Accounting Structure
Chapter 12 - Production, Synergy, and Accounting

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production, synergy, and accounting

In this chapter we will use the quantum formalism to model production, and the accounting for production. There are several reasons why this might be a useful exercise. The most prominent reason is that the structure allows contemplating the phenomenon of synergy, where combining production activities is more efficient that conducting them separately. Furthermore, the benefit is due to an information effect. If accounting is, indeed, an information science, then accounting might well supply some insights into this information activity. In the quantum realm the effect is quite mysterious. In a production setting the same effect is perhaps not quite so mysterious. And double entry accounting offers a unified and useful representation, particularly in the directed graph format.

The quantum formalism is rich enough to run experiments, and there are implications for how to do the accounting measurement: how many cost pools to use, for example, or what activities to measure separately. The general implication is that the accounting measurement activity is, itself, an input to the production process. Information synergy is a delicate phenomenon, in the quantum world as well as in a production environment, and improper measurement can easily lead to its destruction.

12.1 Euler's formula

Before proceeding to a production setting, we require a really quite remarkable result known as Euler's formula.

Theorem 1 *Euler's formula:*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\text{where } i = \sqrt{-1}$$

We noted an expansion for e^x in chapter 5.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

There exist similar expansions for the trigonometric functions sine and cosine.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \end{aligned}$$

Euler worked out that using i raised to the appropriate powers allowed fitting the expressions together. A familiar case of Euler's formula is when it is evaluated at

$$\theta = \pi$$

The expression then contains the five most important numbers in mathematics, 0, 1, e, π, and i, as well as addition, exponentiation, and equality.

$$e^{i\pi} + 1 = 0$$

12.2 quantum operations

Quantum units (qubits) are the basic building blocks of nature, and include particles like electrons and photons. In the previous chapter we defined qubits as two element vectors. In this chapter we exploit the Dirac notation where the two basic qubits are denoted as $|0\rangle$ and $|1\rangle$. The basic operations on qubits are transformation and measurement.

In the previous chapter quantum processes were derived from an axiomatic development. In this chapter the development is simplifed somewhat, and it *might* be possible to do this chapter without studying the previous one, although checking with chapter 11 on occasion certainly would not hurt.

12.2.1 transformation

	objects	
transformations	$\|0\rangle$	$\|1\rangle$
X	$\|1\rangle$	$\|0\rangle$
Y	$\|0\rangle$	$-\|1\rangle$
H	$(\|0\rangle\| + \|1\rangle)/\sqrt{2}$	$(\|0\rangle\| - \|1\rangle)/\sqrt{2}$
Θ	$e^{i\theta}\|0\rangle$	$\|1\rangle$

The transformations X, bit flip, Y, phase flip, and H, Hadamard or beam splitter, were introduced in chapter 11. Θ is new for this chapter. We will require it for our production functions. The angle θ will be interpreted as the amount of labor put into production: the larger the angle, the greater the amount of labor.

12.2.2 measurement

In the previous chapter measurement of a qubit was accomplished by projection into a space defined by an eigenvector. The result of the measurement was uncertain, and the probabilities were computed from the projection. Dirac notation simplifies the measurement calculation in a large variety of cases. Physically, measurement can establish the position of an electron which can be uncertain before the measurement is conducted.

A general qubit, before it is measured (or looked at carefully), can be a combination of the elementary objects.

$$\alpha|0\rangle + \beta|1\rangle$$

Once measurement occurs, the qubit will reduce to one or the other elementary object. It can no longer be an uncertain combination of the two. All qubits have unit length, meaning

$$\alpha^2 + \beta^2 = 1$$

So the probability that the resultof the measurement will be $|0\rangle$ is α^2 and the probability a $|1\rangle$ will be the result is β^2. Since the numbers sum to one, this is a sensible probability measure.

A particular problem arises when a coefficient contains the imaginary number i. As $i^2 = -1$, negative numbers show up, and, of course, are unacceptable as probabilities. The problem is solved by using $i(-i)$ to "square" the i term, called the complex conjugate.

$$i(-i) = -(-1) = 1$$

In a production function a qubit of the following form will occur.

$$\frac{e^{i\theta}+1}{2}|0\rangle + \frac{e^{i\theta}-1}{2}|1\rangle$$

Euler's formula is used to compute the probability of $|0\rangle$ being the result from measurement. When "squaring" the coefficient, change the sign on the i term.

$$\begin{aligned} & \left(\frac{e^{i\theta}+1}{2}\right)\left(\frac{e^{-i\theta}+1}{2}\right) \\ = & \frac{1+e^{i\theta}+e^{-i\theta}+1}{4} \end{aligned}$$

Now substitute Euler.

$$\frac{1+\cos\theta+i\sin\theta+\cos(-\theta)+i\sin(-\theta)+1}{4}$$

and recalling that

$$\begin{aligned} \cos(-\theta) &= \cos\theta \quad \text{and} \\ \sin(-\theta) &= -\sin\theta \end{aligned}$$

The squared coefficient is

$$\frac{2+2\cos\theta}{4} = \frac{1+\cos\theta}{2}$$

The derived expression is a fine probability measure, as it is always positive between 0 and 1. We can, however, convert to a more convenient form using Euler once more.

$$\begin{aligned} e^{i(2\theta)} &= \left(e^{i\theta}\right)^2 \\ \cos(2\theta)+i\sin(2\theta) &= (\cos\theta+i\sin\theta)^2 \\ &= \cos^2\theta+2i\sin\theta-\sin^2\theta \end{aligned}$$

Restricting attention to the real terms (terms without an i),

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

As a side note, it is easy to generate trigonometric identities like the above using Euler's formula. In this case, we know from Pythagoras that, for all angles

$$1 = \cos^2 + \sin^2$$

Adding the two equations together

$$1+\cos(2\theta) = 2\cos^2\theta$$

Or rescaling the angle

$$\frac{1+\cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right)$$

And that is a nice expression for the probability that $|0\rangle$ is the result of the measurement. If the angle θ is zero, $|0\rangle$ will always be the result; as θ increases the probability $|1\rangle$ appears as the result increases. As the probabilities add to one, the probability of $|1\rangle$ being the result is

$$\sin^2\left(\frac{\theta}{2}\right)$$

We will use these results in the production function in the next section.

12.3 single unit production

In this section we develop a quantum production function. There are two productive inputs: material and labor. For the simplest process, let the material input be the $|0\rangle$ qubit. Further, let the labor input be an angle, θ, where, up to a point, the greater the angle, the greater or more productive the labor input. The one qubit production function is $H\Theta H|0\rangle$.

$$H\Theta H|0\rangle = \frac{\left(e^{i\theta}+1\right)|0\rangle + \left(e^{i\theta}-1\right)|1\rangle}{2}$$

If there is no labor input ($\theta = 0$), there is no production, and the output is $|0\rangle$: the input unit is what comes out. Successful production is when a different unit $|1\rangle$ emerges from the production process. Recall from the previous section the measurement probabilities are

qubit	*probability*
$\lvert 0\rangle$	$\cos^2(\theta/2)$
$\lvert 1\rangle$	$\sin^2(\theta/2)$

Restricting attention to angles between zero and 180 degrees,[1] the measurement is more likely to be $|1\rangle$ for greater θ. $\theta = 0$ yields no productive change: the measurement result is always $|0\rangle$. A 180 degree angle will produce $|1\rangle$ with certainty. For intermediate angles the probability is monotonocally increasing. If production and measurement are conducted several times, greater effort will in probability yield a larger number of success ($|1\rangle$) measurements. An example allows using the production function to generate some accounting numbers.

Example 12.1 *Let the cost of labor be* $.6\theta^2$*, and, for simplicity, let the cost of the material input* $|0\rangle$ *be zero. Furthermore, let the revenue be* 3 *for a successful production* $|1\rangle$ *(and revenue of zero if output is* $|0\rangle$*). For*

[1] Notice this leaves unexplored the possibility of supplying "too much" labor.

an input angle of 60 *degrees* ($\frac{\pi}{3}$) *the expected profit is computed as follows.*

$$\begin{aligned} &\textit{direct labor cost} &: \\ .6\theta^2 &= .6\left(\frac{\pi}{3}\right)^2 = \frac{\pi^2}{15} \end{aligned}$$

$$\begin{aligned} &\textit{expected revenue} &: \\ 3\sin^2\frac{\theta}{2} &= 3\left(\frac{1}{4}\right) \end{aligned}$$

$$\begin{aligned} &\textit{expected profit} &: \\ &\frac{3}{4} - \frac{\pi^2}{15} \end{aligned}$$

12.4 multiple qubits and entanglement

Dirac notation eases the transition to multiple qubit analysis. In fact, moving to two qubits is relatively painless. Instead of two objects, $|0\rangle$ and $|1\rangle$, they are now combined into four objects of two qubit pairs: $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. It is possible to operate on one of the qubits and leave the other unchanged. For example, H_1means conduct a Hadamard operation on the first qubit only, leaving the second part of the pair unchanged. Some two qubit operations are tabulate below.

	objects			
transformations	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
H_1	$(\|00\rangle + \|10\rangle)/\sqrt{2}$	$(\|01\rangle + \|11\rangle)/\sqrt{2}$	$(\|00\rangle - \|10\rangle)/\sqrt{2}$	$(\|01\rangle + \|11\rangle)/\sqrt{2}$
$CNOT$	$\|00\rangle$	$\|01\rangle$	$\|11\rangle$	$\|10\rangle$

$CNOT$, called controlled not, is an important two qubit operation. When $CNOT$ operates on a two qubit system, the first qubit in the pair is the control qubit, and the second is the target. The target is flipped if, and only if, the control qubit is $|1\rangle$.

The use of $CNOT$, in conjunction with H_1, can produce a two qubit system so important it has its own name.

$$\begin{aligned} CNOT\ H_1|00\rangle &= CNOT\frac{|00\rangle + |10\rangle}{\sqrt{2}} \\ &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\ &= |\beta_{00}\rangle \end{aligned}$$

In general,

$$CNOT\ H_1|ij\rangle = |\beta_{ij}\rangle$$

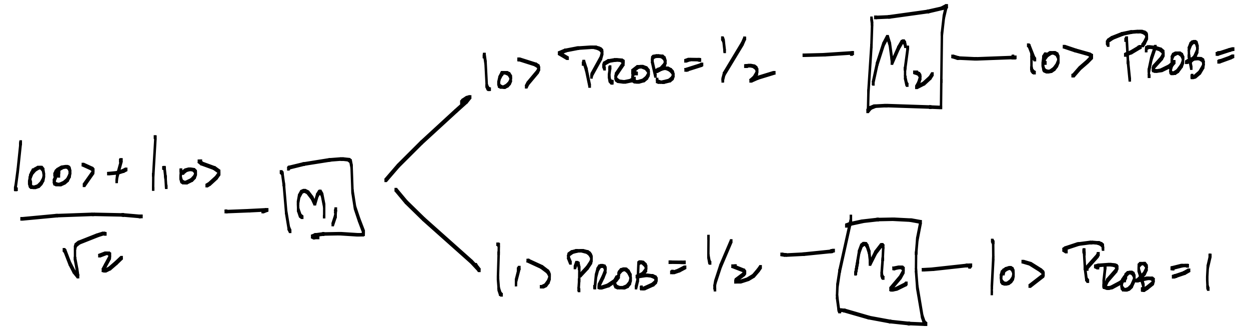

FIGURE 12.1
SEQUENTIAL MEASUREMENT

The resulting two qubit system $|\beta_{ij}\rangle$ is referred to as a Bell or EPR[2] state.

It is also possible to measure one qubit at a time. Consider the two qubit system

$$H_1|00\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

and measure the two qubits in sequence. When the first qubit in the pair is measured, the result is either $|0\rangle$ or $|1\rangle$, each with probability of one-half, as that is the coefficient squared. Then, the second qubit is measured as $|0\rangle$ with probability one. The result of the second measurement is independent of the first measurement, as seems natural. The measurement sequence is illustrated in figure 12.1.

Now conduct the sequential measurement exercise on $|\beta_{00}\rangle$. The results are similar, but there is a striking and surprising difference as illustrated in figure 12.1.

The surprising thing is that the second measurement is no longer independent of the first. In fact, if the result of the first measurement is $|0\rangle$, then that is the result of the second with certainty. Similarly, a result of $|1\rangle$ in the first guarantees a $|1\rangle$ from the second. Einstein, among others, was skeptical of what he called "spooky action at a distance." In particular, the "at a distance" part was disconcerting, since there is nothing in the theory which requires the two qubits to be anywhere near each other. Something seems to happen to the far away one when measurement is applied to the close by one. But this and other surprising phenomena associated with en-

[2] For Einstein, Podalsky, and Rosen, the authors of a paper describing the mysterious and (to them) unlikely properties of these mysterious qubits.

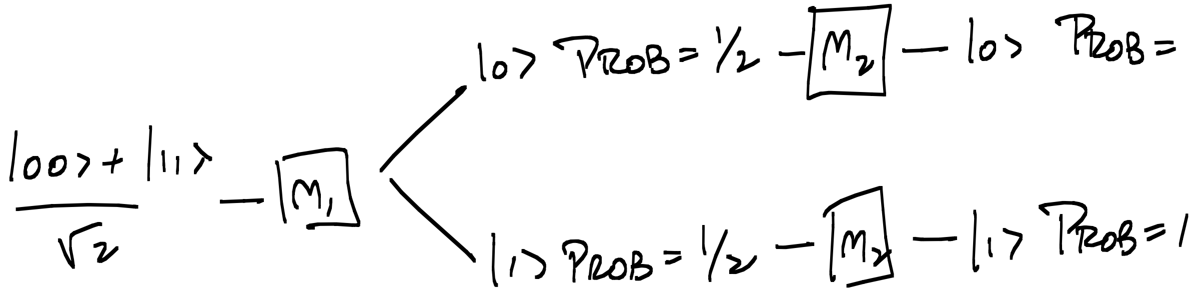

FIGURE 12.2
SEQUENTIAL MEASUREMENT
ENTANGLED QUBITS

tangled qubits have been consistently verified empirically. Nature appears to possess efficient resources to exploit interactions, and this one, termed entanglement, is a quite powerful one.

As mysterious as entanglement is in nature, perhaps it is not quite so surprising in a production context. It is, after all, the transfer of information from one unit to another. When production occurs, it is not beyond comprehension that when one person or department learns something about the production process, that knowledge is quickly available to other related units. In fact, the production process can be designed with fast and efficient transfer of information in mind. Entanglement is a mechanism to incorporate this phenomenon into the production process in a formal way.

12.5 synergy and multiple unit production

Synergy occurs when activities are combined in a particularly efficient way: when two activities are combined the output is strictly more than the sum of the outputs when the two activities are performed separately. To get synergy to appear, we turn, not surprisingly, to entangled qubits, wherein exist powerful interaction effects. Consider a production process with two labor inputs denoted, as before as angles, θ_1 and θ_2, as well as direct material inputs of two $|0\rangle$ qubits, that is $|00\rangle$ This time, however, there is a common input cost which entangles the two qubits prior to production.

$$|\beta_{00}\rangle = CNOT\ H_1|00\rangle$$

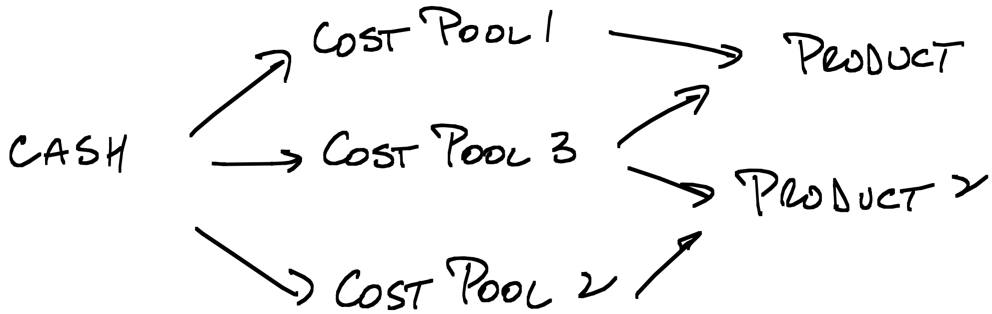

FIGURE 12.3
PRODUCTION WITH
COMMON INPUT FACTOR

The cost of entanglement is another cost pool in addition to the direct costs of labor and material. In the directed graph in figure 12.3 the common input factor is denoted cost pool 3.

A directed graph depicting parallel production of two outputs without the common input factor is in figure 12.4.

The entire production function has twice the direct costs as in the single product process, as well as the entanglement cost.

$$H_2 \Theta_2 H_2 H_1 \Theta_1 H_1 CNOT \; H_1 |00\rangle$$

Successful production occurs when the input, $|\beta_{00}\rangle$ is transformed to an orthogonal state, namely $|\beta_{01}\rangle$. Of course, the probability of success is a function of the input angles.

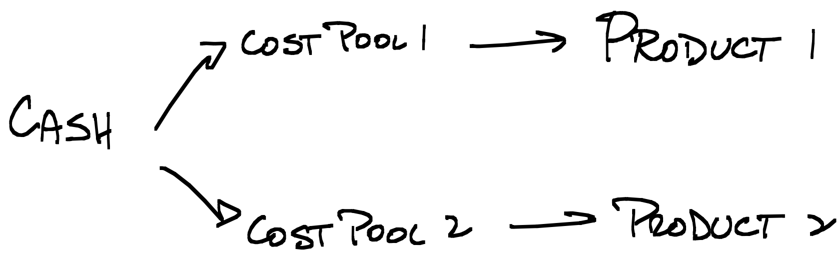

FIGURE 12.4
PRODUCTION WITHOUT
COMMON INPUT FACTOR

The computation of the success probability is not complicated, but a little bit tedious. The first part is derived below.

$$\begin{aligned}
H_1\Theta_1 H_1|\beta_{00}\rangle &= H_1\Theta_1 H_1 \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
&= H_1\Theta_1 \frac{|00\rangle + |10\rangle + |01\rangle - |11\rangle}{2} \\
&= H_1 \frac{e^{i\theta_1}\left(|00\rangle + |01\rangle\right) + |10\rangle - |11\rangle}{2} \\
&= \frac{e^{i\theta_1}\left(|00\rangle + |10\rangle + |01\rangle + |11\rangle\right) + \left(|00\rangle - |10\rangle - |01\rangle + |11\rangle\right)}{2\sqrt{2}} \\
&= \frac{\left(e^{i\theta_1}+1\right)}{2}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) + \frac{\left(e^{i\theta_1}-1\right)}{2}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) \\
&= \frac{\left(e^{i\theta_1}+1\right)}{2}|\beta_{00}\rangle + \frac{\left(e^{i\theta_1}-1\right)}{2}|\beta_{01}\rangle
\end{aligned}$$

Similarly we have

$$H_2\Theta_2 H_2|\beta_{00}\rangle = \frac{\left(e^{i\theta_2}+1\right)}{2}|\beta_{00}\rangle + \frac{\left(e^{i\theta_2}-1\right)}{2}|\beta_{01}\rangle$$

and

$$H_2\Theta_2 H_2|\beta_{01}\rangle = \frac{\left(e^{i\theta_2}+1\right)}{2}|\beta_{01}\rangle + \frac{\left(e^{i\theta_2}-1\right)}{2}|\beta_{00}\rangle$$

Using the three relationships we can evaluate the production process.

$$
\begin{aligned}
H_2\Theta_2H_2H_1\Theta_1H_1|\beta_{00}\rangle &= H_2\Theta_2H_2\left[\frac{\left(e^{i\theta_1}+1\right)}{2}|\beta_{00}\rangle+\frac{\left(e^{i\theta_1}-1\right)}{2}|\beta_{01}\rangle\right] \\
&= \frac{\left(e^{i\theta_1}+1\right)}{2}\frac{\left(e^{i\theta_2}+1\right)}{2}|\beta_{00}\rangle+\frac{\left(e^{i\theta_1}+1\right)}{2}\frac{\left(e^{i\theta_2}-1\right)}{2}|\beta_{01}\rangle \\
&\quad +\frac{\left(e^{i\theta_1}-1\right)}{2}\frac{\left(e^{i\theta_2}+1\right)}{2}|\beta_{01}\rangle+\frac{\left(e^{i\theta_1}-1\right)}{2}\frac{\left(e^{i\theta_2}-1\right)}{2}|\beta_{00}\rangle \\
&= |\beta_{00}\rangle\left(\frac{e^{i(\theta_1+\theta_2)}+e^{i\theta_1}+e^{i\theta_2}+1+e^{i(\theta_1+\theta_2)}-e^{i\theta_1}-e^{i\theta_2}+1}{4}\right) \\
&\quad +|\beta_{01}\rangle\left(\frac{e^{i(\theta_1+\theta_2)}-e^{i\theta_1}+e^{i\theta_2}-1+e^{i(\theta_1+\theta_2)}+e^{i\theta_1}-e^{i\theta_2}-1}{4}\right) \\
&= \left(\frac{e^{i(\theta_1+\theta_2)}+1}{2}\right)|\beta_{00}\rangle+\left(\frac{e^{i(\theta_1+\theta_2)}-1}{2}\right)|\beta_{01}\rangle
\end{aligned}
$$

Notice the similarity in form to the single input production function. In both cases the input qubit remains the same or is transformed to an orthogonal qubit. The coefficients for the single and multiple qubit production processes are of the same form: for two agent production the coefficients are written with the sum of the two agent angles. The probabilities, then, are of the same form as the single input production. The only difference is the sum of the angles appears in the probabilities.

measurement	*probability*
$\|\beta_{01}\rangle$	$\sin^2\{(\theta_1+\theta_2)/2\}$
$\|\beta_{00}\rangle$	$\cos^2\{(\theta_1+\theta_2)/2\}$

The orthogonal flip to $|\beta_{01}\rangle$ is considered a success in the multi-input (group) production function. The question is whether synergy exists, and that is easy to check. Whenever the individual angles sum to less than 180 degrees, the group success rate exceeds the sum of the individual probabilities. Some example angles are tabulated below.

θ_1	probability success θ_1	θ_2	probability success θ_2	sum individual probabilities	group success probability
30°	.067	30°	.067	.134	.250
30°	.067	45°	.146	.213	.371
30°	.067	60°	.250	.317	.500
30°	.067	90°	.500	.567	.750
45°	.146	30°	.067	.213	.371
45°	.146	45°	.146	.293	.500
45°	.146	60°	.250	.396	.629
45°	.146	90°	.500	.646	.854
60°	.250	30°	.067	.317	.500
60°	.250	45°	.146	.396	.629
60°	.250	60°	.250	.500	.750
60°	.250	90°	.500	.750	.933
90°	.500	30°	.067	.567	.750
90°	.500	45°	.146	.646	.854
90°	.500	60°	.250	.750	.933
90°	.500	90°	.500	1.00	1.00

Another way to see the synergy effect is to compute expected income in an extension of the production example.

Example 12.2 *Extend the previous example to a joint production setting. Append a cost of entanglement (common factor cost) be K. Let the cost of labor be $.6\theta_1^2 + .6\theta_2^2$, and, for simplicity, let the cost of the material input $|00\rangle$ be zero. Furthermore, let the revenue be* 3 *for a successful production $|\beta_{01}\rangle$ (and revenue of zero if output is $|\beta_{00}\rangle$). For two input angles of* 60 *degrees ($\theta_1 = \theta_2 = \frac{\pi}{3}$) the expected profit is computed as follows.*

direct labor cost :

$$.6\theta_1^2 + .6\theta_2^2 = 1.2\left(\frac{\pi}{3}\right)^2 = \frac{2\pi^2}{15}$$

expected revenue :

$$3\sin^2\frac{(\theta_1+\theta_2)}{2} = 3\sin^2\frac{(\pi)}{3} = 3\left(\frac{3}{4}\right)$$

expected profit :

$$3\left(\frac{3}{4}\right) - \frac{2\pi^2}{15} - K$$

Recall expected profit in the single product setting was

$$\frac{3}{4}-\frac{\pi^2}{15} \qquad \textit{or for two products}$$
$$2\left(\frac{3}{4}\right)-\frac{2\pi^2}{15}$$

So joint production is superior to two applications of single production as long as

$$K<\frac{3}{4}.$$

12.6 measurement implications

Now we come to the problem of assessing the responsibility (or blame) for the results of the production process. That is, we are particularly interested in the labor input angles performed by the production workers. If they are directly observable, there is no real problem. However, it is often the case that the labor inputs are not easily observable, and inferences about them must be made by measuring the observable outputs of production. As we have seen, when output is measured using the entangled Bell states,the probabilities a particular state is the measurement result is a function of the sum of the input angles:$\theta_1+\theta_2$. We can not, in other words, determine the individual input angles.

It is tempting to consider measuring individual qubits. After all, the first input angle θ_1 operates on the first qubit only, and similarly for θ_2. So if we observe the individual qubits, we might be able to infer something about the individual input angles. But, when synergy is important, this is a very dangerous path. In the first place, measuring individual qubits tells us *nothing* about the individual input angles. But it gets worse than that. Individual measures corrode the production process, and the synergy benefits are lost.

To examine the effects of individual measures, restate the production output in terms of individual qubits, as opposed to entangled Bell qubits.

$$\begin{aligned}
H_2\Theta_2H_2H_1\Theta_1H_1|\beta_{00}\rangle &= \frac{e^{i(\theta_1+\theta_2)}+1}{2}|\beta_{00}\rangle+\frac{e^{i(\theta_1+\theta_2)}-1}{2}|\beta_{01}\rangle \\
&= \frac{e^{i(\theta_1+\theta_2)}+1}{2}\frac{|00\rangle+|11\rangle}{\sqrt{2}}+\frac{e^{i(\theta_1+\theta_2)}-1}{2}\frac{|00\rangle-|11\rangle}{\sqrt{2}} \\
&= |00\rangle\frac{e^{i(\theta_1+\theta_2)}}{\sqrt{2}}+\frac{|11\rangle}{\sqrt{2}}
\end{aligned}$$

Now measure the first qubit by "squaring" the coefficients. The probability the first qubit is $|0\rangle$ is $\frac{1}{2}$. Likewise for the second qubit. It does not matter what θ_1 or θ_2 are; the measurement results are entirely independent of the input angles chosen.

But it gets worse. When the measurements are completed, we have qubits in the form $|00\rangle$ or $|11\rangle$. Entanglement has disappeared. To restart the production process, the cost of entanglement must be incurred. If the measures yielded Bell states in the first place, the post measurement qubits are already entangled: there is no need entangle once again.

The conclusion is clear: measuring individual contributions in a synergistic environment, at least the synergy under consideration here, is a silly exercise. To some extent that squares with experience. An example which is fun to think about is Abbott and Costello's classic comedy routine entitled "Who's on First." Bud Abbott and Lou Costello were both funny individuals, but combined they were far funnier than the sum of two funny guys. Trying to determine their individual contributions to the act is futile. On average Costello got bigger laughs, but the set-up from Abbott was indispensable. But assigning results, in this case laughs, to individuals is almost certainly worse than futile: it can be corrosive if it causes the individuals to compete for greater individual measures. Attempting to eke out an extra laugh from a set-up can seriously degrade the punch line.

There are many examples of inappropriate use of individual measures in a synergy environment. For example, in the early days of Enron, real assets (like natural gas pipelines, paper mills, etc.) were combined with trading desks. Synergy seemed to be the result, possibly from information transfers across activities. As time went by, however, Enron invoked ruthless individual measures, people become reluctant to share information with colleagues, and troubles ensued.[3]

Closer to home, contemplate a scholarly environment. It is often easier and more efficient to learn in a group. That's synergy. However, the imposition of individual measures like grades or pay raises, combined with the perception that relative performance is important, might very well inhibit an individuals inclination to contribute to the group. If we want people to act as a team, then team measurements are sensible; individual measures are not sensible, and endanger teamwork. The larger message is synergy is both powerful and delicate, and can be destroyed by ill-advised individual measures.

12.7 summary

Efficient (synergistic) combination of resources is an important and difficult problem. Efficiency in combination often relies on efficient use of

[3] Two good renderings of the Enron chronicle, to my taste, are *The Smartest Guys in the Room* by McLean and Elkind, and *Conspiracy of Fools* by Eichenwald.

information. Nature is particularly clever, and mysterious, in the use of information when resources are combined. In particular, entanglement can create synergies.

We apply the thinking an formalism of combinations in nature, with a view toward illuminating both physics and accounting. Accounting double entry (directed graph representation) is useful for organizing combinations, and loops appear naturally in the presence of synergy. Entanglement, whether in nature or in an economic environment, is powerful though delicate, and can easily be destroyed with excessive individual measurements.

12.8 exercises

Exercise 12.1 *a. Use the production function*

$$H\Theta H|0\rangle$$

The input costs are

	cost
raw material $\lvert 0\rangle$	0
labor θ	$.8\theta^2$

Revenue from an output of $|1\rangle$ *is 5. Suppose the labor input* θ *is 90 degrees* $(\frac{\pi}{2})$*. What is the expected income?*

b. Suppose θ *is 60 degrees* $(\frac{\pi}{3})$*. What is the expected income?*

Exercise 12.2 *Use the multiproduct production function*

$$H_2\Theta_2H_2H_1\Theta_1H_1CNOT\ H_1|00\rangle$$

The input costs are the same as in exercise 12.1. Output $|\beta_{01}\rangle$ *can be sold for 5, and output* $|\beta_{00}\rangle$ *can not be sold. Suppose* θ_1 *and* θ_2 *are both 90 degrees* $(\frac{\pi}{2})$*. Not counting the cost of entanglement, what is the expected income? Suppose the cost of entanglement is .7. Comparing with single production in exercise 12.1, is single or multiple production preferred?*

Exercise 12.3 *a. Redo the previous exercise with*

$$\theta_1 = \frac{\pi}{3} \text{ and } \theta_2 = \frac{\pi}{2}$$

What is the expected income? If the cost of entanglement is still .7, is single or multiple production preferred?

b. Finally, use

$$\theta_1 = \theta_2 = \frac{\pi}{3}$$

What is the expected income?

Exercise 12.4 *a. Refer to exercise 12.1. What is the optimal angle θ to maximize expected profit?*

b. Refer to exercises 12.2 and 12.3. What are the optimal angles, θ_1 and θ_2 to maximize expected income?

Exercise 12.5 *Verify the math involved in the construction of EPR entangled qubits.*

$$CNOT\ H_1|ij\rangle = |\beta_{ij}\rangle$$

Exercise 12.6 *Show*

$$H_1\Theta_1 H_1|\beta_{11}\rangle = \frac{1-e^{i\theta_1}}{2}|\beta_{10}\rangle + \frac{1+e^{i\theta_1}}{2}|\beta_{11}\rangle$$

Exercise 12.7 *Show*

$$H_2\Theta_2 H_2|\beta_{10}\rangle = \frac{1+e^{i\theta_2}}{2}|\beta_{10}\rangle + \frac{-1+e^{i\theta_2}}{2}|\beta_{11}\rangle$$

Exercise 12.8 *Show*

$$\begin{aligned} & H_2\Theta_2 H_2 H_1\Theta_1 H_1|\beta_{11}\rangle \\ = \ & \frac{-e^{i\theta_1}+e^{i\theta_2}}{2}|\beta_{10}\rangle + \frac{e^{i\theta_1}+e^{i\theta_2}}{2}|\beta_{11}\rangle \end{aligned}$$

12.9 appendix

In the appendix we connect some of the processes used in this chapter with the axiomatic development in chapter 11. For quantum encryption in chapter 11 we worked one qubit at a time. But to consider production, especially synergistic production, we require multiple qubit capability. There is a fourth axiom describing how multiple qubits are formed, operated upon, and measured.

Axiom 4 *Tensor multiplication, denoted $\otimes$, combines vectors and matrices. Two element vectors (single qubits) and 2×2 matrices (for operation and measurement of two qubit systems) are tensor combined as follows.*

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \otimes \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11}\beta_{11} & \alpha_{11}\beta_{12} & \alpha_{12}\beta_{11} & \alpha_{12}\beta_{12} \\ \alpha_{11}\beta_{21} & \alpha_{11}\beta_{22} & \alpha_{12}\beta_{21} & \alpha_{12}\beta_{22} \\ \alpha_{21}\beta_{11} & \alpha_{21}\beta_{12} & \alpha_{22}\beta_{11} & \alpha_{22}\beta_{12} \\ \alpha_{21}\beta_{21} & \alpha_{21}\beta_{22} & \alpha_{22}\beta_{21} & \alpha_{22}\beta_{22} \end{bmatrix}$$

Axiom 2 ***Example 12.3*** *Combine* $|0\rangle$ *and* $|1\rangle$.

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Dirac notation can be used for multiple qubit processes; indeed, it often supplies a most convenient shortcut. The Dirac representation of two qubit systems -

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

Using Dirac notation, tensor combinations can be written so simply the tensor part is hardly noticed.

$$|0\rangle \otimes |1\rangle = |01\rangle$$

It is important to realize the Dirac shortcut can be deceptively simple; verifying the operation the long tensor way is not a bad idea, at least sometimes.

Multiple qubit operations are, like single qubit operations, accomplished with matrix multiplication, and Dirac representation offers a similar shortcut.

Example 12.4 *Operate on the first qubit of* $|00\rangle$ *with an* X *transformation. Recall,* X *is a bit flip:*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In Dirac notation

$$X_1|00\rangle = |10\rangle$$

The subscript on X indicates the first qubit is flipped, while the second remains unchanged.

The previous example can be verified using the long tensor way. As the second qubit is unchanged, the identity matrix is used thereon.

$$\begin{aligned} X_1 &= X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned} X_1|00\rangle &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \end{aligned}$$

Multiple qubit measurement exhibits the same feature. That is, measurement is usually fairly easy to do using Dirac, and worth verifying the long tensor way.

Example 12.5 *Measure the first qubit of $H_1|00\rangle$ using Z. The eigenvectors of Z are $|0\rangle$ and $|1\rangle$. The projections of the first qubit onto the eigenvectors have coefficients equal to $1/\sqrt{2}$. Hence, the measurements are $|00\rangle$ with probability $1/2$, and $|10\rangle$ with probability $1/2$. The second qubit is unchanged by the measurement.*

To verify the previous example using tensor combinations

$$\begin{aligned} Z_1 &= Z \otimes I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

The spectral decomposition of Z_1 is

$$Z_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which can be verified by the usual eigenvector and eigenvalue calculations.

The four Bell states form an orthonormal basis in four dimensional space. One way to see that is to notice the Bell states are orthonormal eigenvectors of a particular matrix.

$$\begin{aligned} XZ \otimes XZ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \end{aligned}$$

Therefore, we can measure any 2 qubit system using the Bell basis, and we will do so for synergistic production. Also, we can, and will, measure Bell states one qubit at a time. Individual qubit measurement demonstrates the "entangled" property of entangled qubits.