

Ralph's Natural Synergy

by

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1 Introduction

The accounting problem under consideration in this paper is the trade-off between individual and aggregate measures. That is, is it more efficient to measure the productivity of each member of a group separately? Or is it more efficient to extract only one measure: a group result? Absent any interesting interactions among the members of the group, the answer is perhaps clear. Obtaining only a group measure cannot increase the "amount" of usable information. On the other hand, when subtle and complex (and hopefully productive) interactions exist, the problem is more difficult. An individual measure may not capture that individual's contribution to group productivity. Individually the measure may not look impressive, but taking into account the abilities and contributions of the other members may yield an entirely different assessment. When intricate teamwork is a key force, whether the team won or lost is quite informative about the contributions of each team member. Furthermore, productive interactions are the hallmark of successful organizations. It seems sensible to analyze accounting for organizations in a setting in which there is a demand for the organization to arise in the first place.

The line of attack in this paper is to model interactions and measurement thereof the way that Nature does. In particular, the model is Nature's activities at the subatomic level: quantum probabilities. There are some advantages to this approach. Interaction (interference) is an important physical phenomenon, and Nature exploits interactions in a remarkably efficient manner. The analysis is subject to some rigor as it is axiomatic; four quantum axioms are presented in the appendix. Finally, it is suggestive that the quantum approach is information based; the concept of information is always central in the analysis.

The paper begins with a standard agency problem. Cost of control is the metric used to evaluate alternative information structures. The first case presented has no interactions. Accordingly, individual measures yield a lower cost of control than an aggregate measure. The next case is maximum coordination. (Synergy is the term we use for positive interaction effects.) The maximum synergy probabilities are derived from the maximum interaction case in Nature. When there is maximum synergy, an aggregate measure is more efficient. This result holds even though there is only one signal available when the aggregate measure is used, as opposed to multiple individual signals under individual measurement. With maximum synergy the coordination effect is stronger than the effect associated with increasing the number of signals.

The axiomatic structure allows confronting other questions as well. Of particular interest is the behavior of control costs when synergy is neither at its maximum level or absent. When synergy is partial, there are regions where aggregate measurement is more or less efficient than individual, and the regions

are characterized. A control problem may become more severe due to changes in the information environment, technology, or preferences. As the severity of the control problem increases, the region of aggregate measurement superiority also increases. Aggregate measurement actually becomes relatively more efficient (relative to individual measures) as the agency problem increases.

2 No Synergy Case

Start with a simple agency problem constructed around the following act-signal probability matrix (information structure).

	<i>Success</i>	<i>Failure</i>
<i>Work</i>	p	$1 - p$
<i>Shirk</i>	0	1

Many of the comparisons in what follows can be made directly from the information structure. When additional context is necessary, a standard control problem will be used. The principal is risk neutral and the risk averse agent exhibits multiplicative separable CARA expected utility representation.

$$U(I - c) = -e^{-r(I-c)}$$

where I is the dollar payment, c the personal cost of the unobservable act, and r the Arrow-Pratt risk aversion parameter. Assuming the principal always wishes to induce the high act (work), the design program takes on a familiar appearance:

$$\begin{aligned} \min \quad & E(pmt) = pI_s + (1 - p)I_f \\ \text{st} \quad & pU(I_s - c_h) + (1 - p)U(I_f - c_h) \geq U(RW) \\ & pU(I_s - c_h) + (1 - p)U(I_f - c_h) \geq U(I_f - c_l) \end{aligned}$$

RW is the reservation wage, and the payments and personal costs are appropriately subscripted. Since with multiplicative CARA the structure of the solution is invariant with respect to reservation wage and the minimum personal cost, set $RW = c_l = 0$. The solution is parameterized by p , r , and c_h . Exploiting the multiplicative CARA preference representation and the fact both constraints are binding, the problem can be written in matrix form, where the information structure is highlighted.

$$\begin{bmatrix} p & 1 - p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U(I_s) \\ U(I_f) \end{bmatrix} = \begin{bmatrix} U(c_h) \\ U(0) \end{bmatrix}$$

For $p = \frac{1}{4}$, $r = .01$, and $c_h = 10$, we have

$$\begin{aligned} I_f &= 0 \\ I_s &= 47.91 \\ E(pmt) &= 11.98 \end{aligned}$$

In order to examine synergy the model must contain another agent, who, to keep things simple, is identical to the first agent. If they are measured individually, the signal probabilities for both agents are the same. These probabilities, as are all subsequent ones, are derived from a quantum mechanical process in Nature. For the individual measurement/ no synergy case, the probabilities are straightforward; the probability of success for one agent is independent of the effort level of the other agent. For example,

$$\text{prob}(\text{success}_1 \mid \text{work}_1, \text{work}_2) = p = \text{prob}(\text{success}_1 \mid \text{work}_1, \text{shirk}_2)$$

If, on the other hand, measurement is on an aggregate basis, there is one, and only one, signal for the joint output: whether the output of their joint effort was measured as a success or a failure. The issue is to assign probabilities to the joint success and failure signals. These probabilities, and all subsequent ones, are derived from a quantum mechanical process in Nature described in the Appendix, and are reported in Proposition 1.

Proposition 1 *Restrict attention to no synergy. For the inputs and production function in the Appendix, the quantum axioms imply the following success probabilities for aggregate measurement.*

$$\begin{aligned} \text{prob}(\text{success} \mid \text{work}_1, \text{work}_2) &= 2p(1-p) \\ \text{prob}(\text{success} \mid \text{work}_1, \text{shirk}_2) &= \text{prob}(\text{success} \mid \text{shirk}_1, \text{work}_2) = p \\ \text{prob}(\text{success} \mid \text{shirk}_1, \text{shirk}_2) &= 0 \end{aligned}$$

One interpretation is that a success signal arises if there is one, and only one, success for the individual agents when both agents work hard. If but one or neither agents works hard, the success probability collapses to the single agent case. The aggregate measurement control problem can be written in best response form. That is, assume the other agent will work hard. Collusion, the shirk-shirk strategy, is not a pressing concern as will be discussed shortly. This provides the following design program.

$$\begin{aligned} \min E(\text{pmt}) &= 2p(1-p)I_s + (1-2p(1-p))I_f \\ \text{st} \quad &2p(1-p)U(I_s - c_h) + (1-2p(1-p))U(I_f - c_h) \geq U(RW) \\ &2p(1-p)U(I_s - c_h) + (1-2p(1-p))U(I_f - c_h) \geq pU(I_s - c_l) + (1-p)U(I_f - c_l) \end{aligned}$$

The two constraints, as is readily verified, are binding. This implies the program can also be written as a matrix equation highlighting the aggregate information structure.

$$\begin{bmatrix} 2p(1-p) & 1-2p(1-p) \\ p & 1-p \end{bmatrix} \begin{bmatrix} U(I_s) \\ U(I_f) \end{bmatrix} = \begin{bmatrix} U(c_h) \\ U(0) \end{bmatrix}$$

The information structure representation allows the comparison in Corollary 1.

Corollary 1 For the no synergy case, aggregate measurement probabilities

$$\begin{bmatrix} 2p(1-p) & 1-2p(1-p) \\ p & 1-p \end{bmatrix}$$

are a Blackwell garbling of the individual probabilities $\begin{bmatrix} p & 1-p \\ 0 & 1 \end{bmatrix}$. The garbling

matrix is $\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$.

Since individual measurement Blackwell dominates aggregate, and since we are dealing with an agency problem, we know from Grossman and Hart [1983] that it is superior for all parametric versions of our setting. To illustrate this for the numerical example parameters, we have an aggregate solution of

$$\begin{aligned} I_f &= -17.42 \\ I_s &= 84.62 \\ E(pmt) &= 20.84 \end{aligned}$$

The control cost, the expected payment, increases from 11.98 to 20.84 when aggregate measurement is used.

One way to see that shirk/shirk is not a dominant equilibrium is to note that, since $\text{prob}(\text{success} \mid \text{shirk}_1, \text{shirk}_2) = 0$, the certainty equivalent is negative for both agents, not even achieving the reservation wage.

It should be noted that there are other ways to define success for the group. For example, a success could be defined as at least one individual experiences success (as opposed to exactly one success), in which case the probabilities are as follows.

$$\begin{aligned} \text{prob}(\text{success} \mid \text{work}_1, \text{work}_2) &= 1 - (1-p)^2 \\ \text{prob}(\text{success} \mid \text{work}_1, \text{shirk}_2) &= \text{prob}(\text{success} \mid \text{shirk}_1, \text{work}_2) = p \\ \text{prob}(\text{success} \mid \text{shirk}_1, \text{shirk}_2) &= 0 \end{aligned}$$

And, since the collusion line can be ignored, the information structure is $\begin{bmatrix} 1 - (1-p)^2 & (1-p)^2 \\ p & 1-p \end{bmatrix}$. It is interesting to note that the "exactly one success" information structure is a garbling of the "at least one success" structure (for $p \leq \frac{2}{1+\sqrt{5}}$). So the control cost could be reduced by using the alternative structure, as is easily checked. Since we are committed to using the measurement systems Nature uses, we will use the less informative "exactly one" structure. When aggregating measure, Nature is not very efficient in the no synergy case. It is when synergy exists that Nature is quite clever, as discussed in the next section.

3 Pure Synergy Case

Synergy arises from the interaction among agents. It is defined in terms of the probabilities.

Definition 1 *Synergy is present when the aggregate success probability for the group is greater than the sum of the success probabilities when measured individually.*

In order for synergy to be defined in this manner, the individual success probability must be less than $\frac{1}{2}$. Otherwise we face the possibility the group success probability might exceed unity. This "less than $\frac{1}{2}$ " restriction will be in place for the rest of the analysis.

An important benefit of synergy, of course, is the benefit associated with additional production. The focus in this paper, however, is solely on the cost of control. Basically the question is whether fewer measurements (in an aggregate system) increase the control costs. The previous section verified that, not surprisingly, in a setting without synergy aggregate measurement yields higher control costs than individual measurements.

The benchmark synergy setting is presented in Propositions 2, the aggregate case, and 3, the individual case.

Proposition 2 *Restrict attention to maximum possible synergy (pure synergy). For the inputs and production function in the Appendix, the quantum axioms imply the following success probabilities for aggregate measurement.*

$$\begin{aligned} \text{prob}(\text{success} \mid \text{work}_1, \text{work}_2) &= 4p(1-p) \\ \text{prob}(\text{success} \mid \text{work}_1, \text{shirk}_2) &= \text{prob}(\text{success} \mid \text{shirk}_1, \text{work}_2) = p \\ \text{prob}(\text{success} \mid \text{shirk}_1, \text{shirk}_2) &= 0 \end{aligned}$$

Nature calculates probabilities using angles and projections. In polar coordinates the projection of an angle, θ , onto the vertical (horizontal) axis is $\sin \theta$ ($\cos \theta$). To change into probabilities, the projection is squared; since by Pythagoras $\sin^2 \theta + \cos^2 \theta = 1$, a probability measure is thereby created. Notice that if θ is viewed as effort level, it is sensible to think of $\sin^2 \theta$ as the probability of success - increasing in effort θ as long as θ does not exceed $\frac{\pi}{2}$. When Nature deals with multiple angles, the same probability as squared projection idea is maintained. Now, however, the probability of success is $\sin^2(\theta_1 + \theta_2)$. The salient trigonometric identity (which can be derived directly from Euler's formula) is $\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$. For the identical agent case $\theta_1 = \theta_2 = \theta$, and $p = \sin^2 \theta$ implies $\sin^2(\theta_1 + \theta_2) = (2 \sin \theta \cos \theta)^2 = 4p(1-p)$.

The alternative measurement regime is an individual measurement for each agent. When Nature attempts an individual measurement in a pure synergy context no information is forthcoming. In an economic setting the interpretation is that coordination and cooperation are the only attributes of interest. Measuring one individual's contribution without considering joint effects is worthless.

Proposition 3 *Restrict attention to maximum possible synergy (pure synergy). For the inputs and production function in the Appendix, the quantum axioms imply the following success probabilities for individual measurement.*

$$\begin{aligned} \text{prob}(\text{success} \mid \text{work}_1, \text{work}_2) &= \frac{1}{2} \\ \text{prob}(\text{success} \mid \text{work}_1, \text{shirk}_2) &= \text{prob}(\text{success} \mid \text{shirk}_1, \text{work}_2) = \frac{1}{2} \\ \text{prob}(\text{success} \mid \text{shirk}_1, \text{shirk}_2) &= \frac{1}{2} \end{aligned}$$

Obviously, it is not possible to induce a high effort level in a pure synergy environment using individual measurement.

Turning to aggregate measurement, reconsider the numerical example. The best response program is

$$\begin{aligned} \min \quad & E(\text{pmt}) = 4p(1-p)I_s + (1-4p(1-p))I_f \\ \text{st} \quad & 4p(1-p)U(I_s - c_h) + (1-4p(1-p))U(I_f - c_h) \geq U(RW) \\ & 4p(1-p)U(I_s - c_h) + (1-4p(1-p))U(I_f - c_h) \\ & \geq pU(I_s - c_l) + (1-p)U(I_f - c_l) \end{aligned}$$

For $p = \frac{1}{4}$ the solution is

$$\begin{aligned} I_f &= -4.65 \\ I_s &= 15.40 \\ E(\text{pmt}) &= 10.39 \end{aligned}$$

Here, as well, collusion is not a problem, as the certainty equivalent for shirk/shirk is less than the reservation wage.

It is suggestive that in the example the aggregate measurement pure synergy case has a lower control cost than individual measurement with no synergy. But as stated in the following corollary, the comparison is context specific.

Corollary 2 *The individual/no synergy and aggregate/pure synergy signal act probabilities are not Blackwell comparable.*

Even though they are not Blackwell comparable, they are comparable when embedded in the CARA multiplicative agency problem. For this particular setting the cost of control for individual/no synergy always exceeds the cost of control for aggregate/synergy as illustrated in the numerical example (See Fellingham and Schroeder, 2006).

So far the results are that when synergy is at its maximum (pure synergy), aggregate measurement is better than individual measurement in the sense that control costs are less. When there is no synergy, the ranking is reversed. Presumably, there will occur situations where there is some, but not pure, synergy. The intermediate setting is the subject of the next section.

4 Partial Synergy

Nature is quite clever at processing information when there is some, but not pure, synergy. Let α denote the probability the input exhibits synergy. The conditional success probabilities are those derived in the preceding section. By the laws of probability, the unconditional probabilities are linear combinations of the benchmark cases. The measurement probabilities are reported in Proposition 4 for individual and aggregate measurement systems, given a best response contracting problem with the other agent choosing to work.

Proposition 4 *For the inputs and production function in the Appendix, the quantum axioms imply the following success probabilities.*

		<i>Individual Measurement</i>	
		<i>success</i>	<i>failure</i>
<i>No synergy</i>	<i>work</i>	p	$1 - p$
	<i>shirk</i>	0	1
		<i>success</i>	<i>failure</i>
<i>Pure synergy</i>	<i>work</i>	$\frac{1}{2}$	$\frac{1}{2}$
	<i>shirk</i>	$\frac{1}{2}$	$\frac{1}{2}$
		<i>success</i>	<i>failure</i>
<i>Partial synergy</i>	<i>work</i>	$(1 - \alpha)p + \frac{1}{2}\alpha$	$1 - (1 - \alpha)p - \frac{1}{2}\alpha$
	<i>shirk</i>	$\frac{1}{2}\alpha$	$1 - \frac{1}{2}\alpha$
		<i>Aggregate Measurement</i>	
		<i>success</i>	<i>failure</i>
<i>No synergy</i>	<i>work</i>	$2p(1 - p)$	$1 - 2p(1 - p)$
	<i>shirk</i>	p	$1 - p$
		<i>success</i>	<i>failure</i>
<i>Pure synergy</i>	<i>work</i>	$4p(1 - p)$	$1 - 4p(1 - p)$
	<i>shirk</i>	p	$1 - p$
		<i>success</i>	<i>failure</i>
<i>Partial synergy</i>	<i>work</i>	$2p(1 - p)(1 + \alpha)$	$1 - 2p(1 - p)(1 + \alpha)$
	<i>shirk</i>	p	$1 - p$

Not surprisingly, for extreme values of α , one information structure is superior to the other in the Blackwell sense. For example, for "little" synergy (α close to zero), individual measurement is Blackwell superior. On the other hand, for large values of α individual measurement is a garbling of aggregate. The bounds defining the regions of dominance are provided in Corollary 3.

Corollary 3 *Aggregate measurement is a Blackwell garbling of individual measurement when $2p - \alpha(5 - 4p) + 2\alpha^2(1 - p) \geq 0$. Individual measurement is a Blackwell garbling of aggregate when $\alpha^2(1 - p) - p + \frac{\alpha}{2} \geq 0$.*

Return to the numerical example with $p = \frac{1}{4}$. Individual is superior to aggregate when $\frac{1}{2} - 4\alpha + \frac{3}{2}\alpha^2 \geq 0$ which is satisfied for $0 \leq \alpha \leq \frac{4 - \sqrt{13}}{3} \simeq .1315$,

that is, for "small" values of α . At the other end of the α continuum, aggregate is Blackwell superior when $\frac{3}{16}\alpha^2 + \frac{1}{8}\alpha - \frac{1}{16} \geq 0$, that is for $\frac{1}{3} \leq \alpha \leq 1$. It is interesting to note that, regardless of the value of p , aggregate is Blackwell superior whenever $\alpha \geq \frac{1}{\lambda}$ where λ is the "golden ratio," an important number in the history of mathematics. This follows from the fact that the greatest lower bound on α occurs for the upper bound on p : $p = \frac{1}{2}$; substitute into the condition in the Corollary. $\frac{1}{\lambda}$ is approximately .61803.

Continuing with $p = \frac{1}{4}$, for values of the synergy parameter, α , between $\frac{4-\sqrt{13}}{3}$ and $\frac{1}{3}$ the measurement systems are not Blackwell comparable. In order to establish a ranking, it is necessary to embed the problem in more structure. As before, let $RW = c_l = 0$, $c_h = 10$, and $r = .01$. We find the control costs are equal for $\alpha = .206364$. That is, for α levels above the cut-off, aggregate measurement yields a lower cost of control for the specific control problem. At the cut-off the expected payment is 13.5868 for both measurement systems. The signal act probabilities at the cutoff are displayed in the table.

Individual			Aggregate		
	<i>success</i>	<i>failure</i>		<i>success</i>	<i>failure</i>
<i>work</i>	.3016	.6984	<i>work, work</i>	.4524	.5476
<i>shirk</i>	.1032	.8968	<i>shirk, work</i>	.25	.75

This suggests comparative statics. Denote α^* as the cut-off synergy level where the control costs are equal for the two different measurement systems. Intuitively, it follows from the Blackwell rankings that for synergy values exceeding α^* , aggregate measurement has lower control cost than individual measurement. In other words, α^* bounds the region of aggregate measurement superiority. The comparative statics noted below demonstrate the general idea that, when the control problem becomes more severe, aggregate measurement becomes relatively more efficient. Though numerically confirmed as of this writing, a formal proof eludes us.

Conjecture 1

$$\begin{aligned} \frac{\partial \alpha^*}{\partial r} &< 0 \\ \frac{\partial \alpha^*}{\partial p} &> 0 \\ \frac{\partial \alpha^*}{\partial c_h} &< 0 \end{aligned}$$

The reduced agency problem is parameterized by the agent risk aversion parameter, r , the private cost of work, c_h , and the success signal probability, p . As r increases, the additional agent risk aversion will increase the cost of control. In that case, the cutoff synergy amount declines, implying the region in which aggregate measures are preferred to individual increases. The relative performance of the two measurement systems goes the same way as the agency problem increases with the private cost of work, c_h . As p is a measure of the informativeness of the information structure, the agency problem eases as p increases. A decline in p is accompanied by a decline in α^* .

5 Concluding Remarks

We have enjoyed working on this project, and, by revealed preference, think Nature's production function is a useful setting in which to explore long-standing accounting issues. This reminds us, among other things, that measurement matters. Nature's process does not allow us to measure simultaneously individual and aggregate measures, just as a production process populated by a variety of individuals is apt to behave in fundamentally different fashion depending on the measures that are overlaid.

6 Appendix

(The appendix contains derivations for propositions 1-4 and corollaries 1-3 by applying the axioms in 6.1 to the production process in 6.2 along with a little algebraic manipulation.)

6.1 Axioms

There are four quantum axioms governing the behavior of quantum probabilities.

1. The superposition axiom:

A quantum unit (qubit) is specified by a two element vector, say $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$,

with $|\alpha|^2 + |\beta|^2 = 1$.

Dirac notation is a useful descriptor.

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $\langle\psi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^H$ where H is Hermitian (conjugate transpose). Following standard, but unfortunate, notation, H in a superscript is Hermitian, but H as a matrix is the Hadamard transformation as in the examples for axiom 2.

2. The transformation axiom:

A transformation of a quantum unit is accomplished by matrix multiplication.

Useful single qubit transformations are $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\Theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}$. Examples of the transformations in Dirac notation:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\Theta |0\rangle = e^{i\theta} |0\rangle; \quad \Theta |1\rangle = |0\rangle$$

3. The measurement axiom:

Measurement of a quantum state is accomplished by a linear projection from a set of projection matrices which add to the identity matrix. The probability of a particular measurement occurring is the squared absolute value of the projection. (A part of the axiom not explicitly used in the paper is that the post-measurement state is the projection appropriately normalized; this effectively rules out multiple measurement.)

For an example, let the projection matrices be $M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Note that M_0 projects onto the $|0\rangle$ vector, since $M_0 = |0\rangle\langle 0|$. For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the projection onto $|0\rangle$ is $M_0|\psi\rangle$. The probability of $|0\rangle$ being the result of the measurement is $\langle\psi|M_0|\psi\rangle = |\alpha|^2$.

4. The combination axiom:

Qubits are combined by tensor multiplication.

For example, two $|0\rangle$ qubits are combined as $|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ denoted

$|00\rangle$. It is often useful to transform one qubit in a combination and leave another unchanged; this can also be accomplished by tensor multiplication. Let H_1 denote a Hadamard transformation on the first qubit. Then if it is

applied to a two qubit system, $H_1 = H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and

$$H_1 |00\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

Another example of a two qubit transformation is the controlled not.

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

An example of important two qubit states are the Bell states defined as follows.

$$|\beta_{00}\rangle = CNOT H_1 |00\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\beta_{ij}\rangle = CNOT H_1 |ij\rangle \text{ for } i, j = 0, 1$$

The four two qubit Bell states form an orthonormal basis.

6.2 Production process

The modelled production process consists of inputs, a production function, and a measurement activity. Inputs are two qubit states exhibiting either no synergy or pure synergy. The no synergy input is $|00\rangle$; the pure synergy input is $|\beta_{00}\rangle$. The production function consists of a sequence of transformations: $H_2\Theta_2H_2H_1\Theta_1H_1$ which is denoted U .

Measurement can be either individual or aggregate. Individual measurement requires a "success" or "failure" signal for both agents. The success

projection matrix for agent one is $M_{S1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Hence, $M_{F1} =$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Similarly, for agent two $M_{S2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $M_{F2} =$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Aggregate measurement requires only a success or failure sig-

nal for the group: $M_S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $M_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

There are four sets of probabilities to calculate: individual measurement with no synergy; individual measurement with synergy; aggregate measurement with no synergy; and aggregate measurement with synergy. For the first case for agent one,

$$\begin{aligned} \text{prob}(\text{success}|\theta_1, \theta_2) &= \langle 00|U^H M_{S1}U|00\rangle \\ &= \sin^2\left(\frac{\theta_1}{2}\right) \end{aligned}$$

Assign the value $p = \sin^2\left(\frac{\theta_1}{2}\right)$.

For individual measurement and synergy (also for agent one):

$$\begin{aligned} \text{prob}(\text{success}|\theta_1, \theta_2) &= \langle \beta_{00}|U^H M_{S1}U|\beta_{00}\rangle \\ &= \frac{1}{2} \end{aligned}$$

Aggregate measurement and no synergy:

$$\begin{aligned}
\text{prob}(\text{success}|\theta_1, \theta_2) &= \langle 00|U^H M_S U|00\rangle \\
&= \sin^2\left(\frac{\theta_1}{2}\right)\cos^2\left(\frac{\theta_2}{2}\right) + \sin^2\left(\frac{\theta_2}{2}\right)\cos^2\left(\frac{\theta_1}{2}\right)
\end{aligned}$$

For $\theta_1 = \theta_2 = \theta$, $\text{prob}(\text{success}|\theta_1, \theta_2) = 2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) = 2p(1-p)$.
Aggregate measurement and synergy:

$$\begin{aligned}
\text{prob}(\text{success}|\theta_1, \theta_2) &= \langle \beta_{00}|U^H M_{S1} U|\beta_{00}\rangle \\
&= \sin^2\left(\frac{\theta_1 + \theta_2}{2}\right)
\end{aligned}$$

For $\theta_1 = \theta_2 = \theta$, $\text{prob}(\text{success}|\theta_1, \theta_2) = \left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^2 = 4p(1-p)$.

7 References