_1(y) = y$ are both solutions to (3.1). Given $a_n(y)$ for some integer $n \geq 0$, define

$$a_{n+2}(y) \equiv \int_{y_0}^{y} \int_{y_0}^{v} \frac{a_n(u)}{\sigma^2(u)} du dv$$

for some $y_0 \in [c, d]$. Note that

$$h_{0,n}(\Delta, y) = \sum_{i=0}^{n} \frac{a_{2i}(y)}{(n-i)!} \left(\frac{\Delta}{2}\right)^{(n-i)}$$
$$h_{1,n}(\Delta, y) = \sum_{i=0}^{n} \frac{a_{2i+1}(y)}{(n-i)!} \left(\frac{\Delta}{2}\right)^{(n-i)}$$

are solutions to (3.1) with final conditions specified by $h_{0,n}(0, y) = a_{2n}(y)$ and $h_{1,n}(0, y) = a_{2n+1}(y)$. Then $h_{0,n}(\Delta, y)$ and $h_{1,n}(\Delta, y)$ are polynomials in Δ, and therefore everywhere analytic. Finite linear combinations of the $h_{0,n}(\Delta, y)$ and $h_{1,n}(\Delta, y)$

$$h(\Delta, y) = \sum_{j=0}^{k} c_j h_{0,j}(\Delta, y) + \sum_{j=0}^{k} d_j h_{1,j}(\Delta, y)$$

are also solutions to (3.1) with the final condition

$$h(0, y) = \sum_{j=0}^{k} c_j a_{2j}(y) + \sum_{j=0}^{k} d_j a_{2j+1}(y).$$

Such functions are also everywhere analytic in Δ. The $a_n(y)$ form an infinite-dimensional space of functions, as can be shown by the following argument. Consider only $a_n(y)$ for even n. Then suppose there is some linear combination of a_{2i} for $0 \leq i \leq n$ such that

$$\sum_{j=0}^{n/2} c_j a_{2j}(y) = 0. \tag{3.2}$$

Then

$$h(\Delta, y) = \sum_{j=0}^{n/2} c_j h_{0,j}(\Delta, y) \tag{3.3}$$

is a solution to the PDE (3.1) for all Δ, with final condition $h(0, y) = 0$. However, by plugging the power series representation of $h(\Delta, y)$ into (3.1), it must be the case that $h(\Delta, y) = 0$. If $c_{n/2} \neq 0$, then $c_{n/2} h_{0,n/2}(\Delta, y)$ contains a term of order $\Delta^{n/2}$; none of the other terms on the right-hand side of (3.3) do. Consequently, it must be the case that $c_{n/2} = 0$. In other words, there is no linear combination with $c_{n/2} \neq 0$ that satisfies (3.2), and $a_n(y)$ is linearly independent of the $a_{2i}(y)$ for $0 \le i < \frac{n}{2}$. By similar argument, the $a_n(y)$ for odd n form an infinite-dimensional set of linearly independent final conditions, which give rise to an infinite-dimensional family of analytic solutions to the PDE.

Infinite linear combinations of the $h_{0,n}(\Delta, y)$ and $h_{1,n}(\Delta, y)$, provided they converge uniformly on all compact subsets in an open neighborhood of Δ and y, are also solutions to (3.1). Consider some d_i, $i \ge 0$, and define

$$h(\Delta, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{d_{k+l} a_{2l}(y)}{k!} \left(\frac{\Delta}{2}\right)^k + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{d_{k+l} a_{2l+1}(y)}{k!} \left(\frac{\Delta}{2}\right)^k .$$

Provided the d_i are chosen such that the sums converge uniformly on some compact set of Δ and y, it can be seen, by term-by-term differentiation, that $h(\Delta, y)$ is a solution (within that compact set) to the PDE (3.1). From the definition of the $a_n(y)$ and the bounds on $\sigma^2(y)$, it follows that

$$\begin{aligned} \frac{|y - y_0|^{2n}}{(2n)! \sigma_{\max}^{2n}} &\le |a_{2n}(y)| \le \frac{|y - y_0|^{2n}}{(2n)! \sigma_{\min}^{2n}} \\ \frac{|y - y_0|^{2n+1}}{(2n+1)! \sigma_{\max}^{2n}} &\le |a_{2n+1}(y)| \le \frac{|y - y_0|^{2n+1}}{(2n+1)! \sigma_{\min}^{2n}}. \end{aligned}$$

Then

$$|h(\Delta, y)| \le \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{|d_{k+l}|}{k!} \frac{|y - y_0|^{2l}}{(2l)! \sigma_{\min}^{2l}} \left(\frac{\Delta}{2}\right)^k + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{|d_{k+l}|}{k!} \frac{|y - y_0|^{2l+1}}{(2l+1)! \sigma_{\min}^{2l}} \left(\frac{\Delta}{2}\right)^k .$$

If, for example, the d_i are uniformly bounded, then the series for $h(\Delta, y)$ converges for all Δ and all $y \in [c, d]$, and is a solution to the PDE (3.1). However, existence of a solution does not require the d_i to be bounded; if they grow sufficiently slowly (as a function of i), then they still specify a solution with a convergent power series.

The PDE (3.1) may appear to be a very special case of the general scalar PDE,

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \mu(x) \frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2} \frac{\partial^2 f}{\partial x^2}(\Delta, x) - r(x) f(\Delta, x). \tag{3.4}$$

However, this PDE can be converted to (3.1) by changes of variables. If

$$f(\Delta, x) = w(x) h(\Delta, x)$$

where $w(x)$ is a solution to

$$\mu(x) w'(x) + \frac{\sigma^2(x)}{2} w''(x) - r(x) w(x) = 0$$

then (3.4) is equivalent to

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \left[\mu(x) + \sigma^2(x) \frac{w'(x)}{w(x)}\right] \frac{\partial h}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2} \frac{\partial^2 h}{\partial y^2}.$$

This change of dependent variables eliminates the coefficient on $h(\Delta, y)$. A further change of independent variable, using the scale transformation, eliminates the coefficient on the first spatial derivative as well, so the transformed PDE is of the form of (3.1) (but with a different coefficient on the second spatial derivative). Thus, despite its apparently restrictive appearance, the results on analytic solutions to (3.1) are very general; given only modest smoothness properties on the coefficients of (3.4), there exists an infinite-dimensional class of final conditions such that an everywhere analytic (in Δ) solution exists.

We have now characterized a set of final conditions for which the solution to (3.1) (and other scalar PDEs that can be transformed to (3.1) by change of variables) is analytic in Δ. However, this characterization may not always be very useful in practice. It is relatively straightforward to construct final conditions that generate analytic solutions, but it is less obvious how to take a given final condition and determine whether it is in fact spanned by the $a_n(y)$ functions specified above. However, it is often possible, for specific problems, to characterize explicitly the set of final conditions that correspond to analytic (in Δ) solutions. The following section explores these cases.

4 Analytic Solutions to Scalar Diffusion Problems

The results of Section 2 allow construction of the power series representations of solutions to conditional moment or asset pricing problems; the results of Section 3 show that, for essentially any scalar diffusion problem, there exists a large, non-trivial class of such problems for which the power series converge. However, in practice, it may be difficult to determine whether, for a given diffusion, the final condition is such that the series does indeed converge. In this section, we consider the problem of determining when, given only the general PDE (that is, the dynamics of the economy) and the final condition (that is, the particular conditional moment or contingent claim price sought), a solution has the analyticity properties needed to apply these results. Our focus in this section is on scalar diffusion and asset pricing problems, but we note that multiple diffusion problems can sometimes be decomposed into a system of scalar problems; for example, if two state variables follow independent diffusions, and enter into the interest rate function additively, then the bond pricing problem is equivalent to two scalar problems. The results of this section therefore have some applicability to multivariate diffusion problems as well.

We first describes some changes of variables that convert scalar conditional moment or pricing problems into a canonical form. We then explore two particular classes of problems in detail, characterizing explicitly the region of analyticity of the solution, and therefore the range of convergence of its power series. For these two classes of problems, the region of analyticity of the solution depends critically on smoothness and growth conditions. Smoothness (i. e., analyticity of the final condition) and growth at a rate bounded by $c \exp\left(kx^2\right)$ for some $c, k > 0$, in all directions in the complex plane, result in a region of analyticity around the origin, with the size of this region determined by the k parameter. If such a bound can be established for all k (with c possibly depending on $k0$, then analyticity (and therefore convergence of the power series) for all values of the time variable can be established.

4.1 Canonical Form PDE

Change of independent variable is a technique frequently used to simplify analysis of a diffusion process (or, equivalently, a parabolic partial differential equation). Less used in the economics and finance literature are changes of time variable and dependent variable, although the latter technique has been used in the partial differential equation literature; see, for example, Colton (1979). By the use of such transformations, solution of the general Feynman-Kac problem can often be reduced to solution of a special case, although, many such transforms cannot easily be applied to multivariate diffusions. Focusing on the scalar diffusion case, we consider the problem

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \mu(x)\frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2}\frac{\partial^2 f}{\partial x^2}(\Delta, x) - r(x) f(\Delta, x). \tag{4.1}$$

with final condition $f(0, x) = g(x)$. We seek solutions to this equation for all $x \in (a, b)$ where a and b are the boundaries of the diffusion process (with $a = -\infty$ or $b = +\infty$ or both possible) and $\Delta \in [0, T]$ for some $T > 0$. We require $\sigma(x) \neq 0$ for all $x \in (a, b)$; as the PDE is motivated by a diffusion process, it is usually the case that $\mu(x)$ and $\sigma(x)$ are such that the boundaries a and b cannot be reached in finite time. However, it is possible to analyze the PDE problem without imposing such restrictions. In general, there are multiple solutions to the PDE problem, even with the given final condition, but at most one of these solutions is also the solution to the probabilistic problem.[9] However, there can be at most one solution which is analytic in the time variable, and, if it exists, this solution is the one that also solves the corresponding probabilistic problem.

Several different changes of variables have been used to simplify stochastic processes and/or partial differential equations. The scale transformation (see, for example, Karlin and Taylor, 1981) is a change of independent variable that eliminates the drift from a diffusion process (or, equivalently, removes the first spatial derivative term from a PDE). Aït-Sahalia (2002) uses a different change of independent variable to normalize the diffusion coefficient of a stochastic differential equation to a constant (or, equivalently, to set the coefficient of the second spatial derivative in a PDE to a constant). Colton (1979) transforms both dependent and independent variables, allowing both elimination of the first spatial derivative term and normalization of the second spatial derivative coefficient to a constant. We use both these transforms

$$y = \int^x \frac{du}{\sigma(u)} \qquad f(\Delta, x) = e^{-\int^x \left[\frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)}\right] du} h(\Delta, y).$$

The lower limits of the integrals are not specified, so these expressions really describe a family of transforms. Positivity and continuity of $\sigma(x)$ on the interior of the state space ensure that y is strictly increasing in x, and can be inverted.

[9] For example, the function that is everywhere zero is a solution to the ordinary heat equation with a final condition of zero, and is also the correct solution to the corresponding probabilistic problem. However, there also exist non-zero solutions to the same PDE with the same final condition; see the construction in Cannon (1984). The alternate solution is necessarily non-analytic at $\Delta = 0$; if it were analytic, the coefficients of the power series would have to satisfy the recursive relation derived in Section 2, and with a final condition equal to zero, this relation can only be satisfied if the coefficients are all zero.

The transformed PDE, in terms of y and $h(\Delta, y)$ instead of x and $f(\Delta, x)$, is

$$\frac{\partial h}{\partial \Delta} = \frac{1}{2}\frac{\partial^2 h}{\partial y^2} - r_h(y)\, h \tag{4.2}$$

with

$$r_h(y) \equiv -\frac{\mu^2(x)}{2\sigma^2(x)} - \frac{\mu'(x)}{2} + \frac{\mu(x)\,\sigma'(x)}{\sigma(x)} - \frac{\sigma'(x)\,\sigma'(x)}{8} + \frac{\sigma''(x)\,\sigma(x)}{4} - r(x)$$

where x is an implicit function of y. The final condition, in terms of h and y, is

$$h(0, y) = g_h(y) \equiv \exp\left(\int^x \left[\frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)}\right] du\right) g(x). \tag{4.3}$$

Provided the diffusion coefficient is bounded away from zero on the interior of the state space, these transforms are always well-defined (although in some cases we may not be able to evaluate the integral in the transformed final condition explicitly). Since y as a function of x can be inverted, the process Y_t, defined by applying the change of independent variables to X_t, inherits the Markov property of X_t. On the interior of the state space of X_t, the ratio between f and h is positive, so, for example, a strictly positive f implies a strictly positive h.

We can also assign probabilistic meaning to the transformed PDE given in (4.2); this same equation (given sufficient regularity conditions) arises as the solution to the probabilistic problem

$$h(\Delta, y) = E\left[g_h(W_{t+\Delta}) \exp\left(-\int_t^{t+\Delta} r_h(W_u) du\right) \middle| W_t = y\right]$$

where W_t is a canonical Brownian motion. Note, however, that although the independent variable in this equation is y, the process Y_t, defined by the change of independent variables applied to X_t, is in general not a Brownian motion, and may have a state space different than the state space of the Brownian motion (i. e., the entire real line). Nonetheless, the changes of variables described show that the original pricing problem is equivalent to the problem of finding a functional of a Brownian motion, even when the state variable is not a Brownian motion. The asset pricing problem is then equivalent to the problem of pricing a different asset in a different economy, in which both the interest rate and the final payoff of the alternate asset are functions of the value of a Brownian motion.

In a term structure context, the pricing PDEs for models that may seem quite distinct at first can sometimes be transformed by change of variables to the same general PDE, with only the final condition differing. For example, in the scalar version of the linear-quadratic model of Ahn, Dittmar, and Gallant (2002), the $r_h(y)$ coefficient is a quadratic function of y; the model of Vasicek (1977) transforms to the same PDE after application of the change of variables; in both cases, the final condition for a zero-coupon bond price is $g(x) = 1$, which implies that $g_h(y)$ is exponential quadratic (but with different coefficients in the two models). The pricing PDEs for the models of Cox, Ingersoll, and Ross (1985) and Ahn and Gao (1999) both transform to the case in which $r_h(y)$ contains a term proportional to y^2, a constant term, and a term proportional to $1/y^2$; the two models then differ (for bond pricing purposes) only in the parameter values and the specification of g_h. The pricing PDE for callable corporate bonds in the model of Jarrow, Li, Liu, and Wu (2006) also transforms to this case; these authors use our technique to approximate callable bond prices. However, there also exist many other models that have not yet appeared in the literature, but that also transform to these cases.

In Section 3, it was shown that, for essentially arbitrary choice of $\mu(x)$, $\sigma(x)$, and $r(x)$ functions, there is an infinite-dimensional family of final conditions $g(x)$ such that the corresponding solution $h(\Delta, y)$ of (4.2), with final condition as in (4.3), is analytic in Δ in some neighborhood of the origin. That section also shows how to construct such a $g(x)$. However, the methods of that section are not particularly useful in solving the reverse problem, that of determining, for a given $g(x)$ function, whether $h(\Delta, y)$ is analytic. Determining whether the solution is analytic around the origin is important, since, if it is not, a power series does not converge anywhere else. Even if the solution is analytic in some neighborhood of $\Delta = 0$, it is important to know the locations of any singularities, since the location of such singularities determines the range of convergence of any power series constructed. There exist at least two forms of (4.2) for which the location of the singularities of the solutions can be characterized explicitly. These two forms encompass many, if not all, of the term structure models commonly used in the literature for which bond prices are known explicitly.[10] However, these two forms also encompass many other potential models that have not as yet been studied. The next two sections examine these two classes of models; see Kimmel (2008) and Jarrow, Li, Liu, and Wu (2006) for applications of these results.

4.2 Brownian Motion

The special cases of (4.2) in which the $r_h(y)$ coefficient is either linear or quadratic

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - (ay + d)\, h(\Delta, y) \tag{4.4}$$

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y - a)^2 + d\right] h(\Delta, y) \tag{4.5}$$

admit particularly straightforward analysis. A constant or zero r_h function is a special case of both (4.4) and (4.5). It is possible, by parameterizing the r_h function in (4.5) slightly differently, to include (4.4) as a special case as well; however, the solutions of the two equations have sufficiently different properties so as to warrant separate treatment. The region of analyticity for a solution $h(\Delta, y)$ of (4.4) and (4.5) can be characterized in a straightforward manner, as shown in the following two theorems. Throughout not only this section but also the next, we take as given a norm $\|z\|$ (over the reals) on the set of all complex numbers z. Such norms

[10] The model of Ahn and Gao (1999) is encompassed by one of the two forms discussed here. However, this model is a rare (and perhaps unique) case in that bond prices are not analytic in maturity at $\Delta = 0$, unless strong (and probably unrealistic) restrictions are imposed on the model parameters. Consequently, power series for bond prices under this model do not converge. To the best of our knowledge, this model is the only interest rate model with closed-form bond prices that cannot be represented by convergent power series, although it should be noted that bond prices are closed-form under this model only if the confluent hypergeometric function (also known as Kummer's function) is taken to be fundamental. Although power series for bond prices do not usually converge in this model, there exist other classes of security prices, within the same model, with convergent power series representations.

include $\|z\| \equiv |z|/k$ for $k > 0$, but also asymmetric norms such as

$$\|z\| \equiv \sqrt{(\mathrm{Re}\, z)^2 / k_1 + (\mathrm{Im}\, z)^2 / k_2}.$$

Use of norms that are asymmetric with respect to direction in the complex plane establishes regions of analyticity (in the time variable) for PDE solutions that extend further in some directions in the complex plane than in others. Power series converge within a circle, so it may seem only symmetric norms are particularly useful. However, it is possible to construct power series not in Δ, but in some non-affine function of Δ; appropriate choice of such a non-affine function can then effectively extend the range of convergence for positive Δ if analyticity of the solution can be established in a non-circular region. See Kimmel (2008) for results and examples. We therefore state most results in terms of a general norm.

The following theorem characterizes the region of analyticity of the solution to (4.4).

Theorem 1. *Let $g(y)$ be analytic for all y, and let there exist some $c > 0$ and some norm (over the reals) $\|y\|$ such that $g(y)$ satisfies*

$$|g(y)| \leq ce^{\frac{\|y\|^2}{2}}.$$

Let a and d be arbitrary numbers. Then there exists an $h(\Delta, y)$, defined and analytic for all complex y and $\|\sqrt{\Delta}\| < 1$, that satisfies

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - (ay + d)\, h(\Delta, y) \tag{4.6}$$

$$h(0, y) = g(y). \tag{4.7}$$

Proof: See appendix.

Although the solution to (4.6) and (4.7) obviously depends on the parameters a and d, the region of existence and analyticity established by the theorem does not. This result can be interpreted as follows: if the theorem establishes that the conditional expectation of some function of a Brownian motion is analytic in a particular region, it also establishes that the discounted expected value of the same function is analytic in the same region, if the instantaneous interest rate is an affine function of the state variable. Note, however, that Section 4.1 shows that the theorem applies to a much broader class of problems than those driven by a Brownian motion. Many non-affine problems are covered by the theorem, by the changes of variables described in that section.

By contrast, the quadratic coefficient b in (4.5) has a strong effect on the region in which the PDE solution can be shown to exist and be analytic in Δ. The following theorem addresses this case.

Theorem 2. *Let $g(y)$ be analytic for all y, and let a, b, and d be arbitrary numbers. Let there exist some $c > 0$ and some norm (over the reals) $\|y\|$ such that $g(y)$ satisfies*

$$\left| e^{-\frac{b}{2}(y-a)^2} g(y) \right| \leq ce^{\frac{\|y\|^2}{2}}.$$

Define

$$\tau(\Delta) \equiv \begin{cases} \frac{e^{2b\Delta}-1}{2b} & b \neq 0 \\ \Delta & b = 0 \end{cases}.$$

Then there exists an $h(\Delta, y)$*, defined and analytic for all complex values of* y *and* $\|\sqrt{\tau(\Delta)}\| < 1$*, that satisfies*

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y-a)^2 + d\right] h(\Delta, y) \tag{4.8}$$

$$h(0, y) = g(y). \tag{4.9}$$

Proof: See appendix.

In probabilistic terms, this theorem describes a large class of functions of a Brownian motion whose conditional expectations are analytic in the time variable, and characterizes the region of analyticity. However, it also applies to many other situations. For example, a process which is not a Brownian motion, but that can be changed to a Brownian motion by change of independent variable, is also covered by applying Theorem 1 or Theorem 2 after the change of variables. Similarly, this theorem effectively characterizes a set of final asset payoffs that generate pricing functions that are analytic in maturity, provided the pricing PDE can be converted to (4.4) or (4.5) by change of dependent and/or independent variables, as described in Section 4.1. For any of these applications, if the conditions of the theorem hold for a symmetric norm of the form $\|z\| \equiv |z| / \sqrt{k_0}$, then the solution to the PDE is analytic for all $|\Delta| < k_0$, and a power series approximation to the solution converges for at least these values of Δ. If the conditions of the theorem hold for an asymmetric norm, then time transformation methods (see Kimmel, 2008) may improve the range of convergence.

It may be useful to characterize those final conditions that correspond to solutions $h(\Delta, y)$ of (4.4) and (4.5) that are defined and analytic for all Δ. The following two corollaries examine these cases.

Corollary 1. *Let* $g(y)$ *be analytic for all* y*, and for each* $k > 0$*, let there exist some* $c_k > 0$ *such that* $g(y)$ *satisfies*

$$|g(y)| \leq c_k e^{\frac{|y|^2}{2k}}.$$

Let a *and* d *be arbitrary complex numbers. Then there exists an* $h(\Delta, y)$*, defined and analytic for all complex* y *and* Δ*, that satisfies*

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - (ay + d)\, h(\Delta, y)$$

$$h(0, y) = g(y).$$

Proof: See appendix.

Corollary 1 extends the result of Theorem 1; given stronger growth restrictions on $g(y)$, the region of analyticity can be extended to all values of Δ. The next result does the same thing for Theorem 2.

Corollary 2. *Let* $g(y)$ *be analytic for all* y*, and let* a*,* b*, and* d *be arbitrary numbers. For each* $k > 0$*, let* *there exist some* $c_k > 0$ *such that* $g(y)$ *satisfies*

$$\left| e^{-\frac{b}{2}(y-a)^2} g(y) \right| \leq c_k e^{\frac{|y|^2}{2k}}.$$

Then there exists an $h(\Delta, y)$*, defined and analytic for all complex* y *and* Δ*, that satisfies*

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y-a)^2 + d\right] h(\Delta, y)$$
$$h(0, y) = g(y).$$

Proof: See appendix.

These corollaries applies to all the same situations described in the discussion of Theorems 1 and 2, provided a stronger growth restrictions on the final condition are imposed. But, if the conditions of either Corollary 1 or 2 are satisfied, then the conditional moment or pricing function is analytic for all complex values of the time variable. The power series representation of the desired function then converges uniformly on the interval $[0, T]$ for any value $0 < T < +\infty$, although in general, this convergence is not uniform on the entire interval $[0, +\infty)$.

Several term structure models that have appeared in the literature are covered by Theorems 1 and 2 and by Corollaries 1 and 2 (as are many models that have not previously appeared in the literature). The model of Vasicek (1977) is covered by Corollary 2, and the model of Ahn, Dittmar, and Gallant (2002) is covered by Theorem 2. Power series for zero-coupon bond prices in the former model therefore converge for all maturities; for the latter model, the series converge for some finite range, and diverge for longer maturities. Even for the Vasicek (1977) model, though, the convergence is not uniform, with the result that, for very long maturities, a large number of terms in the power series may be needed before the truncated series is a good approximation to the bond price. However, Kimmel (2008) applies a non-affine transformation of the time variable to the bond pricing problem in both of these models, and converts them to a problem covered by Corollary 2. Power series for bond prices under both models then converge for all maturities. He goes on to show, under mild assumptions, that the convergence is uniform for all maturities. The value of the results in this section, however, is not the ability to approximate bond prices in models for which they are already known in closed-form, but to approximate price of bonds and other contingent claims, and also conditional moments, in the large class of problems that, by the changes of variables described in Section 4.1, are covered by the two theorems and corollaries. See Section 5 for a discussion of the models of Vasicek (1977) and Ahn, Dittmar, and Gallant (2002), and additional examples.

4.3 General Affine

The partial differential equation

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y). \tag{4.10}$$

is similar to the two cases considered in the previous section, but contains an extra term not included there. If $a = 0$, then this PDE is a special case of (4.4) (for $b = 0$) or (4.5) (for $b \neq 0$). We refer to this equation as the general affine PDE; although there may at first or even second glance seem to be nothing particularly affine about (4.10), the problem of finding conditional moments for the scalar affine diffusion of Feller (1951) can be reduced to the problem of solving (4.10) by changes of variables. The pricing PDE for those scalar affine yield models that are not covered by the results of the previous section can also be transformed to this PDE by change of variables, as can the pricing PDE for some non-affine yield models (e. g., Ahn and Gao, 1999). Jarrow, Li, Liu, and Wu (2006) consider a model in which the problem of pricing callable corporate bonds reduces, after change of variables, to the problem of solving this PDE with a particular final condition. The following theorem characterizes explicitly the final conditions that admit analytic (in Δ) solutions.

Theorem 3. *Let $g_1(y)$ and $g_2(y)$ be even and analytic for all y, and let a, b, and d be arbitrary numbers. Let there exist some $c > 0$ and some norm (over the reals) $\|y\|$ such that $g_1(y)$ and $g_2(y)$ satisfy*

$$\left|e^{-\frac{b}{2}y^2} g_1(y)\right| \leq c e^{\frac{\|y\|^2}{2}} \qquad \left|e^{-\frac{b}{2}y^2} g_2(y)\right| \leq c e^{\frac{\|y\|^2}{2}}. \tag{4.11}$$

Define

$$\tau(\Delta) \equiv \begin{cases} \frac{e^{2b\Delta}-1}{2b} & b \neq 0 \\ \Delta & b = 0 \end{cases}.$$

Then there exist $h_1(\Delta, y)$ and $h_2(\Delta, y)$, defined and analytic for all complex y and $\|\sqrt{\tau(\Delta)}\| < 1$, such that $h(\Delta, y)$, defined by

$$h(\Delta, y) \equiv \begin{cases} h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} & \frac{\sqrt{1+8a}}{2} \notin \mathbb{N} \\ h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y & \frac{\sqrt{1+8a}}{2} \in \mathbb{N} \end{cases},$$

satisfies

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y) \tag{4.12}$$

$$h(0, y) = \begin{cases} g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} & \frac{\sqrt{1+8a}}{2} \notin \mathbb{N} \\ g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y & \frac{\sqrt{1+8a}}{2} \in \mathbb{N} \end{cases} \tag{4.13}$$

for all complex $y \neq 0$ and $\|\sqrt{\tau(\Delta)}\| < 1$. Throughout, $\sqrt{1+8a}$ refers to the positive square root when $a \geq -1/8$.

Proof: See appendix.

If $a = 0$, this PDE reduces to that of either Theorem 1 or Theorem 2, and $h(\Delta, y)$ is a solution to the PDE at $y = 0$ as well. If $a \neq 0$ and $\sqrt{1+8a}$ is an odd integer, then $h(\Delta, y)$ generally has a pole of order $\left(\sqrt{1+8a} - 1\right)/2$ at $y = 0$, but is analytic at this point in the special case $g_1(y) = 0$. If $\sqrt{1+8a}$ is not an odd integer, then the solution is never analytic at $y = 0$ (except for the trivial special case of $g_1(y) = g_2(y) = 0$), and there is also a branch cut discontinuity in the complex plane. In this case, we take $h(\Delta, y)$ to be a so-called *global analytic function*, which is an equivalence class of overlapping branches of the analytic continuation of a function (see, for example, Ahlfors, 1979). The solution can then be considered to be analytic for all values of $y \neq 0$.

The above result, while valid for any value of a, may not be expressed in the most useful form for $a < -1/8$, since the two terms in the expression for $h(0, y)$ are then in general complex, even if $g_1(y)$ and $g_2(y)$ are real functions; most real-world applications have real-valued final conditions. However, the final condition can be restated equivalently as

$$h(0, y) = g_3(y)\sqrt{y}\cos\left(\frac{\sqrt{-8a-1}}{2}\right) + g_4(y)\sqrt{y}\sin\left(\frac{\sqrt{-8a-1}}{2}\right)$$

where $g_3(y) = g_1(y) + g_2(y)$ and $g_4(y) = \imath[g_2(y) - g_1(y)]$ or, equivalently, $g_1(y) = [g_3(y) + \imath g_4(y)]/2$ and $g_2(y) = [g_3(y) - \imath g_4(y)]/2$. If $g_1(y)$ and $g_2(y)$ satisfy the growth conditions of (4.11), then $g_3(y)$ and $g_4(y)$ also satisfy the same growth conditions (possibly with a different value of c); the final condition $g(y)$ is then real, provided $a < -1/8$ and $g_3(y)$ and $g_4(y)$ are real functions.

As in the Brownian motion case, we may interpret solutions to (4.10) as functionals of a Brownian motion; specifically, they are the expected value of the final condition, applied to the terminal value of the Brownian motion, discounted at an interest rate specified by the last term in the PDE,[11]

$$h(\Delta, y) = E\left[g(W_{t+\Delta})\, e^{-\int_t^{t+\Delta}\left(\frac{a}{W_u^2} + \frac{b^2}{2}W_u^2 + d\right)du} \middle| W_t = y\right].$$

The theorem describes a large class of final conditions such that the corresponding solutions $h(\Delta, y)$ are analytic in the time variable, and characterizes the region of analyticity. But as in the Brownian motion case, it also applies to many other situations. For example, conditional moments of the square-root process of Feller (1951) (after change of independent variable) satisfy this PDE. Furthermore, conditional moments of any process that can be changed to the square-root process by a change of independent variable are also covered the theorem. Similarly, the theorem effectively characterizes a set of final asset payoffs that generate pricing functions which are analytic in maturity, for a wide combination of process and interest rate specifications, provided the pricing PDE can be converted to (4.10) by change of dependent and/or independent variables, as in Colton (1979).

It may be useful, as in Section 4.2, to characterize those final conditions that correspond to solutions $h(\Delta, y)$ to (4.10) that are analytic for all values of Δ. The following corollary examines this case.

Corollary 3. *Let $g_1(y)$ and $g_2(y)$ be even and analytic for all y, and for each $k > 0$, let there exist some $c_k > 0$ such that $g_1(y)$ and $g_2(y)$ satisfy*

$$|g_1(y)| \le c_k e^{\frac{|y|^2}{2k}} \qquad |g_2(y)| \le c_k e^{\frac{|y|^2}{2k}}$$

[11] This interpretation is subject to technical conditions that make the probabilistic problem equivalent to the partial differential equation. However, as none of the subsequent analysis depends on the applicability of this interpretation, we do not verify these conditions.

Then for any numbers a, b, and d, there exist $h_1(\Delta, y)$ *and* $h_2(\Delta, y)$*, defined and analytic for all complex y and* Δ*, such that* $h(\Delta, y)$*, defined as*

$$h(\Delta, y) \equiv \begin{cases} h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} & \frac{\sqrt{1+8a}}{2} \notin \mathbb{N} \\ h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y & \frac{\sqrt{1+8a}}{2} \in \mathbb{N} \end{cases}$$

satisfies

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y)$$

$$h(0, y) = \begin{cases} g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} & \frac{\sqrt{1+8a}}{2} \notin \mathbb{N} \\ g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y & \frac{\sqrt{1+8a}}{2} \in \mathbb{N} \end{cases}$$

for all complex $y \neq 0$ *and* Δ*.*

Proof: See appendix.

This corollary applies to all the same situations described in the discussion of Theorem 3, provided the stronger growth restriction on the final condition is imposed. But, if the conditions of the corollary apply, then the conditional moment or pricing function is analytic for all complex values of the time variable, and the power series of the desired function converges for all Δ. However, this convergence is in general not uniform on $\Delta \in [0, +\infty)$.

As in the Brownian motion case, these previous two results can be applied to a broad class of problems. As previously noted (and shown in detail in Section 5), several term structure models that have appeared in the literature reduce to the Brownian motion case after changes of variables; several more reduce to the general affine case (as do many other models that have not previously appeared in the literature). Furthermore, better convergence properties can often be established by change of the time variable before application of Theorem 3 or Corollary 3. This method often extends the range of Δ for which a power series converges, and sometimes even establishes uniform convergence for all positive Δ. See Kimmel (2008).

5 Examples

In this section, we approximate solutions, by truncated power series, to the conditional moment and bond pricing problem in several models. Some of these models have appeared in the literature, and have closed-form solutions; for these cases, the approximate and exact solutions can be compared, to evaluate the accuracy of the approximation. Other cases we examine have not previously appeared in the literature, and exact solutions are unknown. All cases are covered by the results of Section 4, which establish the convergence of the power series solutions to these problems, and which also establish the range of convergence.

5.1 Vasicek Term Structure Model

In the model of Vasicek (1977), the risk-neutral interest rate process follows

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma dW_t.$$

The time t price $P\left(\Delta, r\right)$ of a zero-coupon bond, that pays one unit of account at time $t+\Delta$, with the current short interest rate given by $r_t = r$, satisfies

$$\begin{aligned}\frac{\partial P}{\partial \Delta}\left(\Delta, r\right) &= \kappa\left(\theta - r\right)\frac{\partial P}{\partial r}\left(\Delta, r\right) + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2}\left(\Delta, r\right) - rP\left(\Delta, r\right)\\ P\left(0, r\right) &= 1.\end{aligned}$$

The changes of variables of Section 4.1, which in this case are

$$P\left(\Delta, r\right) = e^{\frac{\kappa}{2}\left[y(r) - \frac{\theta}{\sigma}\right]^2} h\left(\Delta, y\left(r\right)\right) \qquad y\left(r\right) = \frac{r}{\sigma},$$

change the PDE to the canonical form,

$$\begin{aligned}\frac{\partial h}{\partial \Delta}\left(\Delta, y\right) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}\left(\Delta, y\right) - \left[\frac{\kappa^2}{2}\left(y - \frac{\theta}{\sigma}\right)^2 + \sigma y - \frac{\kappa}{2}\right] h\left(\Delta, y\right)\\ h\left(0, y\right) &= e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}.\end{aligned}$$

This canonical problem is covered by Corollary 2, which establishes existence of a solution that is everywhere analytic in Δ, with

$$a = \frac{\theta}{\sigma} - \frac{\sigma}{\kappa^2} \qquad b = -\kappa \qquad d = \theta - \frac{\kappa}{2} - \frac{\sigma^2}{2\kappa^2}.$$

For any $k > 0$, there exists a $c_k > 0$ such that

$$\left|e^{-\frac{b}{2}(y-a)^2} g\left(y\right)\right| = \left|e^{y\frac{\sigma}{\kappa} - \frac{\theta}{\kappa} + \frac{\sigma^2}{2\kappa^3}}\right| \le c_k e^{\frac{|y|^2}{2k}},$$

so the bound of Corollary 2 is satisfied. Note that this bound would not be satisfied for $b = \kappa$ instead of $b = -\kappa$, even though Theorem 2 allows either choice; the choice of $b = -\kappa$ therefore establishes a stronger result.

Corollary 2 (with $b = -\kappa$) establishes that $h\left(\Delta, y\right)$ is everywhere analytic in Δ and in y. From the relation between $P\left(\Delta, r\right)$ and $h\left(\Delta, y\right)$, it follows that $P\left(\Delta, r\right)$ is also analytic in both Δ and r. A power series can be constructed for either $h\left(\Delta, y\right)$ or $P\left(\Delta, r\right)$, with coefficients calculated by (2.6) and (2.7); in either case, the series converges for all Δ, irrespective of the value of y or r. The first few coefficients in the power series for $h\left(\Delta, y\right)$ are

$$\begin{aligned}a_0\left(y\right) &= e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}\\ a_1\left(y\right) &= e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}\left(-y\sigma\right)\\ a_2\left(y\right) &= e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}\left(y^2\sigma^2 + y\kappa\sigma - \theta\kappa\right)\end{aligned}$$

$$a_3(y) = e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left[-y^3\sigma^3 - 3y^2\kappa\sigma^2 + \left(3\theta\kappa - \kappa^2\right)y\sigma + \theta\kappa^2 + \sigma^2\right].$$

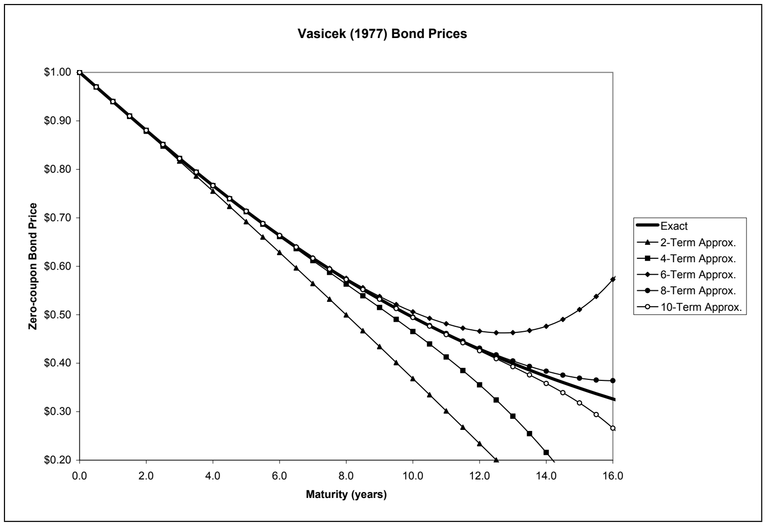

Figure 1: This figure shows prices of zero-coupon bonds as a function of maturity, from the model of Vasicek (1977), both in closed-form and with approximations including varying numbers of terms. The risk-neutral parameters are from Cheridito, Filipović, and Kimmel (2007), and the current instantaneous interest rate is taken to be $r = 6\%$. As shown, the two-term approximate price quickly deviates substantially from the true price as maturity increases beyond a few years. However, approximations with more terms do much better; the error in yield of the 10-term approximation for a bond with maturity of ten years is approximately one basis point. For this model, bond price approximations converge for all maturities; however, the convergence is not uniform, and might therefore be slow for very large maturities.

Figure 1 compares the true prices of zero-coupon bonds under the model of Vasicek (1977), to prices obtained using the approximations from truncated power series as described here. The risk-neutral parameter values used are from Cheridito, Filipović, and Kimmel (2007). As shown, the approximations are quite accurate even with a small number of terms for short maturity bonds. As maturity increases, more terms are needed to achieve a given level of accuracy; ten term approximations are very accurate for maturities up to at least ten years. More terms are needed as maturity increases because, although the series converge for all maturities, they do not do so uniformly. Consequently, a large number of terms may be needed for accuracy when the maturity is very long. By combining our methods with the time transformation techniques of Kimmel (2008), it is possible to improve dramatically the convergence properties; he considers this particular model, and finds a combination of the two methods results in approximations that are extremely accurate for all maturities, even with only a few terms.

5.2 Ahn, Dittmar, and Gallant

Another model in which bond prices are known in closed-form is that of Ahn, Dittmar, and Gallant (2002), so it also serves as an illustrative example. The risk-neutral state variable process,

$$dx_t = \kappa(\theta - x_t)\,dt + \sigma dW_t,$$

is the same as in the Vasicek (1977) model, but the state variable cannot be identified with the instantaneous interest rate. The interest rate process is specified by:[12]

$$r_t = x_t^2 + \phi$$

The pricing PDE, with final condition, for a zero-coupon bond in this model is

$$\begin{aligned}\frac{\partial P}{\partial \Delta}(\Delta, x) &= \kappa(\theta - x)\frac{\partial P}{\partial x}(\Delta, x) + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(\Delta, x) - \left(x^2 + \phi\right)P(\Delta, x) \\ P(0, x) &= 1.\end{aligned}$$

The change of variables needed to put the PDE into the canonical form are exactly the same changes as in the Vasicek (1977) model, but with x in place of r. The canonical form PDE is then

$$\begin{aligned}\frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{\kappa^2}{2}\left(y - \frac{\theta}{\sigma}\right)^2 + \sigma^2 y^2 + \phi - \frac{\kappa}{2}\right]h(\Delta, y) \\ h(0, y) &= e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}.\end{aligned}$$

This canonical form PDE is similar to that derived in the Vasicek (1977) case. The last term is of the same functional form, but has different parameter values,

$$a = \frac{\kappa^2\theta}{\sigma\left(\kappa^2 + 2\sigma^2\right)} \qquad b = -\sqrt{\kappa^2 + 2\sigma^2} \qquad d = -\frac{\kappa}{2} + \frac{\theta^2\kappa^2}{\kappa^2 + 2\sigma^2} + \phi.$$

This PDE and final condition do not satisfy the conditions of Corollary 2; instead, Theorem 2 must be applied, and establishes convergence for a limited range of maturities. In particular, note that

$$e^{-\frac{b}{2}(y-a)^2} g(y) = e^{\frac{\sqrt{\kappa^2+2\sigma^2}}{2}\left(y - \frac{\kappa^2\theta}{\sigma\left(\kappa^2+2\sigma^2\right)}\right)^2} e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}.$$

Since there is a non-zero coefficient on y^2 in the exponent on the right-hand side (whenever $\sigma > 0$), the conditions of Corollary 2 cannot be satisfied here, as they were in the Vasicek (1977) model. However, this quantity satisfies the conditions of Theorem 2 for the norm

$$\|y\| \equiv \left| y\sqrt{\sqrt{\kappa^2 + 2\sigma^2} - \kappa} \right|,$$

so the power series of the PDE solution converges for all

$$|\Delta| < \left| \frac{1}{\sqrt{\kappa^2 + 2\sigma^2} - \kappa} \right|.$$

In the case of Vasicek (1977), our results guarantee convergence of a power series approximation to the PDE solution for all Δ. Here, convergence is guaranteed only for some finite range of values, whose size depends on the κ and σ parameters. In the typical case of $\kappa > 0$ and $\sigma > 0$, the choice of $b = -\sqrt{\kappa^2 + 2\sigma^2}$ establishes a larger radius of convergence than $b = +\sqrt{\kappa^2 + 2\sigma^2}$, although the conditions of the theorem allow either choice.

[12] Our parameterization is different, and also slightly less general, than that of Ahn, Dittmar, and Gallant (2002), but suffices for our purposes.

Theorem 2 (with $b = -\sqrt{\kappa^2 + 2\sigma^2}$) establishes that $h(\Delta, y)$ is analytic for all y, and for Δ within a neighborhood of the origin. If follows immediately that $P(\Delta, r)$ is also analytic for all r, and for the same region of Δ. Power series coefficients for either $h(\Delta, y)$ or $P(\Delta, r)$ can therefore be found using (2.6) and (2.7). The first few coefficients in the power series for $h(\Delta, y)$ are

$$
\begin{aligned}
a_0(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2} \\
a_1(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left(-y^2\sigma^2 - \phi\right) \\
a_2(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left(y^4\sigma^4 + 2y^2\sigma^2(\kappa+\phi) - 2y\theta\kappa\sigma + \phi^2 - \sigma^2\right) \\
a_3(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\begin{bmatrix} -y^6\sigma^6 - 3\sigma^4(2\kappa+\phi)y^4 + 6\theta\kappa\sigma^3y^3 \\ -\sigma^2\left(4\kappa^2 - 7\sigma^2 + 6\kappa\phi + 3\phi^2\right)y^2 \\ +6\theta\kappa\sigma(\kappa+\phi)y \\ -2\theta^2\kappa^2 + 2\kappa\sigma^2 + 3\sigma^2\phi - \phi^3 \end{bmatrix}.
\end{aligned}
$$

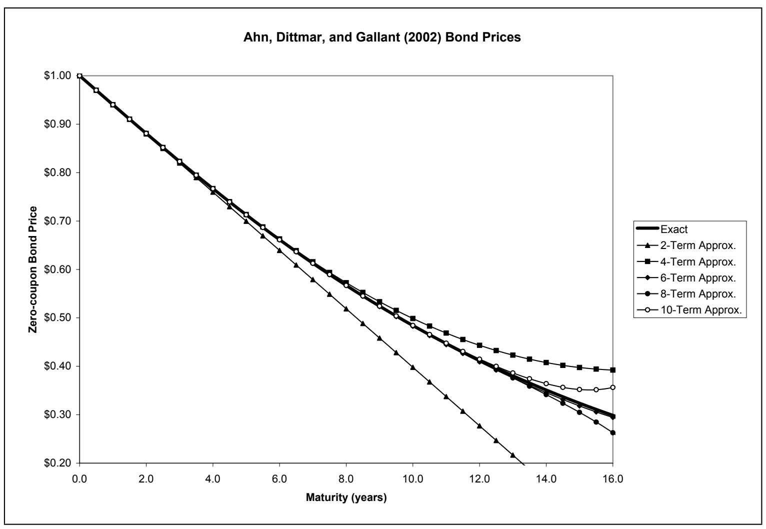

Figure 2: This figure shows prices of zero-coupon bonds as a function of maturity, from the model of Ahn, Dittmar, and Gallant (2002), both in closed-form and with approximations including varying numbers of terms. The risk-neutral parameters match the risk-neutral speed of mean reversion and unconditional mean, variance, and kurtosis of the Cox, Ingersoll, and Ross (1985) model estimated by Cheridito, Filipović, and Kimmel (2007). The current instantaneous interest rate is taken to be $r = 6\%$. As shown, the two-term approximate price begins to deviate substantially from the true price as maturity increases beyond a few years, but approximations with more terms do much better. With these parameters, the series for the bond price function converges for maturities of up to approximately 20.8 years; for longer maturities, the series diverges.

Figure 2 compares the true prices of zero-coupon bonds under the model of Ahn, Dittmar, and Gallant (2002). These authors do not estimate a single-factor model, so we use parameters that match the unconditional mean, variance, and kurtosis of the interest rate process implied by the CIR model estimated by Cheridito, Filipović, and Kimmel (2007). These three constraints do not identify the four parameters of the model, so we also require that the speed of mean reversion be the same as in the CIR model estimates. As shown, the approximations are highly accurate with even a small number of terms, provided the maturity of the bond is short. As maturity increases, however, more terms are needed to achieve a given level of accuracy. Unlike bond prices in the model of Vasicek (1977), bond prices in this model are not everywhere analytic in maturity, and consequently, the approximations converge only for a finite range of maturities. For the particular parameter values chosen, the bond price function has singularities in Δ with modulus of approximately $|\Delta| = 20.8$, which is not coincidentally the range of maturity established by Theorem 2. The power series therefore diverges for longer maturities. It is possible to improve dramatically the convergence properties of the series by combining our methods with a non-affine transformation of the time variable. For examples, see Kimmel (2008), who uses both methods to derive series that converge uniformly for all maturities, even when the state variable process is not stationary. He further finds that the approximations are extremely accurate for all maturities, even with only a few terms.

5.3 Cox, Ingersoll, and Ross

Bond prices are known in closed-form in the model of Cox, Ingersoll, and Ross (1985), which we use as another example. The risk-neutral interest rate process,

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}dW_t,$$

has the same drift as in the Vasicek (1977) model, but a different diffusion term. See Feller (1951) for restrictions on the parameters that ensure existence of the process, and also non-attainment of the boundary value of zero. The pricing PDE, with final condition, for a zero-coupon bond is

$$\begin{aligned} \frac{\partial P}{\partial \Delta}(\Delta, r) &= \kappa(\theta - r)\frac{\partial P}{\partial r}(\Delta, r) + \frac{\sigma^2 r}{2}\frac{\partial^2 P}{\partial r^2}(\Delta, r) - rP(\Delta, r) \\ P(0, r) &= 1. \end{aligned}$$

The change of variables needed to put the PDE in the canonical form,

$$P(\Delta, r) = r^{\frac{1}{4} - \frac{\theta\kappa}{\sigma^2}} e^{\frac{\kappa r}{\sigma^2}} h(\Delta, y(r)) \qquad y(r) = \frac{2\sqrt{r}}{\sigma},$$

are different than those used in the previous two cases. The canonical form PDE is then

$$\begin{aligned} \frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \begin{bmatrix} \frac{\kappa^2 + 2\sigma^2}{8} y^2 - \frac{\theta\kappa^2}{\sigma^2} \\ + \frac{(4\theta\kappa - \sigma^2)(4\theta\kappa - 3\sigma^2)}{8\sigma^4 y^2} \end{bmatrix} h(\Delta, y) \\ h(0, y) &= \left(\frac{\sigma y}{2}\right)^{-\frac{1}{2} + \frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}}. \end{aligned}$$

This PDE is of the form specified by Theorem 3, with

$$a = \frac{(4\theta\kappa - \sigma^2)(4\theta\kappa - 3\sigma^2)}{8\sigma^4} \qquad b = -\frac{\sqrt{\kappa^2 + 2\sigma^2}}{2} \qquad d = -\frac{\theta\kappa^2}{\sigma^2}.$$

The boundedness condition of Theorem 3 is imposed on

$$e^{-\frac{b}{2}y^2}g_2(y) = e^{\frac{-\kappa+\sqrt{\kappa^2+2\sigma^2}}{4}y^2}.$$

The theorem conditions are satisfied for

$$\|y\| \equiv \left|y\sqrt{\sqrt{\kappa^2+2\sigma^2}-\kappa}\right|.$$

A power series approximation to the PDE solution therefore converges for all

$$|\Delta| < \left|\frac{1}{\sqrt{\kappa^2+2\sigma^2}-\kappa}\right|.$$

As with the case of Ahn, Dittmar, and Gallant (2002), our results guarantee convergence of the power series of the solution only for a finite range of values of Δ, whose size depends on the κ and σ parameters. Also as in the previous case, it is possible to apply the theorem with the value of b being the negative of the choice above; however, for typical parameter values, the results are stronger with the choice above.

The function $h(\Delta, y)$ is everywhere analytic for all y, and for all Δ within a neighborhood of the origin. Analyticity of $P(\Delta, r)$ follows immediately from its relation to $h(\Delta, y)$, so we can therefore construct the power series of either function, calculating the coefficients with (2.6) and (2.7). The first few coefficients in the power series of $h(\Delta, y)$ are

$$\begin{aligned}
a_0(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \\
a_1(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \left(-y^2\sigma^2/4\right) \\
a_2(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \left(y^4\sigma^4 + 4\kappa\sigma^2y^2 - 16\kappa\theta\right)/16 \\
a_3(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \begin{bmatrix} -\sigma^6y^6/64 - 9\kappa\sigma^4y^4/48 \\ +\sigma^2\left(3\theta\kappa - \kappa^2 + \sigma^2\right)y^2/4 + \theta\kappa^2 \end{bmatrix}.
\end{aligned}$$

Figure 3 compares the true prices of zero-coupon bonds under the model of Cox, Ingersoll, and Ross (1985) to prices obtained using the approximations derived here, using the risk-neutral parameters from Cheridito, Filipović, and Kimmel (2007). As shown, the approximations are very accurate for maturities of eight years or more even with a small number of terms. However, the series do not converge, with these parameter values, for maturities greater than approximately 9.17 years. See Kimmel (2008) for the combination of our technique with the method of time transformations, which extends the range of convergence to arbitrarily long maturities, and makes the convergence uniform.

5.4 Other Models

In all of the examples considered so far, the quantity sought (the price of a zero-coupon bond) is already known in closed-form; these examples therefore serve as a benchmark to evaluate our methods. The value of an approximation method, however, is not to approximate things that are already known, but things that

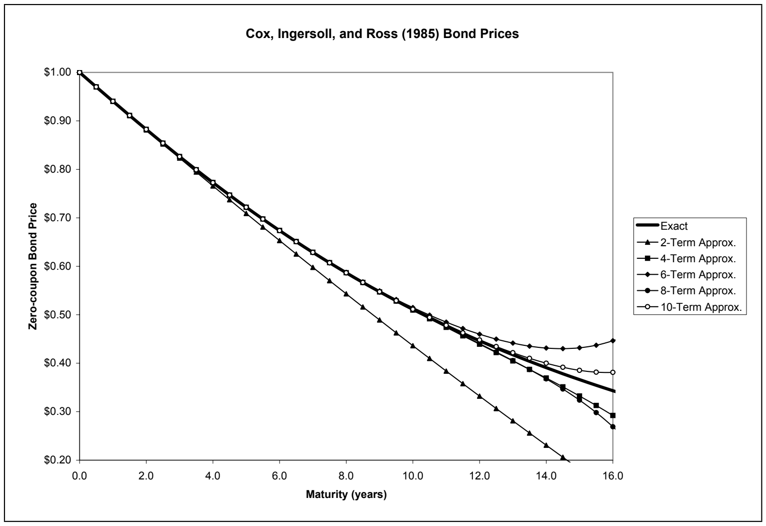

Figure 3: This figure shows prices of zero-coupon bonds as a function of maturity, from the model of Cox, Ingersoll, and Ross (1985), both in closed-form and with approximations including varying numbers of terms. The risk-neutral parameters are from Cheridito, Filipović, and Kimmel (2007), and the current instantaneous interest rate is taken to be $r = 6\%$. As shown, the two-term approximate price is accurate for maturities up to a few years; approximations with more terms do much better, matching the true price accurately for substantially longer maturities. For the parameter values chosen, the series diverge for maturities beyond approximately 9.17 years. But for shorter maturities, solutions can be approximated as accurately as desired by including sufficiently many terms in the power series.

are unknown. The tools developed here allow construction of a wide variety of models in which prices or conditional moments are not known in closed-form, but can nonetheless be approximated accurately. All that is needed is to choose one of the two versions of the canonical PDE examined in Section 4, and specify a final condition that satisfies the conditions of Theorems 1, 2, or 3, or Corollaries 1, 2, or 3. Solutions to these canonical form problems can then be found by series approximation; furthermore, by reversing the changes of dependent and independent variables[13] used to construct the canonical form PDE, a wide variety of non-canonical problems can be constructed, all of which can be solved by series approximation.

Continuing with term structure models as an example, we can construct many models from the same general PDE that underlies Vasicek (1977) and Ahn, Dittmar, and Gallant (2002),

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y-a)^2 + d\right] h(\Delta, y).$$

A final condition $h(0, y) = g(y)$ that satisfies the smoothness and growth conditions of either Theorem 2 or Corollary 2 can be used to construct a bond pricing model. By reversing the change of variables used in Section 4.1 to convert a general scalar PDE into the canonical form, bond prices, defined as

$$P(\Delta, y) = \frac{h(\Delta, y)}{g(y)},$$

[13] Change of independent variable is essentially aesthetic in latent variable models. However, change of dependent variables has real implications for the solution.

then solve

$$\begin{aligned}\frac{\partial P}{\partial \Delta}(\Delta, y) =& \frac{g'(y)}{g(y)}\frac{\partial P}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta, y) \\ &- \left[\frac{b^2}{2}(y-a)^2 + d - \frac{g''(y)}{2g(y)}\right] P(\Delta, y) \\ P(0, y) =& 1.\end{aligned}$$

so that the solution $P(\Delta, y)$ is the price of a bond with underlying state variable dynamics (under risk-neutral probabilities)

$$dY_t = \frac{g'(Y_t)}{g(Y_t)} dt + dW_t^Q$$

with an interest rate specification[14]

$$r(Y_t) = \frac{b^2}{2}(Y_t - a)^2 + d - \frac{g''(Y_t)}{2g(Y_t)}.$$

If $g(y)$ is strictly positive (or strictly negative) for real values of y, Y_t can take on any real value; if $g(y)$ has zeros for real y, then there are boundaries away from $\pm\infty$.

Compared with the relative sparsity of models for which bond prices are known in closed-form, there are many specifications of $g(y)$ that allow approximation, although it is convenient to consider $w(y)$, which relates to $g(y)$ as follows

$$g(y) = e^{\frac{b}{2}(y-a)^2} w(y).$$

For example, if $w(y)$ is a polynomial, exponential function, a polynomial multiplied by an exponential function, or a sum of all three of these types of functions, then it specifies a term structure models in which bond prices are everywhere analytic, since the final condition satisfies the conditions of Corollary 2. Functions such as $\exp(ky^2)$ satisfy the conditions of Theorem 2, so that bond prices are analytic in some region that includes the origin (and the smaller k is, the larger the region). Such functions can be multiplied by polynomials or exponential functions, added together, etc., to specify still more non-affine term structure models in which bond prices can be approximated by convergent series for some range of maturities.

Even some functions that may appear at first glance to be non-analytic also satisfy the corollary conditions; for example, consider

$$g(y) = e^{\frac{b}{2}(y-a)^2} \cosh(ky^{\frac{3}{2}}).$$

Despite the fractional exponent, this condition is uniquely defined and analytic in y, since the hyperbolic cosine function is even; analyticity is evident from its power series. This specification is an intermediate case between the model of Vasicek (1977), in which $w(y)$ grows at a rate proportional to $\exp(ky)$ for some k, and the model of Ahn, Dittmar, and Gallant (2002), in which the corresponding quantity grows at a rate

[14] The equivalence of the PDE problem and the probabilistic problem must be verified for specific choices of $g(y)$.

proportional to $\exp\left(ky^2\right)$ for some k. Here, $w(y)$ grows at a rate proportional to $\exp\left(ky^{3/2}\right)$ for some k. As such, Corollary 2 establishes analyticity of the solution for all values of Δ. The first few terms in the power series expansion of $h(\Delta, y)$ around $\Delta = 0$ are given by

$$a_0(y) = e^{\frac{b}{2}(y-a)^2} \cosh\left(ky^{\frac{3}{2}}\right)$$

$$a_1(y) = e^{\frac{b}{2}(y-a)^2} \frac{1}{8} \begin{bmatrix} (9ky + 4b - 8d) \cosh\left(ky^{\frac{3}{2}}\right) \\ +3k\left(4by^2 - 4aby + 1\right) \sinh\left(ky^{\frac{3}{2}}\right) / \sqrt{y} \end{bmatrix}$$

$$a_2(y) = e^{\frac{b}{2}(y-a)^2} \begin{bmatrix} \begin{bmatrix} 144b^2k^2y^3 - 9k^2/y \\ +9k^2\left(9k^2 - 32ab^2\right) y^2 \\ +144k^2\left(3b + a^2b^2 - d\right) y \\ +8\left(2(b-2d)^2 - 27abk^2\right) \end{bmatrix} \dfrac{\cosh\left(ky^{\frac{3}{2}}\right)}{128} \\ + \begin{bmatrix} 72bk^2y^4 + 8ab + 3/y \\ +8b\left(10b - 8d - 9ak^2\right) y^3 \\ +2 \begin{pmatrix} 32abd \\ -48ab^2 \\ +27k^2 \end{pmatrix} y^2 \\ +16\left(b + a^2b^2 - d\right) y \end{bmatrix} \dfrac{3k \sinh\left(ky^{\frac{3}{2}}\right)}{128 y^{\frac{3}{2}}} \end{bmatrix}.$$

This specification can be combined with others; for example, it can be multiplied by a polynomial, an exponential function, or a function of the form $\exp\left(ky^2\right)$, added to other such functions, etc.

With a little creativity, one can easily choose specifications of $g(y)$ that lead to some interesting features in the implied term structure model. For example, the specifications of the $g(y)$ function corresponding to both the Vasicek (1977) and Ahn, Dittmar, and Gallant (2002) is of the form

$$g(y) = e^{k(y-\phi)^2}.$$

A function of this form can be multiplied by an analytic function that grows more slowly in y. The term structure model specified by the resulting $g(y)$ then has similar interest rate behavior to these models for extreme values of the state variable, but at intermediate values, the state variable dynamics will be somewhat different than those of these two models. In other words, local deformations in the drift and the diffusion functions can be introduced, while retaining the dynamic behavior of either of these two models for extreme interest rates.

Other than the model of Cox, Ingersoll, and Ross (1985), all the cases we have considered are based on a canonical form PDE that meets the conditions of Theorem 1 or 2, or their corresponding corollaries. It is also possible to construct non-affine term structure models based on Theorem 3 or Corollary 3. Consider

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2} \frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2} y^2 + d\right) h(\Delta, y).$$

Given some $g_2(y)$ that satisfies the conditions of Theorem 3 or Corollary 3 (with $g_1 = 0$), we can reverse the change of variables used to construct the canonical PDE. Then bond prices are given by

$$P(\Delta, y) = \frac{h(\Delta, y)}{y^{\frac{1+\sqrt{1+8a}}{2}} g_2(y)}.$$

The bond price function then satisfies

$$\begin{aligned} \frac{\partial P}{\partial \Delta}(\Delta, y) =& \left[\frac{\alpha}{y} + \frac{g_2'(y)}{g_2(y)}\right] \frac{\partial P}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta, y) \\ & - \left[\frac{b^2}{2} y^2 + d - \frac{\alpha}{y}\frac{g_2'(y)}{g_2(y)} - \frac{1}{2}\frac{g_2''(y)}{g_2(y)}\right] P(\Delta, y) \\ P(0, y) =& 1. \end{aligned}$$

Thus, every $g_2(y)$ that satisfies the conditions of either Theorem 3 or Corollary 3 implicitly specifies a term structure model. We have already seen the choice of $g_2(y)$ that gives rise to the model of Cox, Ingersoll, and Ross (1985); other choices give rise to non-affine models. In all cases, bond prices are analytic in a region of maturities including the origin, and can therefore be approximated with power series.

5.5 Other Applications

The applications discussed above all concern pricing zero-coupon bonds in term structure models; nearly all extant models with closed-form bond prices are covered by the results of Section 4, so bond prices can be approximated with a convergent power series. However, many additional models that have not previously appeared in the literature are also covered by these results. Although a natural application, the pricing of non-defaultable bonds is not the only problem that can be solved using our results. For example, pricing of credit derivatives requires modeling not only the interest rate process, but a default process. Given an interest rate process r_t and default intensity process λ_t, pricing of defaultable bonds involves evaluation of expressions such as $\mathrm{E}\left[\exp\left(-\int_t^T (r_u + \lambda_u)\, du\right)\right]$, where the expectation is taken under a risk-neutral probability measure. Evaluation of this quantity is difficult for non-affine specifications of the r_t and λ_t processes. However, our methods establish convergence of a power series representation of this quantity for many non-affine models, greatly expanding the class of models that can be considered in practice.

Other applications are also possible. For example, Jarrow, Li, Liu, and Wu (2006) use the methods of Section 4 (specifically, Theorem 3) to price callable bonds. Furthermore, for any arbitrary diffusion process, our results establish an infinite-dimensional family of moment conditions that can be approximated with power series, and that can therefore be used in an estimation procedure.

6 Conclusion

We have developed a method for closed-form approximation of conditional moments and bond prices for a wide variety of diffusion problems, and derived conditions to establish a minimum range of convergence of the approximations. Our method can be augmented by time transformation methods, which sometimes extends the range of convergence to include all positive time horizons, sometimes uniformly. These methods make feasible the rapid calculation of bond prices for many models in which such calculation would otherwise not be practical, and therefore make feasible estimation techniques for non-affine models based on likelihood or minimum distance searches. Kimmel (2008) has further examined the performance of our technique, and found rapid convergence of bond price approximations for very long maturities for many models in the literature, and also for many non-linear models that have not previously appeared. Jarrow, Li, Liu, and Wu (2006) have applied our technique to the problem of pricing callable corporate bonds in a structural model.

Potential future work includes extension of the method to multivariate diffusions. For some multivariate cases (e. g., independent state variable processes that enter additively into the interest rate function), the pricing or conditional moment problem can be broken into several independent univariate problems. Nonetheless, in the general multivariate case, it is not even possible to express the pricing PDE in the canonical form in all cases. However, the method of change of dependent variable can also lead to construction of new term structure models in which (after the change of variable) the pricing PDE is the same as the PDE that arises in multivariate affine models. Since expectations of polynomials of affine diffusions are analytic in the time horizon, at least a partial characterization of the final conditions with analytic moments is possible; as in the univariate case, each final condition with analytic moments corresponds to a term structure model with analytic bond prices. Other potential future work includes expanding the class of diffusions and interest rate specifications for which the class of problems with analytic (in time) solutions can be characterized explicitly. Other possible avenues of future research include development of methods for approximating conditional moments or asset prices that have singularities at a time horizon of zero (such as standard put and call options); if the nature of the singularity is sufficiently well understood, it may be possible to develop a series with an initial term that captures the singularity, such that the difference between the true (but unknown) solution and the initial term is analytic in the time variable, and can then be approximated by a power series. Such methods remain to be explored in full detail, however.

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A Appendix: Proofs

This appendix includes proofs of all theorems and corollaries in the main text, as well as several auxiliary lemmas (with proofs) not included in the main text.

A.1 Proof of Theorem 1

We express the PDE solution as an integral

$$h(\Delta, y) = e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \begin{bmatrix} g\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right) \\ +g\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right) \end{bmatrix} du. \tag{A.1}$$

The integrand is even in $\sqrt{\Delta}$, so it does not matter (for $\Delta \neq 0$) which square root is chosen. It must be demonstrated that $h(\Delta, y)$ is well-defined, is analytic in Δ and y, and solves the PDE (4.6) with final condition (4.7).

To show the existence of $h(\Delta, y)$ in the specified region, we first note that

$$\left| \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \left[g\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right) + g\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right) \right] \right| \leq \frac{c}{\sqrt{2\pi}} e^{\frac{c_2 u^2 + 2c_1|u| + c_2}{2}} \tag{A.2}$$

where $c_2 = \|\sqrt{\Delta}\|^2 - 1$, $c_1 = \left\|y - a\Delta^2/2\right\| \|\sqrt{\Delta}\|$, and $c_0 = \left\|y - a\Delta^2/2\right\|^2$. So the integral converges, and $h(\Delta, y)$ is well-defined, for all $\|\sqrt{\Delta}\| < 1$.

To establish analyticity, we note that the integrand in (A.1) is continuous in u, y, and Δ, and for each u, is also analytic in y and Δ. These properties follow from the power series of $g(z)$; terms that are odd in the square root of Δ from $g\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right)$ cancel with the odd terms from $g\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right)$. The integrand is therefore analytic in Δ. Furthermore, it is uniformly bounded on compact sets of y and Δ; replacing $\left\|y - a\Delta^2/2\right\|$ and $\|\sqrt{\Delta}\|$ from the right-hand side of (A.2) by their maximum values on a compact set establishes such a bound. With continuity of the integrand, analyticity for each u, and uniform boundedness on compacts, the integral inherits the analyticity of the integrand in y and Δ for $\|\sqrt{\Delta}\| < 1$. (See Lang, 1999, who refers to this result as the "differention lemma.") It follows that $h(\Delta, y)$ is defined and analytic for all y and $\|\sqrt{\Delta}\| < 1$.

Finally, $h(\Delta, y)$ satisfies the partial differential equation with final condition. Satisfaction of the final condition (4.7) is straightforward; for $\Delta = 0$,

$$h(0, y) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} g(y) du = g(y).$$

To show that the proposed solution solves the general partial differential equation, we calculate derivatives. The time derivative is

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \left[a^2\Delta^2/2 - (ay+d)\right] h(\Delta, y) + e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \times \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{4} \begin{bmatrix} \left(u/\sqrt{\Delta} - 2a\Delta\right) g'\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right) \\ -\left(u/\sqrt{\Delta} + 2a\Delta\right) g'\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right) \end{bmatrix} du.$$

The differentiation under the integral sign is justified by the uniform boundedness of the integrand on compact sets. Despite the appearance of $\sqrt{\Delta}$ in the denominator, the integrand on the right-hand side is well-defined when this quantity is zero, as can be seen by taking limits; furthermore, it is analytic in Δ. The second spatial derivative is

$$\frac{\partial^2 h}{\partial y^2}(\Delta, y) = a^2\Delta^2 h(\Delta, y) - e^{\frac{a^2\Delta^3}{6}-(ay+d)\Delta}$$
$$\times \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \begin{bmatrix} (2a\Delta - u/\sqrt{\Delta})\, g'\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right) \\ +(2a\Delta + u/\sqrt{\Delta})\, g'\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right) \end{bmatrix} du.$$

To derive this result, it is necessary to differentiate inside the integral, and then integrate the terms containing the second derivative of $g(z)$ by parts. These operations are justified by uniform boundedness of the integrand and its first and second derivative on compacts.[15] Substituting the time and second spatial derivatives of $h(\Delta, y)$ into the general PDE (4.6), one verifies that $h(\Delta, y)$ is a solution.

A.2 Proof of Theorem 2

The proof proceeds similarly to that of Theorem 1. We express $h(\Delta, y)$ as an integral

$$h(\Delta, y) = e^{\frac{b}{2}(y-a)^2+\left(\frac{b}{2}-d\right)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \begin{bmatrix} \phi\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix} du \tag{A.3}$$

where $\phi(z) = \exp\left(-b(z-a)^2/2\right) g(z)$. The integrand is even in $\sqrt{\tau(\Delta)}$, so it does not matter which square root is chosen. It must be demonstrated that $h(\Delta, y)$ is well-defined, is analytic in Δ and y, and solves the PDE (4.8) with final condition (4.9).

To show existence of $h(\Delta, y)$ in the specified region, we first note that the integrand satisfies

$$\left| \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \begin{bmatrix} \phi\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix} \right| \leq \frac{c e^{\frac{c_2 u^2 + 2c_1|u| + c_0}{2}}}{\sqrt{2\pi}}$$

where $c_2 = \|\sqrt{\tau(\Delta)}\|^2 - 1$, $c_1 = \|a + e^{b\Delta}(y-a)\| \, \|\sqrt{\tau(\Delta)}\|$, and $c_0 = \|a + e^{b\Delta}(y-a)\|^2$. Since the coefficient on u^2 in the exponent on the right-hand side is negative whenever $\|\sqrt{\tau(\Delta)}\| < 1$, the integral converges for these values. The leading exponential factor in (A.3) is defined for all Δ and y, so it follows that $h(\Delta, y)$ is well-defined for all complex y and Δ such that $\|\sqrt{\tau(\Delta)}\| < 1$.

To establish analyticity, note that the integrand is continuous in u, y, and τ, and for each u, is analytic in y and τ; as in the proof of Theorem 1, this follows from the power series of $\phi(z)$. But $\tau(\Delta)$ is an analytic function of Δ, so the integrand is also analytic in Δ. It is also uniformly bounded on compact sets of y and Δ. Then, by the differentiation lemma (see Lang, 1999), the integral inherits the analyticity of the integrand in y and Δ (provided $\|\sqrt{\tau(\Delta)}\| < 1$).

Finally, we show that $h(\Delta, y)$ satisfies the partial differential equation with final condition. Satisfaction of

[15] The bound on $g(z)$ itself is assumed. The bounds on the first and second spatial derivatives follow by application of Cauchy's integral theorem.

the final condition (4.9) is straightforward; for $\Delta = 0$,

$$h(0,y) = e^{\frac{b}{2}(y-a)^2} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \phi(y)\, du = e^{\frac{b}{2}(y-a)^2} \phi(y) = g(y).$$

To show that the proposed solution solves the general partial differential equation, we calculate derivatives as follows. Note that all derivations remain valid in the special case of $b = 0$, in which case $\tau(\Delta) = \Delta$. The derivative with respect to Δ is

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \left(\frac{b}{2} - d\right) h(\Delta, y) + e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \times \left[\begin{array}{l} \dfrac{be^{b\Delta}(y-a)}{2} \displaystyle\int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\begin{array}{c} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{array} \right] du \\ + \dfrac{e^{2b\Delta}}{4\sqrt{\tau(\Delta)}} \displaystyle\int_{-\infty}^{+\infty} \frac{ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\begin{array}{c} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ -\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{array} \right] du \end{array} \right].$$

The presence of $\sqrt{\tau(\Delta)}$ in the denominator on the last line poses no problems, since it is multiplied by an integral that is an odd function of $\sqrt{\tau(\Delta)}$, so the last term can be assigned the limiting value when Δ approaches zero. The integrand in (A.3) is uniformly bounded on compact sets, which justifies differentiation under the integral sign. The second spatial derivative is

$$\frac{\partial^2 h}{\partial y^2}(\Delta, y) = \left[b^2 (y-a)^2 + b\right] h(\Delta, y) + e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \times \left[\begin{array}{l} be^{b\Delta}(y-a) \displaystyle\int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\begin{array}{c} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{array} \right] du \\ + \dfrac{e^{2b\Delta}}{2\sqrt{\tau(\Delta)}} \displaystyle\int_{-\infty}^{+\infty} \frac{ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\begin{array}{c} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ -\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{array} \right] du \end{array} \right].$$

This result follows by differentiating under the integral, and then integrating the terms with the second derivative of $\phi(z)$ by parts; both operations are justified by the uniform boundedness of the integrand, and its first and second spatial derivatives, on compact sets. Substitution of these expressions into the general PDE shows that $h(\Delta, y)$ is a solution.

A.3 Proof of Corollary 1

Choose any $k > 0$, and define the norm $\|y\| \equiv |y| / \sqrt{k}$. Then $g(y)$ satisfies the conditions of Theorem 1 for this norm and for $c = c_k$. So by Theorem 1, there exists a solution $h(\Delta, y)$ to the partial differential equation (with final condition) that is analytic for all y and for all $|\Delta| < k$. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the solution constructed in the proof of Theorem 1 does not depend on c or on the norm, so it is clear that the solutions constructed for different values of k are the same functions. Consequently, the solution is defined and analytic in y and Δ.

A.4 Proof of Corollary 2

Choose any $k > 0$, and define $\|y\| \equiv |y| / \sqrt{k}$. Then $g(y)$ satisfies the conditions of Theorem 2 for this norm and for $c = c_k$. By Theorem 2, there exists a solution $h(\Delta, y)$ to the partial differential equation (with final condition) that is analytic for all y and all $|\tau(\Delta)| < k$. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. The solution constructed in the proof of Theorem 2 does not depend on c or on the norm, so it is clear that the solutions constructed here for different values of k coincide. Consequently, the solution is defined and analytic in y and τ for all $\tau(\Delta)$. Since $\tau(\Delta)$ is well-defined and analytic for all Δ, it follows that $h(\Delta, y)$ is well-defined and analytic for all y and Δ.

A.5 Proof of Theorem 3

The proof of Theorem 3 is much longer and more complicated than the proofs of Theorems 1 and 2, and involves explicit construction of the solution to a partial differential equation as an infinite series. The lengthy construction of the PDE solution is needed only to prove the theorem and establish analyticity of the solution, not to construct its power series representation. For the latter purpose, those willing to trust in the correctness of the proof may ignore it completely, and need only verify that a particular problem satisfies the conditions of the theorem. Construction of the power series then proceeds by the simple recursive procedure.

The construction of the PDE solution uses four different integral operators, and Lemmas 1 through 4 establish various properties of those operators. Lemma 5 constructs a solution to a related PDE that is not in the canonical form, and finally the theorem proof uses Lemma 5 (several times) to construct the PDE solution.

A.5.1 Lemma 1

Lemma 1. *Let $\psi(z)$ be analytic and even for all complex z, and let there exist a $c > 0$ and a norm (over the reals) $\|z\|$ such that $\psi(z)$ satisfies*

$$|\psi(z)| \le c e^{\frac{\|z\|^2}{2}}.$$

Then

$$\nu(\Delta, y) \equiv \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \left[\psi\left(y + u\sqrt{\Delta}\right) + \psi\left(y - u\sqrt{\Delta}\right)\right] du \tag{A.4}$$

is analytic in both variables for all y and all $\|\sqrt{\Delta}\| < 1$, and is also even in y. Furthermore, $\nu(\Delta, y)$ satisfies

$$|\nu(\Delta, y)| \le \frac{2c}{\sqrt{1 - \|\sqrt{\Delta}\|^2}} e^{\frac{\|y\|^2}{2\left(1 - \|\sqrt{\Delta}\|^2\right)}}. \tag{A.5}$$

Finally, $\nu(\Delta, y)$ satisfies the PDE with final condition

$$\frac{\partial \nu}{\partial \Delta}(\Delta, y) = \frac{1}{2} \frac{\partial^2 \nu}{\partial y^2}(\Delta, y) \qquad \nu(0, y) = \psi(y) \tag{A.6}$$

for all complex y and $\|\sqrt{\Delta}\| < 1$.

Theorem 1 establishes the existence of (A.5) and its analyticity for all y and $\|\sqrt{\Delta}\| < 1$, by choosing $a = d = 0$ and $g(z) = \psi(z)$. Theorem 1 also establishes that $\nu(\Delta, y)$ solves (A.6). It remains only to show that $\nu(\Delta, y)$ is even in y, and that it satisfies (A.5). Evenness of the integrand in y follows from the evenness of $\psi(z)$ (which was not assumed in Theorem 1); the integral then inherits the evenness of the integrand. From the bound on $\psi(z)$ and the properties of a norm, it follows that

$$\left|\frac{1}{2}\left[\psi\left(y + u\sqrt{\Delta}\right) + \psi\left(y - u\sqrt{\Delta}\right)\right]\right| \leq c e^{\frac{(\|y\| + |u| \|\sqrt{\Delta}\|)^2}{2}}. \tag{A.7}$$

The bound on $\nu(\Delta, y)$ follows by substituting the right-hand side of (A.7) into (A.4) and performing the integration; the integral derived in this way satisfies (A.5), so by dominated convergence, $\nu(\Delta, y)$ must also.

A.5.2 Lemma 2

Lemma 2. *Let $\psi(z)$ be analytic and even for all complex z, and let there exist some $c > 0$ and some norm (over the reals) $|z| / \sqrt{k_2} \leq \|z\| \leq |z| / \sqrt{k_1}$, such that $\psi(z)$ satisfies the bound*

$$|\psi(z)| \leq c e^{\frac{\|z\|^2}{2}}.$$

Then

$$\phi(\Delta, y) \equiv \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u}{2y\sqrt{\Delta}} \left[\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)\right] du \tag{A.8}$$

is analytic in both variables for all y and all $\|\sqrt{\Delta}\| < 1$, and is also even in y. Furthermore, $\phi(\Delta, y)$ satisfies

$$|\phi(\Delta, y)| \leq \frac{2c\left(e^2 + 1\right)}{k_1\left(1 - \|\sqrt{\Delta}\|^2\right)^{3/2}} e^{\frac{\|y\|^2}{2\left(1 - \|\sqrt{\Delta}\|^2\right)}}. \tag{A.9}$$

The integrand in (A.8) is continuous in u, Δ, and y, and, for each u, analytic in y and Δ. This follows from the power series of $\psi(z)$, which contains only even terms in z; substituting in $z = y \pm u\sqrt{\Delta}$ and expanding with the binomial theorem, the expression in brackets contains only terms which are odd in y and $u\sqrt{\Delta}$. After multiplying by u and dividing by y and $\sqrt{\Delta}$, only even powers remain. It follows from the bound on $\psi(z)$ and the properties of a norm that

$$\left|\frac{u}{2y\sqrt{\Delta}}\left[\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)\right]\right| \leq \frac{c\,|u|}{k_1 \|y\| \|\sqrt{\Delta}\|} e^{\frac{(\|y\| + |u| \|\sqrt{\Delta}\|)^2}{2}}. \tag{A.10}$$

However, we can also establish another bound on this expression using the maximum modulus principle of analytic functions. Specifically, we show that

$$\left|\frac{u}{2y\sqrt{\Delta}}\left[\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)\right]\right| \leq \frac{2c\,|u|^2 e}{k_1} e^{\frac{(\|y\| + |u| \|\sqrt{\Delta}\|)^2}{2}}. \tag{A.11}$$

If $\|y\| \, |u| \, \|\sqrt{\Delta}\| \geq 1/2$, this follows directly from (A.10). If $\|y\| \, |u| \, \|\sqrt{\Delta}\| < 1/2$ and $\|y\| \geq 1/\sqrt{2}$, it follows by replacing $\|\sqrt{\Delta}\|$ in (A.10) by $1/\left(2\|y\| \, |u|\right)$, since the integrand is an analytic function of $u\sqrt{\Delta}$, and since an analytic function takes its maximum value in a closed set on the boundary of that set. The bound is established similarly when $|u| \, \|y\| \, \|\sqrt{\Delta}\| < 1/2$ and $|u| \, \|\sqrt{\Delta}\| \geq 1/\sqrt{2}$; in this case, $\|y\|$ in (A.10) is replaced by $1/\left(2\,|u| \, \|\sqrt{\Delta}\|\right)$. Finally, when $\|y\| < 1/\sqrt{2}$ and $|u| \, \|\sqrt{\Delta}\| < 1/\sqrt{2}$, the bound follows by replacing each of these quantities by $1/\sqrt{2}$.

Convergence of the integral for $\|\sqrt{\Delta}\| < 1$ follows from (A.11). Furthermore, the integrand is continuous, for each value of u, analytic in y and Δ, and uniformly bounded on compacts (simply replace $\|y\|$ and $\|\sqrt{\Delta}\|$ in (A.11) by their maximum values on the compact set). It follows (see, for example, Lang, 1999) that the integral inherits the analyticity of the integrand. Evenness in y follows from the evenness of the integrand in y, which itself follows directly from the evenness of the $\psi(z)$ function.

The only property that remains to be established is satisfaction of the bound (A.9). However, unlike the other properties, for which the bound (A.11) suffices, both (A.10) and (A.11) are needed. Specifically, we divide the range of integration in (A.8) into two pieces, $0 \leq |u| \leq 1/\left(\|y\| \, \|\sqrt{\Delta}\|\right)$ and $1/\left(\|y\| \, \|\sqrt{\Delta}\|\right) \leq |u| \leq +\infty$, apply the bound from (A.11) to the first part, and the bound from (A.10) to the second part. Furthermore, since the bound on the integrand is even in u, we need only integrate over positive values. The required integrals are

$$\left| \int_0^{\frac{1}{\|y\|\|\sqrt{\Delta}\|}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u}{2y\sqrt{\Delta}} \begin{bmatrix} \psi\left(y + u\sqrt{\Delta}\right) \\ -\psi\left(y - u\sqrt{\Delta}\right) \end{bmatrix} du \right| \leq \frac{ce^2 e^{\frac{\|y\|^2}{2}}}{k_1\left(1 - \|\sqrt{\Delta}\|^2\right)^{3/2}}$$

$$\left| \int_{\frac{1}{\|y\|\|\sqrt{\Delta}\|}}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u}{2y\sqrt{\Delta}} \begin{bmatrix} \psi\left(y + u\sqrt{\Delta}\right) \\ -\psi\left(y - u\sqrt{\Delta}\right) \end{bmatrix} du \right| \leq \frac{ce^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{k_1\left(1 - \|\sqrt{\Delta}\|^2\right)^{3/2}}.$$

The required bound (A.9) follows by summing these two, and then doubling the sum so that the entire range of integration (both positive and negative values of u) is included.

A.5.3 Lemma 3

Lemma 3. *Let $\psi(\tau, z)$ be defined and analytic for all complex z and $\|\tau\| < 1$ for some norm (over the reals) $|z|/\sqrt{k_2} \leq \|z\| \leq |z|/\sqrt{k_1}$, and let $\psi(\tau, z)$ be even in z. Let there exist some $c > 0$ and some integer $n \geq 0$ such that $\psi(\tau, z)$ satisfies*

$$|\psi(\tau, z)| \leq c\left[-\ln\left(1 - \|\sqrt{\tau}\|^2\right)\right]^n \frac{e^{\frac{\|z\|^2}{2\left(1-\|\sqrt{\tau}\|^2\right)}}}{\left(1 - \|\sqrt{\tau}\|^2\right)^{3/2}}.$$

Then

$$\nu(\Delta, y) \equiv \Delta \int_0^1 \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{1}{2} \begin{bmatrix} \psi\left(s\Delta, y + u\sqrt{(1-s)\,\Delta}\right) \\ +\psi\left(s\Delta, y - u\sqrt{(1-s)\,\Delta}\right) \end{bmatrix} duds \tag{A.12}$$

is analytic in both variables for all y and all $\|\sqrt{\Delta}\| < 1$, and is also an even function of y. Furthermore, $\nu(\Delta, y)$ satisfies

$$|\nu(\Delta, y)| \leq \frac{2c}{n+1}\left[-\ln\left(1 - \|\sqrt{\Delta}\|^2\right)\right]^{n+1} \frac{e^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{\sqrt{1 - \|\sqrt{\Delta}\|^2}}. \tag{A.13}$$

Finally, $\nu(\Delta, y)$ satisfies the PDE with final condition

$$\frac{\partial \nu}{\partial \Delta}(\Delta, y) - \frac{1}{2}\frac{\partial^2 \nu}{\partial y^2}(\Delta, y) = \psi(\Delta, y) \qquad \nu(0, y) = 0 \tag{A.14}$$

for all complex y and $\|\sqrt{\Delta}\| < 1$.

From the bound on $\psi(\tau, z)$ and the properties of a norm, it follows that

$$\left|\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\right| \leq 2c\left[-\ln\left(1 - s\|\sqrt{\Delta}\|^2\right)\right]^n \frac{e^{\frac{\left(\|y\| + |u|\sqrt{1-s}\|\sqrt{\Delta}\|\right)^2}{2\left(1-s\|\sqrt{\Delta}\|^2\right)}}}{\left(1 - s\|\sqrt{\Delta}\|^2\right)^{3/2}}. \tag{A.15}$$

Since s takes values in $[0, 1]$, the integral converges for any $\|\sqrt{\Delta}\| < 1$. The integrand is also continuous, and for each value of s and u, analytic in y and Δ. This follows from the power series of $\psi(\tau, z)$; the terms that are odd in the square root of $(1-s)\Delta$ cancel. Furthermore, the integrand is uniformly bounded on compact sets of y and Δ. The integral is therefore analytic in y and Δ as well. Evenness of the integral in y follows immediately from evenness of the integrand, which itself is a consequence of the evenness of $\psi(\tau, z)$ in z. The bound (A.13) follows by substituting the left-hand side of (A.15) in for the bracketed expression in (A.12), and performing the integration.

The uniform convergence of the integrand on compacts justifies operations such as differentiation under the integral sign, and integration by parts. It remains to show that (A.14) is satisfied. The time derivative is

$$\frac{\partial \nu}{\partial \Delta}(\Delta, y) = \psi(\Delta, y) + \int_0^1 \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u\sqrt{\Delta}}{4\sqrt{1-s}} \begin{bmatrix} \psi_z\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) \\ -\psi_z\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right) \end{bmatrix} duds. \tag{A.16}$$

The second spatial derivative is

$$\frac{\partial^2 \nu}{\partial y^2}(\Delta, y) = \int_0^1 \int_{-\infty}^{+\infty} \left(\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u\sqrt{\Delta}}{2\sqrt{1-s}} \begin{bmatrix} \psi_z\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) \\ -\psi_z\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right) \end{bmatrix} \right) duds. \tag{A.17}$$

The last step follows from integration by parts. Summing (A.16) and (A.17), the general PDE in (A.14) is satisfied. Satisfaction of the final condition follows by evaluating (A.12) at $\Delta = 0$.

A.5.4 Lemma 4

Lemma 4. *Let $\psi(\tau, z)$ be defined and analytic for all z and $\|\tau\| < 1$ for some norm (over the reals) $|z|/\sqrt{k_2} \leq \|z\| \leq |z|/\sqrt{k_1}$, and let $\psi(\tau, z)$ be even in z. Let there exist some $c > 0$ and some integer $n \geq 0$ such that*

$\psi(\tau, z)$ satisfies

$$|\psi(\tau, z)| \leq c\,[-\ln(1 - \|\sqrt{\tau}\|^2)]^n \frac{e^{\frac{\|z\|^2}{2\left(1-\|\sqrt{\tau}\|^2\right)}}}{(1-\|\sqrt{\tau}\|^2)^{3/2}}. \tag{A.18}$$

Then

$$\phi(\Delta, y) \equiv \Delta \int_0^1 \int_{-\infty}^{+\infty} \left(\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u}{2y\sqrt{(1-s)\,\Delta}} \begin{bmatrix} \psi\left(s\Delta, y + u\sqrt{(1-s)\,\Delta}\right) \\ -\psi\left(s\Delta, y - u\sqrt{(1-s)\,\Delta}\right) \end{bmatrix} \right) duds \tag{A.19}$$

is analytic in both variables for all y and all $\|\sqrt{\Delta}\| < 1$, and is also an even function of y. Furthermore, $\phi(\Delta, y)$ satisfies

$$|\phi(\Delta, y)| \leq \frac{2c\left(e^2+1\right)k_2}{k_1\,(n+1)}\,[-\ln(1-\|\sqrt{\Delta}\|^2)]^{n+1} \frac{e^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{(1-\|\sqrt{\Delta}\|^2)^{3/2}}.$$

The integrand in (A.19) is continuous, and, for each u and s, analytic in y and Δ. This follows from the power series of $\psi(\tau, z)$ (which contains only even terms in z). Substituting in the arguments for z and expanding using the binomial theorem, the numerator contains only odd powers of y and $u\sqrt{(1-s)\,\Delta}$, which cancel with the denominator, leaving only even powers. From (A.18) and the properties of a norm,

$$\left| \frac{1}{2yu\sqrt{(1-s)\,\Delta}} \begin{bmatrix} \psi\left(s\Delta, y + u\sqrt{(1-s)\,\Delta}\right) \\ -\psi\left(s\Delta, y - u\sqrt{(1-s)\,\Delta}\right) \end{bmatrix} \right| \leq \frac{c\,[-\ln(1-s\|\sqrt{\Delta}\|^2)]^n}{k_1\,\|y\|\,|u|\,\sqrt{1-s}\,\|\sqrt{\Delta}\|} \frac{e^{\frac{\|y\|+|u|\sqrt{(1-s)}\|\sqrt{\Delta}\|^2}{2\left(1-s\|\sqrt{\Delta}\|^2\right)}}}{(1-s\|\sqrt{\Delta}\|^2)^{3/2}}. \tag{A.20}$$

Proceeding as in the proof of Lemma 2, the maximum modulus principle gives another bound,

$$\left| \frac{1}{2yu\sqrt{(1-s)\,\Delta}} \begin{bmatrix} \psi\left(s\Delta, y + u\sqrt{(1-s)\,\Delta}\right) \\ -\psi\left(s\Delta, y - u\sqrt{(1-s)\,\Delta}\right) \end{bmatrix} \right| \leq \frac{2ce\,[-\ln(1-s\|\sqrt{\Delta}\|^2)]^n}{k_1} \frac{e^{\frac{\|y\|+|u|\sqrt{(1-s)}\|\sqrt{\Delta}\|^2}{2\left(1-s\|\sqrt{\Delta}\|^2\right)}}}{(1-s\|\sqrt{\Delta}\|^2)^{3/2}}. \tag{A.21}$$

Breaking the inner integral in (A.19) into two, and applying the bound from (A.21) on $u \in [0, 1/(\|y\|\sqrt{(1-s)\,\Delta})]$, and the bound from (A.20) on $u \in [1/(\|y\|\sqrt{(1-s)\,\Delta}), +\infty)$, establishes the required bound.

A.5.5 Lemma 5

Lemma 5. *Let $\psi(y)$ be analytic and even for all y, and let $\eta(\Delta, y)$ be even in y, and analytic in both variables for all y and all Δ such that $\|\sqrt{\tau(\Delta)}\| < 1$, where $|z|/\sqrt{k_2} \leq \|z\| \leq |z|/\sqrt{k_1}$ is a norm over the reals. Further let there exist some $c > 0$ and $d > 0$ such that $\psi(z)$ and $\eta(\tau, z)$ satisfy*

$$|\psi(z)| \leq ce^{\frac{\|z\|^2}{2}} \qquad |\eta(\Delta, z)| \leq de^{\frac{\|z\|^2}{2(1-\|\sqrt{\tau(\Delta)}\|^2)}}.$$

Then for any complex γ, there exists a function $w(\Delta, y)$ that is defined and analytic for all complex y and Δ such that $\|\sqrt{\tau(\Delta)}\| < 1$, even in y, and that satisfies

$$\frac{\partial w}{\partial \Delta}(\Delta, y) = \frac{\gamma}{y}\frac{\partial w}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}(\Delta, y) - \left(\frac{b^2}{2}y^2 + d\right) w(\Delta, y) + \gamma\eta(\Delta, y) \tag{A.22}$$

$$w(0,y) = \psi(y) \,. \tag{A.23}$$

Furthermore, there exist continuous functions $d_1(\Delta)$ *and* $d_2(\Delta)$*, defined for all* $\|\sqrt{\tau(\Delta)}\| < 1$*, such that* $w(\Delta, y)$ *and its first spatial derivative satisfy*

$$|w(\Delta, y)| \leq d_1(\Delta)\, e^{\frac{\left\|e^{b\Delta} y\right\|^2}{2\left(1-\|\sqrt{\tau(\Delta)}\|^2\right)}} \tag{A.24}$$

$$\left|\frac{\partial w}{\partial y}(\Delta, y)\right| \leq d_2(\Delta)\,|y|\, e^{\frac{\left\|e^{b\Delta} y\right\|^2}{2\left(1-\|\sqrt{\tau(\Delta)}\|^2\right)}} \,. \tag{A.25}$$

Proof:

We construct the solution explicitly, by the parametrix method of Levi (1907), as described in Friedman (1964). Note that the assumptions of Friedman (1964) are not satisfied in this case, because the coefficients of the PDE are not bounded. We therefore must modify the method.

First, define

$$\nu_0(\Delta, y) \equiv \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \, \frac{1}{2}\left[\psi\left(y + u\sqrt{\Delta}\right) + \psi\left(y - u\sqrt{\Delta}\right)\right] du \tag{A.26}$$

$$\phi_1(\Delta, y) \equiv \eta(\Delta, y) + \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{u}{2y\sqrt{\Delta}}\left[\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)\right] du.$$

Then for each integer $i \geq 1$, define

$$\nu_i(\Delta, y) \equiv \int_0^1 \int_{-\infty}^{+\infty} \frac{\Delta e^{-\frac{u^2}{2}}}{2\sqrt{2\pi}} \begin{bmatrix} \phi_i\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) \\ +\phi_i\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right) \end{bmatrix} du ds$$

$$\phi_{i+1}(\Delta, y) \equiv \int_0^1 \int_{-\infty}^{+\infty} \frac{\Delta u e^{-\frac{u^2}{2}}}{y\sqrt{2\pi}} \frac{\begin{bmatrix} \phi_i\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) \\ -\phi_i\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right) \end{bmatrix}}{2\sqrt{(1-s)\Delta}} du ds.$$

Finally, for each $i \geq 0$, define

$$\xi_i(\Delta, y) = e^{\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)\Delta} \nu_i\left(e^{b\Delta} y, \tau(\Delta)\right).$$

The solution to the PDE is then given by

$$w(\Delta, y) \equiv \xi_0(\Delta, y) + \sum_{n=1}^{\infty} \gamma^n \xi_n(\Delta, y). \tag{A.27}$$

Four things must be proven. First, it must be shown that $w(\Delta, y)$ is well-defined. Second, it must be shown that $w(\Delta, y)$ is analytic in Δ and y, and even in y. Third, it must be shown that it solves the PDE with final condition. Finally, it must be shown that $w(\Delta, y)$ and its spatial derivative satisfy the stated bounds.

To show existence of the integrals and convergence of the infinite sum, we first note that $\psi(y)$ satisfies the conditions of Lemmas 1 and 2. Together with the assumptions on $\eta(\Delta, y)$, it follows by these lemmas that $\nu_0(\Delta, y)$ and $\phi_1(\Delta, y)$ are defined and analytic for all y and $\|\sqrt{\Delta}\| < 1$, and are also even in y. The lemma also establishes bounds on $\nu_0(\Delta, y)$ and the second term in the definition of $\phi_1(\Delta, y)$; together with the assumptions on $\eta(\Delta, y)$, we have

$$|\nu_0(\Delta, y)| \leq \frac{2ce^{\frac{\|y\|^2}{2(1-\|\sqrt{\Delta}\|^2)}}}{\sqrt{1-\|\sqrt{\Delta}\|^2}}$$

$$|\phi_1(\Delta, y)| \leq \left(d + \frac{2c(e^2+1)}{k_1}\right) \frac{e^{\frac{\|y\|^2}{2(1-\|\sqrt{\Delta}\|^2)}}}{(1-\|\sqrt{\Delta}\|^2)^{3/2}}.$$

But $\phi_1(\Delta, y)$ itself then satisfies the conditions of Lemmas 3 and 4. Lemma 4 can thus be applied recursively, with each $\phi_i(\Delta, y)$ for $i \geq 1$ satisfying the conditions of the lemma, and the construction in the lemma producing $\phi_{i+1}(\Delta, y)$, which itself satisfies the conditions of both Lemmas 3 and 4. Proceeding in this way, by repeated application of this lemma, it is established that for all values of $i \geq 1$, the functions $\phi_i(\Delta, y)$ and $\nu_i(\Delta, y)$ are defined and analytic in both variables (with $\|\sqrt{\Delta}\| < 1$), even in y, and satisfy

$$|\phi_i(\Delta, y)| \leq \frac{\frac{2c(e^2+1)+dk_1}{k_1} e^{\frac{\|y\|^2}{2(1-\|\sqrt{\Delta}\|^2)}} \left[-\frac{2ck_2(e^2+1)\ln(1-\|\sqrt{\Delta}\|^2)}{k_1}\right]^{i-1}}{(1-\|\sqrt{\Delta}\|^2)^{3/2}(i-1)!} \tag{A.28}$$

$$|\nu_i(\Delta, y)| \leq \frac{\left(\frac{dk_1}{k_2(e^2+1)} + \frac{c}{k_2}\right) e^{\frac{\|y\|^2}{2(1-\|\sqrt{\Delta}\|^2)}}}{\sqrt{1-\|\sqrt{\Delta}\|^2}} \frac{\left[-\frac{2(e^2+1)k_2\ln(1-\|\sqrt{\Delta}\|^2)}{k_1}\right]^i}{i!}. \tag{A.29}$$

Note that the $\nu_i(\Delta, y)$ are bounded by the coefficients of an exponential power series, and therefore converge uniformly on compact subsets of y and Δ, provided $\|\sqrt{\Delta}\| < 1$. It follows that the $\xi_i(\Delta, y)$ are also analytic in both y and Δ, and converge uniformly on compact subsets of y and Δ, where $\|\sqrt{\tau(\Delta)}\| < 1$. Since each term in the sum that defines $w(\Delta, y)$ is analytic for all y and $\|\sqrt{\tau(\Delta)}\|$, uniform convergence on compacts establishes the analyticity of the sum. It is also evident that $w(\Delta, y)$ is even in y, since each term in the sum is even in y.

It remains to establish that $w(\Delta, y)$ is a solution to the PDE. Satisfaction of the final condition is established in a straightforward way. For every $i \geq 1$, $\nu_i(0, y) = 0$. From (A.26), it follows by application of Lemma 1 that $\nu_0(0, y) = \psi(y)$, which establishes the final condition.

To show that $w(\Delta, y)$ is a solution to the general PDE, we first note that

$$\begin{aligned}\frac{\partial w}{\partial \Delta}(\Delta, y) - \frac{1}{2}\frac{\partial^2 w}{\partial \Delta^2}(\Delta, y) + \left(\frac{b^2}{2}y^2 + d\right) w(\Delta, y) \\ = b\gamma w(\Delta, y) + e^{\frac{b}{2}(y^2+\Delta)+(b\gamma-d)} e^{2b\Delta} \sum_{n=1}^{\infty} \gamma^n \phi_n\left(\tau(\Delta), e^{b\Delta} y\right)\end{aligned} \tag{A.30}$$

where $\nu_i^{(\tau)}(\tau, z)$ denotes partial differentiation with respect to the first argument, and partial differentiation with respect to the second argument once and twice are denoted by $\nu_i^{(z)}(\tau, z)$ and $\nu_i^{(zz)}(\tau, z)$, respectively. Differentiation term-by-term in (A.30) is justified by the uniform convergence of (A.27) on compacts. Evaluation of the derivatives comes from Lemmas 1 and 3. Next, we note that

$$\frac{\partial w}{\partial y}(\Delta, y) = byw\left(\tau(\Delta), e^{b\Delta}y\right) + ye^{b\Delta}e^{\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)}\begin{pmatrix} \phi_1\left(\tau(\Delta), e^{b\Delta}y\right) - \eta\left(\tau(\Delta), e^{b\Delta}y\right) \\ +\sum_{n=1}^{\infty}\gamma^n\phi_{n+1}\left(\tau(\Delta), e^{b\Delta}y\right)\end{pmatrix}. \tag{A.31}$$

By substituting (A.30) and (A.31) into (A.22), it can be seen that $w(\Delta, y)$ satisfies the general PDE.

The bounds on $w(\Delta, y)$ and its derivative follow because, except for $\nu_0(\Delta, y)$, the bounds in (A.29) are the coefficients of an exponential power series. It follows that

$$|w(\Delta, y)| \leq \left(\frac{dk_1}{k_2\left(e^2+1\right)} + \frac{2c}{\min(1, k_2)}\right)e^{\left|\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)\Delta\right|} \\ \times \frac{e^{\frac{\left\|e^{b\Delta}y\right\|^2}{2\left(1-\|\sqrt{\tau(\Delta)}\|^2\right)}}}{\sqrt{1-\|\sqrt{\tau(\Delta)}\|^2}}\left(1-\|\sqrt{\tau(\Delta)}\|^2\right)^{-\frac{2|\gamma|\left(e^2+1\right)k_2}{k_1}},$$

which establishes (A.24). (A.25) follows similarly, by applying (A.28) to (A.31).

A.5.6 Proof of Theorem 3

The general PDE of Lemma 5 (i. e., (A.22) and (A.23)) is not in the canonical form (4.2). However, the general PDE of the theorem (i. e., (4.12) and (4.13)) can be converted to the PDE of the lemma by a change of variables, and we can therefore express the solution of this PDE in terms of the solution to the PDE of the Lemma.

We begin with the case $\sqrt{1+8a}/2 \notin \mathbb{N}$. We first apply Lemma 5 with $\gamma = \left(1-\sqrt{1+8a}\right)/2$, $\psi(y) = g_1(y)$, and $\eta(\Delta, y) = 0$, and find a solution $h_1(\Delta, y) = w(\Delta, y)$. By applying Lemma 5 a second time, with $\gamma = \left(1+\sqrt{1+8a}\right)/2$ and $\psi(y) = g_2(y)$, we find another solution $h_2(\Delta, y) = w(\Delta, y)$. We now can construct a solution to the original PDE,

$$h(\Delta, y) = y^{\frac{1-\sqrt{1+8a}}{2}}h_1(\Delta, y) + y^{\frac{1+\sqrt{1+8a}}{2}}h_2(\Delta, y).$$

By inspection, this solution satisfies the PDE with final condition (i. e., (4.12) and (4.13)).

For $\left(\sqrt{1+8a}\right)/2 \in \mathbb{N}$, we apply Lemma 5 with $\gamma = \left(1-\sqrt{1+8a}\right)/2$, $\psi(y) = g_1(y)$, and $\eta(\Delta, y) = 0$, and then again with $\gamma = \left(1+\sqrt{1+8a}\right)/2$, $\psi(y) = g_2(y)$, and $\eta(\Delta, y) = 0$, obtaining solutions $h_1(\Delta, y)$ and $h_2(\Delta, y)$, respectively. We apply Lemma 5 again, this time with $\gamma = \left(1-\sqrt{1+8a}\right)/2$, $\psi(y) = 0$, and,

$$\eta(\Delta, y) = -y^{-1+\sqrt{1+8a}}\frac{\partial h_2}{\partial y}(\Delta, y) - \frac{\sqrt{1+8a}}{2}y^{-2+\sqrt{1+8a}}h_2(\Delta, y).$$

Since $h_2(\Delta, y)$ is itself the result of an application of Lemma 5, it is analytic in both variables, even in y, and satisfies the bounds in the lemma statement, (A.22) and (A.23). It follows that $\eta(\Delta, y)$ is analytic in both variables, and even in y. Since $h_2(\Delta, y)$ is analytic and even in y, its derivative with respect to y is analytic and odd in y. Since $\sqrt{1+8a}$ is a non-negative even integer, the derivative of $h_2(\Delta, y)$ is premultiplied by y raised to a power that is either -1 or a positive odd integer. If it is a positive odd integer, the first term is analytic and even in y; if it is -1, then the result of dividing an analytic and odd function by y is analytic (and even) everywhere except $y = 0$, but extends by analytic continuation to this value. If $\sqrt{1+8a} = 0$, then the second term is zero, and is trivially analytic in y; if $\sqrt{1+8a}$ is some positive even integer, then the second term is the analytic function $h_2(\Delta, y)$ multiplied by a constant and a non-negative even power of y, which is also analytic. So both terms are analytic and even in y.

The $\eta(\Delta, y)$ in the third application of Lemma 5 does not satisfy the lemma conditions for the same norm $\|y\|$ used in the first two applications. Define $\|y\|_\epsilon \equiv \|y\| / (1-\epsilon)$ for any $0 < \epsilon < 1$. For each ϵ, $\eta(\Delta, y)$ satisfies the conditions of Lemma 5 for some d. Applying the lemma a third time and denoting the result by $h_3(\Delta, y)$, the solution to the PDE is

$$h(\Delta, y) = [h_1(\Delta, y) + h_3(\Delta, y)]\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y.$$

By inspection, $h(\Delta, y)$ solves the general PDE with final condition. The construction of $h_3(\Delta, y)$ is valid only for $\|\sqrt{\tau(\Delta)}\|_\epsilon < 1$, not $\|\sqrt{\tau(\Delta)}\| < 1$. But ϵ can be chosen to be arbitrarily small, so analyticity of $h_3(\Delta, y)$ for any particular value of $\|\sqrt{\tau(\Delta)}\| < 1$ can be established by applying Lemma 5 with the norm $\|y\|_\epsilon$ for sufficiently small ϵ. The bounds from Lemma 5 then do not apply to $h_3(\Delta, y)$ uniformly for all $\|\sqrt{\tau(\Delta)}\| < 1$, but there are no boundedness conditions in the theorem statement to be established; the boundedness conditions in the lemma are only needed for $h_2(\Delta, y)$, constructed in the second application, to show that the third application is justified. The theorem is now proven for the case of $\sqrt{1+8a}/2 \in \mathbb{N}$.

A.6 Proof of Corollary 3

Choose any $k > 0$, and define $\|y\| \equiv |y| / \sqrt{k}$. Then $g_1(y)$ and $g_2(y)$ satisfy the conditions of Theorem 3 for this norm and for $c = c_k$. So by application of Theorem 3, there exist functions $h_1(\Delta, y)$ and $h_2(\Delta, y)$, defined and analytic for all y and $|\Delta| < k$, such that $h(\Delta, y)$, as defined in the corollary statement, satisfies the partial differential equation with final condition. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the construction in the proof of Theorem 3 does not depend on c or on the choice of the norm, so it is clear that the solutions constructed for different values of k are the same function. Consequently, the solution is defined and analytic for all complex Δ.