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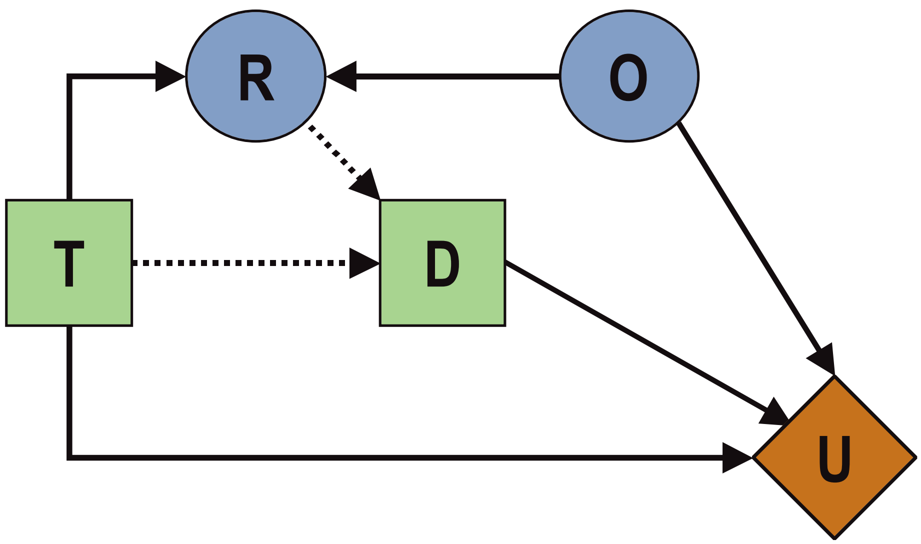
A Qualitative Linear Utility Theory for Spohn's Theory of Epistemic Beliefs



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Outline

- **Introduction**
- **Spohn's Theory of Epistemic Beliefs**
- **Utility Theory**
  - **Axioms**
  - **Main Result**
- **Oil Wildcatter's Problem**
- **Summary and Future Work**

Introduction

- Spohn's theory of epistemic beliefs (a.k.a. kappa calculus) is viewed as a qualitative counterpart of probability theory.
- Kappa values have probabilistic semantics—order of magnitude probabilities.
- For decision making with kappa functions, there is a need for a qualitative utility theory for decision making.
- Goal of this study is to propose a qualitative utility theory analogous to von Neumann-Morgenstern's utility theory
- Main theorem: There exists a qualitative linear utility function such that the utility of a Spohnian lottery is equal to the "expected value" of the Spohnian lottery. Furthermore, such a utility function is unique.

Introduction

- Related Work:
  - Pearl, J. (1993), “From conditional oughts to qualitative decision theory,” UAI-93, 12–20.
  - Tan S.-W & J. Pearl (1994), “Qualitative decision theory,” AAAI-94, 928–933.
  - Dubois, D. & H. Prade (1995), “Possibility theory as a basis for qualitative decision theory,” IJCAI-95, 1924–30.
  - Dubois, D., H. Prade, & R. Sabbadin (1998), “Possibility theory as a basis for qualitative decision theory,” UAI-98, 121–128.
  - Brafman, R. I. & M. Tennenholtz (1996), “On the foundation of qualitative decision theory,” AAAI-96.
  - Brafman, R. I. & M. Tennenholtz (1997), “On the axiomatization of qualitative decision criterion,” AAAI-97.

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Spohn's Theory of Epistemic Beliefs

- Representation: Disbelief function (kappa function)

  $\kappa: 2^{\Omega} \setminus \varnothing \to \mathbb{Z}^+$ such that

  - there exists $\omega \in \Omega$ s.t. $\kappa(\{\omega\}) = 0$
  - for $A \subseteq \Omega$, $\kappa(A) = \text{Min}\{\kappa(\{\omega\}) \mid \omega \in A\}$
- Semantics:
  - If $\kappa(A) > 0$, then we disbelieve proposition A to degree $\kappa(A)$.
  - If $\kappa(A) = 0$, and $\kappa(A^c) > 0$, then we believe proposition A to degree $\kappa(A^c)$.
  - If $\kappa(A) = 0$, and $\kappa(A^c) = 0$, then we neither believe nor disbelieve proposition A.
  - $\kappa(A) = \lfloor -\log_b P(A) \rfloor$

Spohn's Theory of Epistemic Beliefs

- Example:
  - Amount of oil: $\Omega = \{dr, we, so\}$,
    $\kappa(dr) = 0$, $\kappa(we) = 1$, $\kappa(so) = 2$
  - We believe *dr* to degree 1
    disbelieve *we* to degree 1
    disbelieve *so* to degree 2,
    believe *dr* or *we* to degree 2, etc.
  - Using b = 2,
    $0.5 < P(dr) \leq 1$
    $0.25 < P(we) \leq 0.5$
    $0.125 < P(so) \leq 0.25$

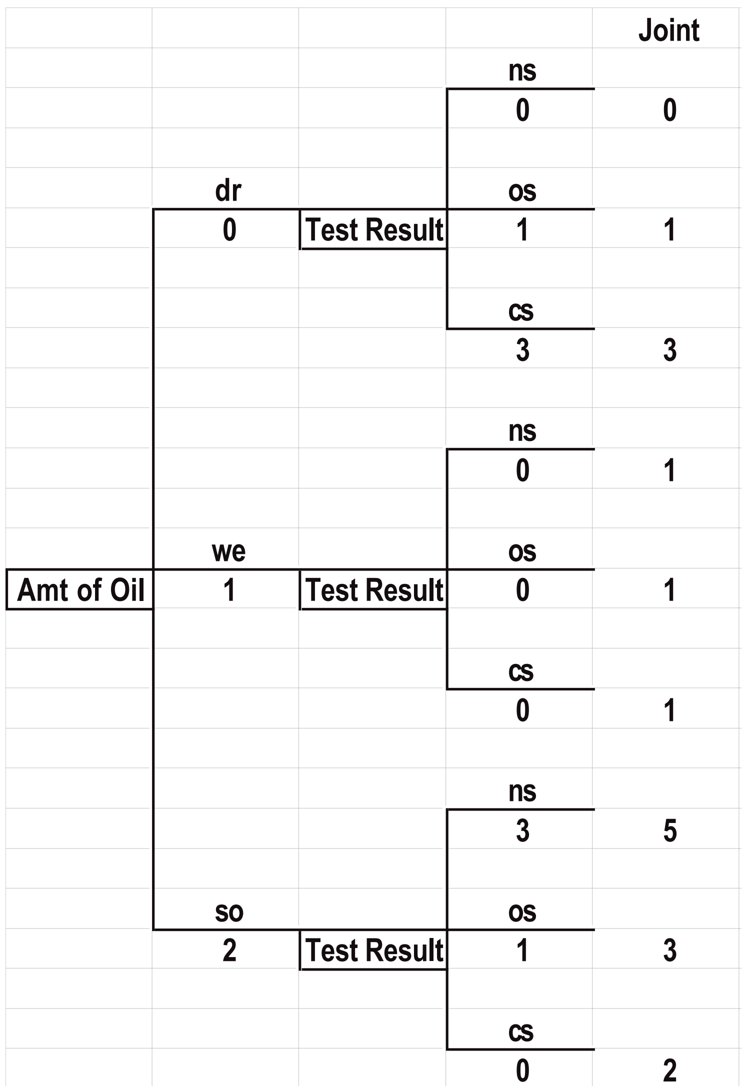
Spohn's Theory of Epistemic Beliefs

- Inference:
  - Combination is pointwise addition (followed by normalization)
  - Marginalization is minimization
  - Removal is pointwise subtraction
- Example:

  - Amount of oil: Ω_O = {dr, we, so}, κ(dr) = 0, κ(we) = 1, κ(so) = 2
  - Test Result: Ω_R = {ns, os, cs}
    κ(ns|dr) = 0, κ(os|dr) = 1, κ(cs|dr) = 3
    κ(ns|we) = 0, κ(os|we) = 0, κ(cs|we) = 0
    κ(ns|so) = 3, κ(os|so) = 1, κ(cs|so) = 0

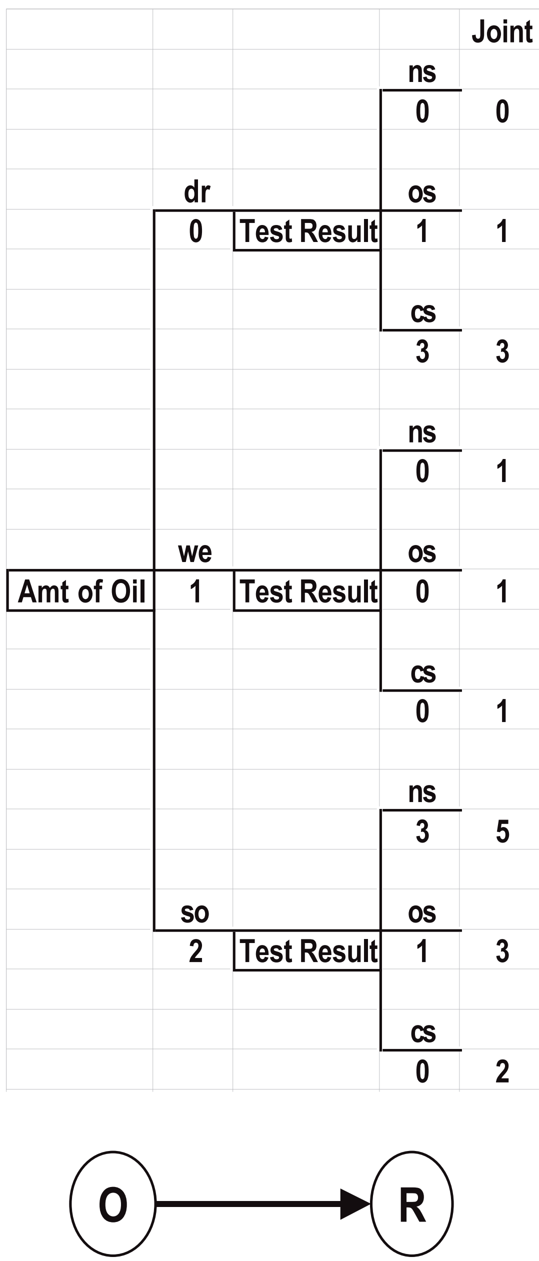
Spohn's Theory of Epistemic Beliefs

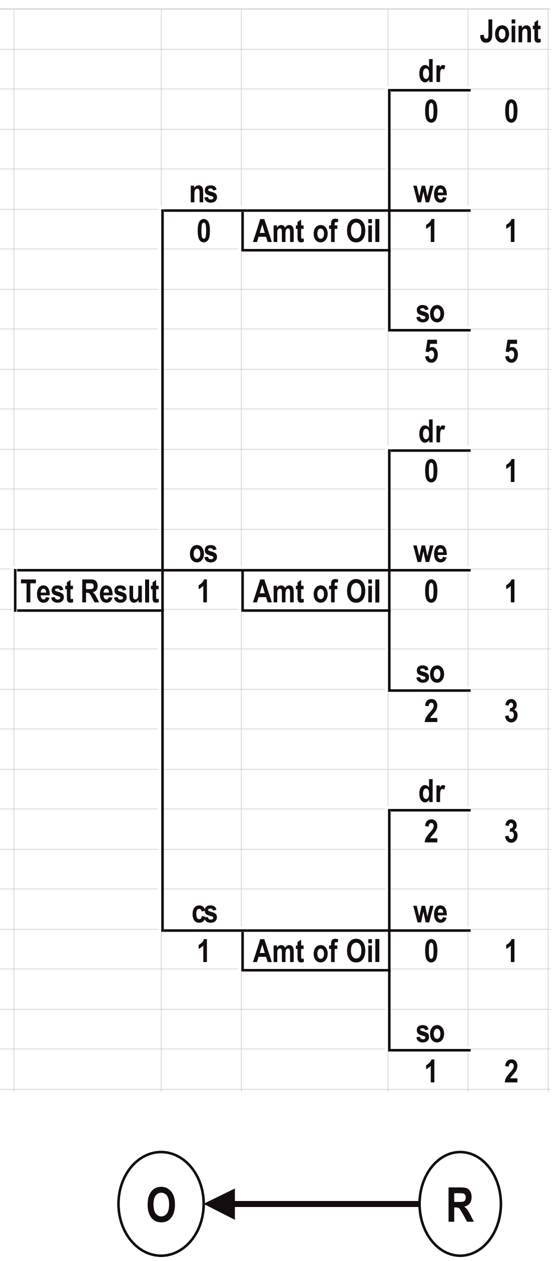
- Example (continued)



Spohn's Theory of Epistemic Beliefs

- Example (continued)





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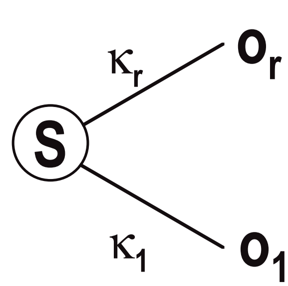
Utility Theory

- Set of prizes $O = \{o_1, \ldots, o_r\}$
- Preference relation $\succcurlyeq$ on O where $o_i \succcurlyeq o_j$ means o_i is qualitatively preferred or indifferent to o_j
- We write $o_i \succ o_j$ (read as o_i is qualitatively strictly preferred to o_j) iff $o_i \succcurlyeq o_j$ and $o_j \not\succcurlyeq o_i$
- We write $o_i \sim o_j$ (read as o_i is qualitatively indifferent to o_j) iff $o_i \succcurlyeq o_j$ and $o_j \succcurlyeq o_i$
- We assume the prizes are ordered so that $o_1 \succcurlyeq o_2 \succcurlyeq \ldots \succcurlyeq o_r$ and $o_1 \succ o_r$
- Let $\mathscr{L}$ denote the set of all Spohnian lotteries on O.
- A standard lottery S has only two outcomes, o_1 and o_r.



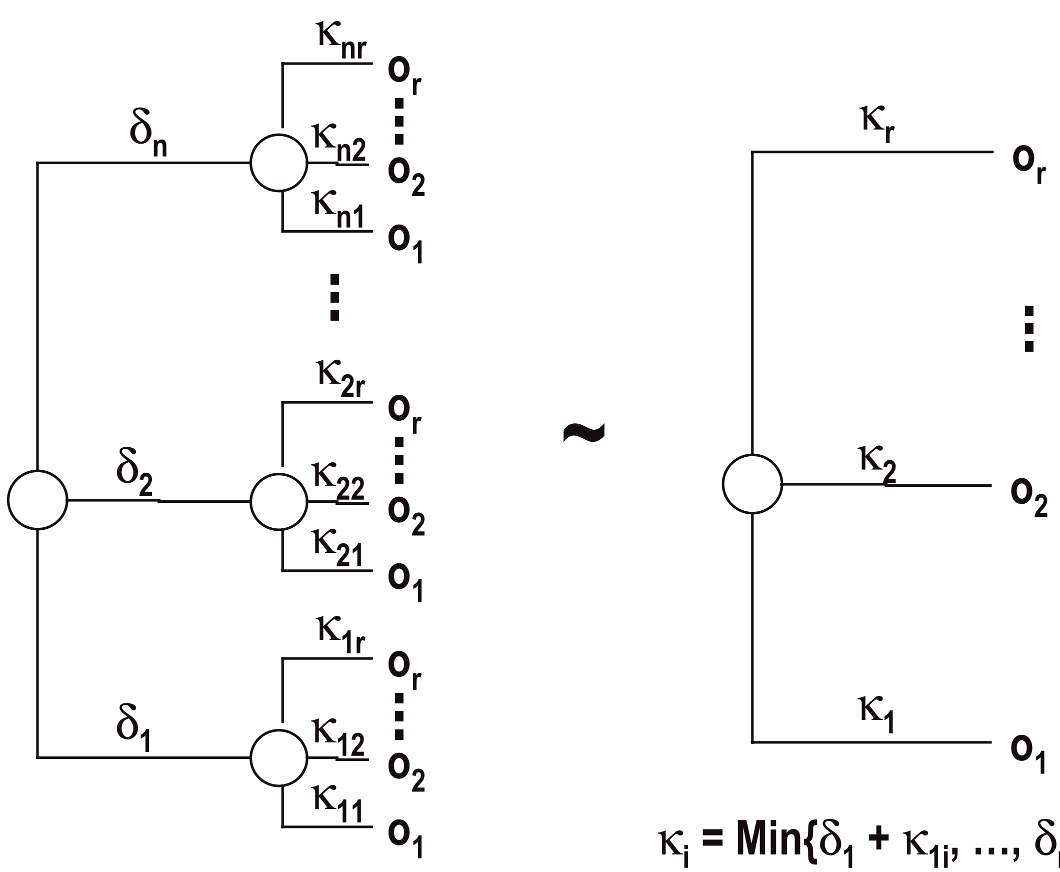
Utility Theory

- **Axiom 1 (Ordering of prizes)**

  The preference relation $\succcurlyeq$ on the set of prizes O is complete and transitive.

- **Axiom 2 (Reduction of compound lotteries)**

  A compound lottery is indifferent to a simple lottery who disbelief degrees are calculated as per Spohn's calculus.

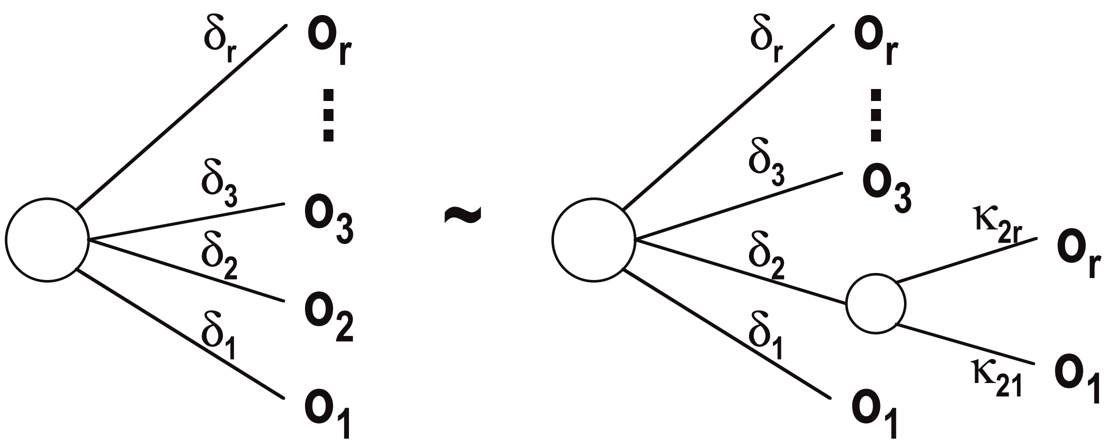


$\kappa_i = \text{Min}\{\delta_1 + \kappa_{1i}, \ldots, \delta_n + \kappa_{ni}\}$

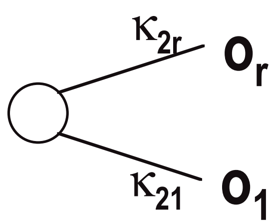
Utility Theory

- Axiom 3 (Substitutability)

  Indifferent lotteries are substitutable.



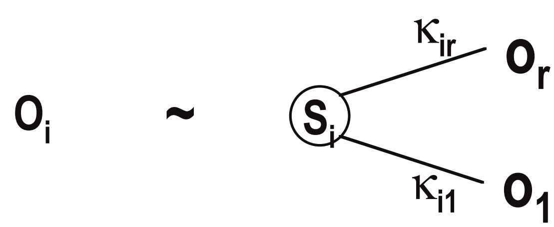
if o_2 ~



Utility Theory

- **Axiom 4 (Continuity)**

  For each prize o_i, there is a standard lottery S_i that is indifferent to it.



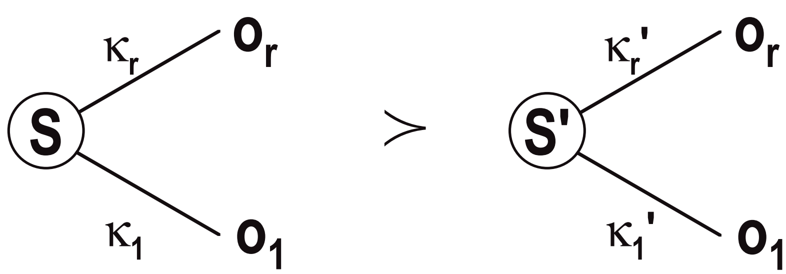
- **Axiom 5 (Transitivity)**

  Preference relation $\succcurlyeq$ over the set of Spohnian lotteries $\mathscr{L}$ is complete and transitive.

Utility Theory

- Axiom 6 (Qualitative monotonicity)

  Preference relation $\succcurlyeq$ over the set of standard lotteries satisfies the following condition:



  iff either

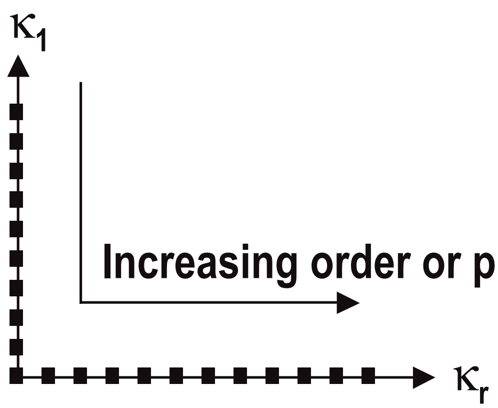
  $\kappa_1 = \kappa_1' = 0$, $\kappa_r > \kappa_r'$, or

  $\kappa_1 = 0$, $\kappa_1' > 0$, or

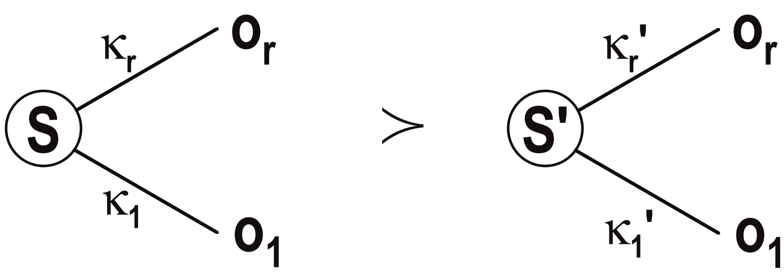
  $\kappa_1 < \kappa_1'$, $\kappa_r = \kappa_r' = 0$

Utility Theory

- Define $B_0 = \{(\kappa_r, \kappa_1) \mid \kappa_r, \kappa_1$ P $\mathbb{Z}^+, \text{Min}\{\kappa_r, \kappa_1\} = 0\}$



- From Axiom 6, we have an ordering > in B_0



iff either

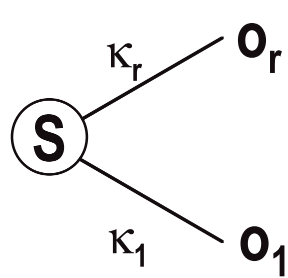
$\kappa_1 = \kappa_1' = 0, \kappa_r > \kappa_r'$, or

$\kappa_1 = 0, \kappa_1' > 0$, or

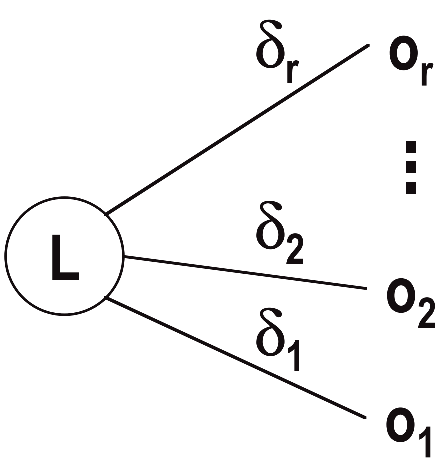
$\kappa_1 < \kappa_1', \kappa_r = \kappa_r' = 0$

Utility Theory

- Define qualitative utility function $U: \mathscr{L} \rightarrow B_0$ such that $U(L_1) \geq U(L_2)$ if and only if $L_1 \succcurlyeq L_2$.
- For standard lottery S, define $U(S) = (\kappa_r, \kappa_1)$



- Given Spohnian lottery L,



$U(o_r) = (\kappa_{rr}, \kappa_{r1})$

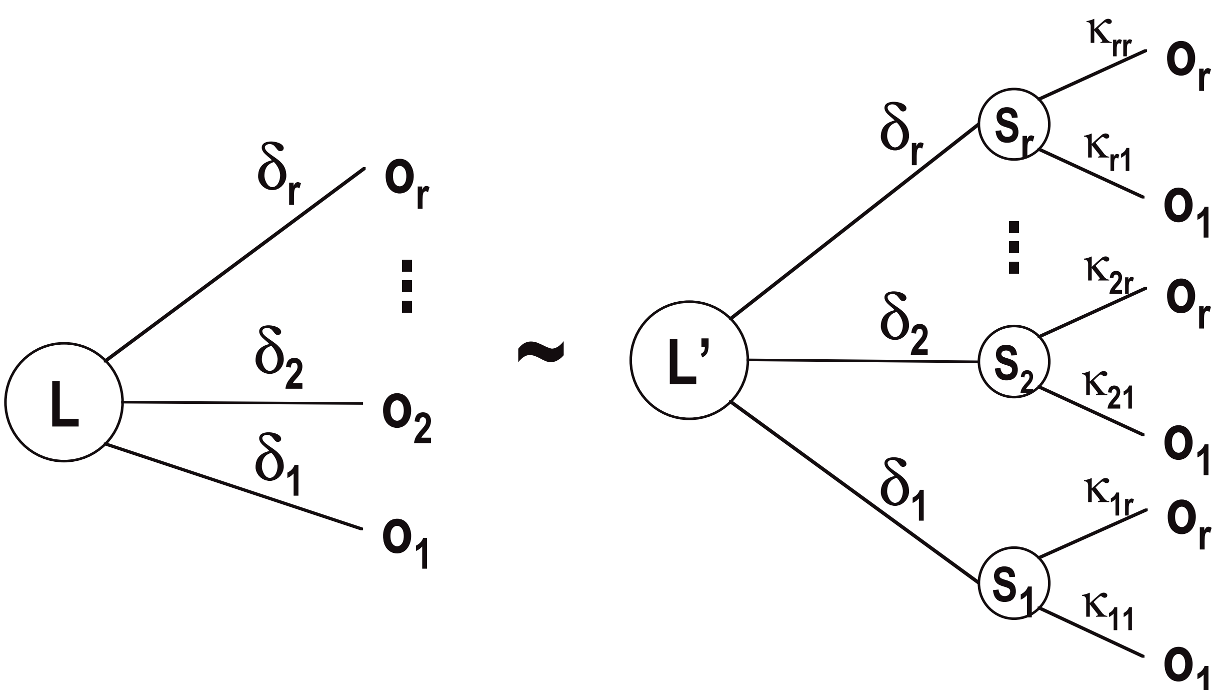
$U(o_2) = (\kappa_{2r}, \kappa_{21})$

$U(o_1) = (\kappa_{1r}, \kappa_{11})$

- define
  $EU(L) = \text{Min}\{\delta_1 + U(o_1), \delta_2 + U(o_2), \ldots, \delta_r + U(o_r)\}$
  $= \text{Min}\{\delta_1 + (\kappa_{1r}, \kappa_{11}), \delta_2 + (\kappa_{2r}, \kappa_{21}), \ldots, \delta_r + (\kappa_{rr}, \kappa_{r1})\}$
  $= (\text{Min}\{\delta_1 + \kappa_{1r}, \delta_2 + \kappa_{2r}, \ldots, \delta_r + \kappa_{rr}\},$
  $\text{Min}\{\delta_1 + \kappa_{11}, \delta_2 + \kappa_{21}, \ldots, \delta_r + \kappa_{r1}\})$

Utility Theory

- **MAIN THEOREM**
  Suppose $\succcurlyeq$ is a preference relation on the set of all Spohnian lotteries $\mathscr{L}$ that satisfies axioms 1 to 6. Then there exists a qualitative utility function U such that U(L) = EU(L). Furthermore, such a qualitative utility function is unique.
- **Proof**:

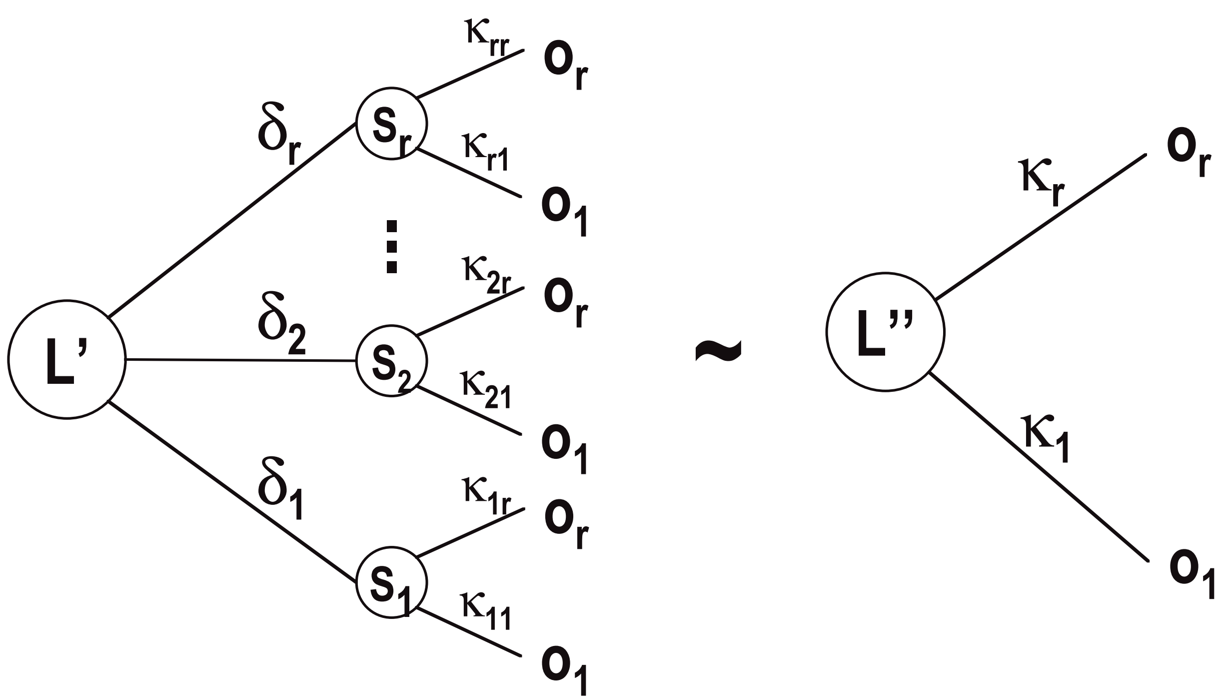


U(L) = U(L')

$$EU(L) = (\text{Min}\{\delta_1+\kappa_{1r}, \ldots, \delta_r+\kappa_{rr}\}, \text{Min}\{\delta_1+\kappa_{11}, \ldots, \delta_r+\kappa_{r1}\})$$

Utility Theory

- **Proof** (continued)



$\kappa_r = \text{Min}\{\delta_1+\kappa_{1r}, \ldots, \delta_r+\kappa_{rr}\}$

$\kappa_1 = \text{Min}\{\delta_1+\kappa_{11}, \ldots, \delta_r+\kappa_{r1}\}$

$$U(L) = U(L') = U(L'') = (\kappa_r, \kappa_1) = EU(L)$$

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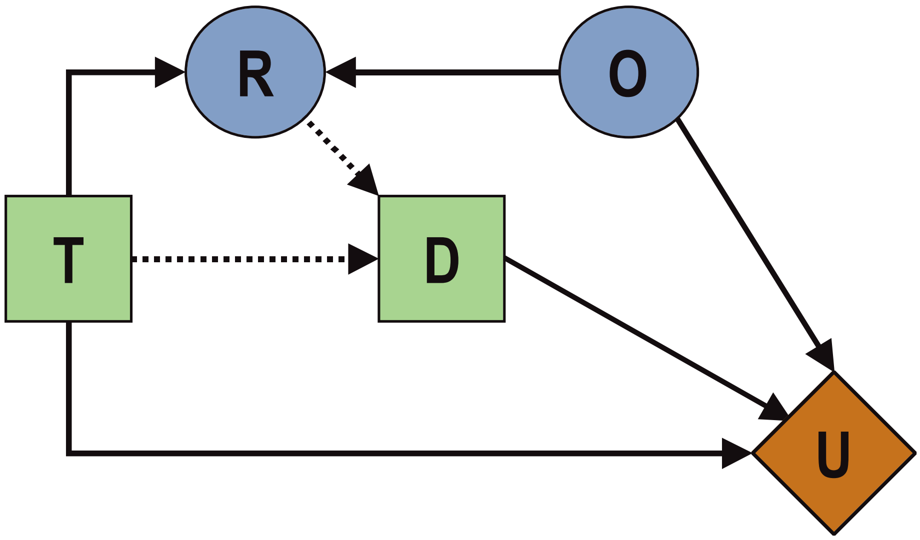
Oil Wildcatter's Problem

- Influence Diagram



- Uncertainty Model:
  - Amount of oil: Ω_O = {dr, we, so},
    κ(dr) = 0, κ(we) = 1, κ(so) = 2
  - Test Result: Ω_R = {ns, os, cs}
    κ(ns|dr) = 0, κ(os|dr) = 1, κ(cs|dr) = 3
    κ(ns|we) = 0, κ(os|we) = 0, κ(cs|we) = 0
    κ(ns|so) = 3, κ(os|so) = 1, κ(cs|so) = 0

Oil Wildcatter's Problem

- Approximate $ Payoffs

T	D	O	Approx. $
t	d	dr	-80,000
t	d	we	40,000
t	d	so	190,000
t	nd	-	-10,000
nt	d	dr	-70,000
nt	d	we	50,000
nt	d	so	200,000
nt	nd	-	0

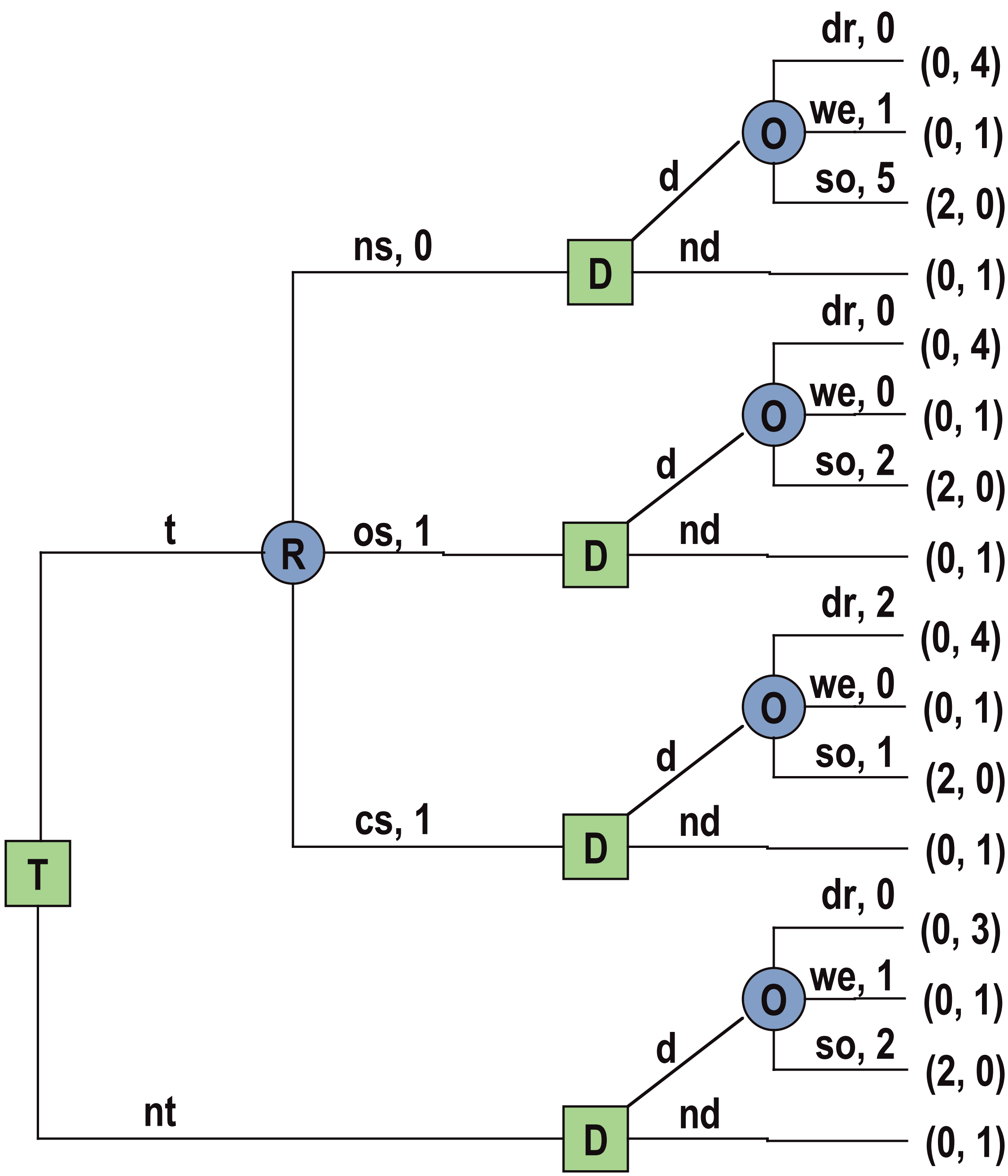
Oil Wildcatter's Problem

- Fix standard lottery with $o_r = -100{,}000$, $o_1 = +250{,}000$
- Assume wildcatter is "risk neutral."
- Utilities for the approx. $ amounts are as follows:

Approx. $	1-p	p	κ_r	κ_1
200,000	0.14	0.86	2	0
190,000	0.17	0.83	2	0
50,000	0.57	0.43	0	1
40,000	0.60	0.40	0	1
0	0.71	0.29	0	1
-10,000	0.74	0.26	0	1
-70,000	0.91	0.09	0	3
-80,000	0.94	0.06	0	4

Oil Wildcatter's Problem

♦ **Decision Tree Representation:**



Oil Wildcatter's Problem

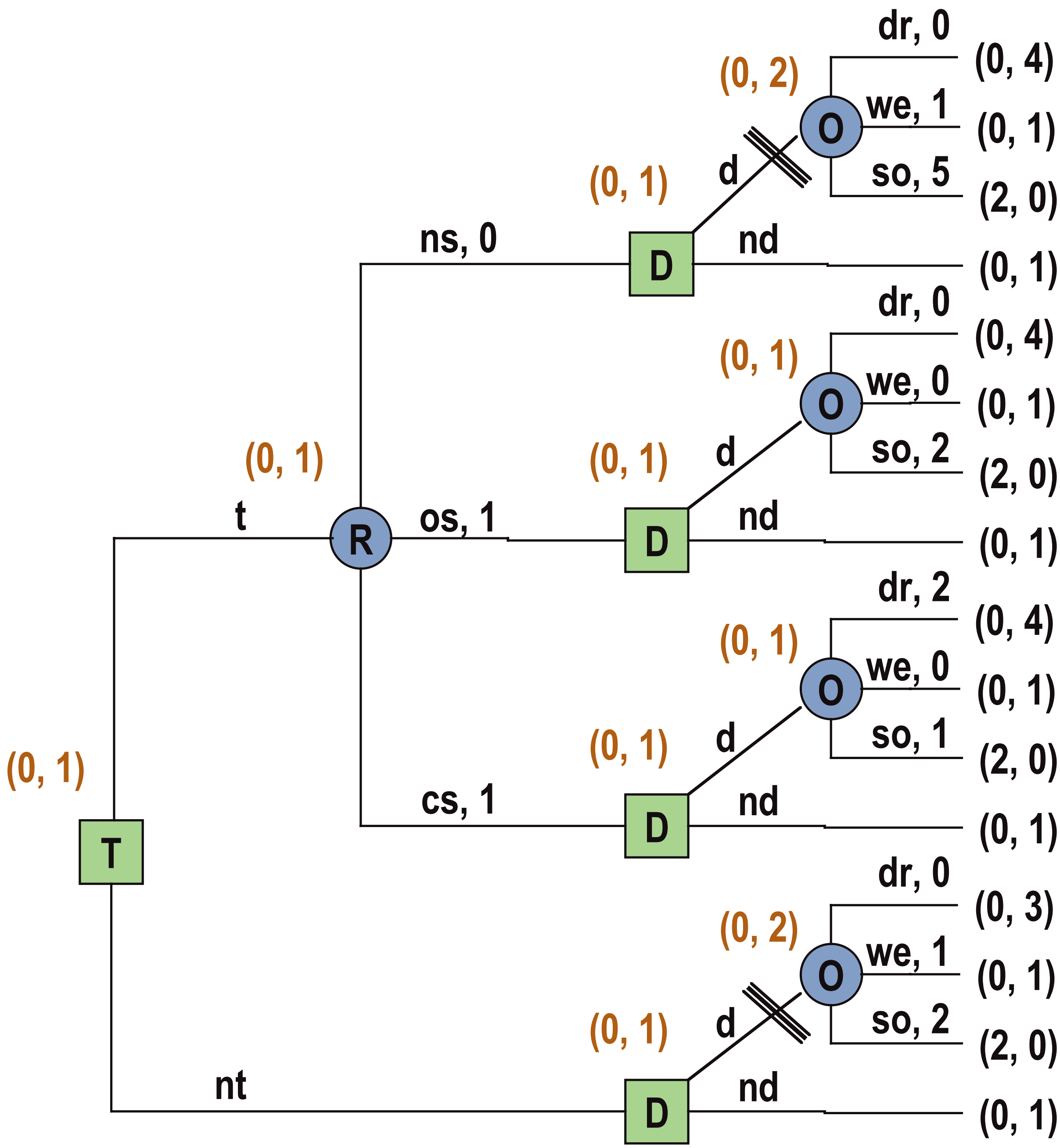
♦ Strategic Form

Profit, $000	(dr, ns)	(dr, os)	(dr, cs)	(we, ns)	(we, os)	(we, cs)	(so, ns)	(so, os)	(so, cs)	
(t, d, d, d, –)	–80	–80	–80	40	40	40	190	190	190	
(t, d, d, nd, –)	–80	–80	–10	40	40	–10	190	190	–10	
(t, d, nd, d, –)	–80	–10	–80	40	–10	40	190	–10	190	
(t, d, nd, nd, –)	–80	–10	–10	40	–10	–10	190	–10	–10	
(t, nd, d, d, –)	–10	–80	–80	–10	40	40	–10	190	190	
(t, nd, d, nd, –)	–10	–80	–10	–10	40	–10	–10	190	–10	
(t, nd, nd, d, –)	–10	–10	–80	–10	–10	40	–10	–10	190	
(t, nd, nd, nd, –)	–10	–10	–10	–10	–10	–10	–10	–10	–10	
(nt, –, –, –, d)	–70	–70	–70	50	50	50	200	200	200	
(nt, –, –, –, nd)	0	0	0	0	0	0	0	0	0	
Kappa	0	1	3	1	1	1	5	3	2	

Utility	(dr, ns)	(dr, os)	(dr, cs)	(we, ns)	(we, os)	(we, cs)	(so, ns)	(so, os)	(so, cs)	EU
(t, d, d, d, –)	0, 4	0, 4	0, 4	0, 1	0, 1	0, 1	2, 0	2, 0	2, 0	0, 2
(t, d, d, nd, –)	0, 4	0, 4	0, 1	0, 1	0, 1	0, 1	2, 0	2, 0	0, 1	0, 2
(t, d, nd, d, –)	0, 4	0, 1	0, 4	0, 1	0, 1	0, 1	2, 0	0, 1	2, 0	0, 2
(t, d, nd, nd, –)	0, 4	0, 2	0, 2	0, 1	0, 2	0, 2	2, 0	0, 2	0, 2	0, 2
(t, nd, d, d, –)	0, 1	0, 4	0, 4	0, 1	0, 1	0, 1	0, 1	2, 0	2, 0	**0, 1**
(t, nd, d, nd, –)	0, 1	0, 4	0, 1	0, 1	0, 1	0, 1	0, 1	2, 0	0, 1	**0, 1**
(t, nd, nd, d, –)	0, 1	0, 1	0, 4	0, 1	0, 1	0, 1	0, 1	0, 1	2, 0	**0, 1**
(t, nd, nd, nd, –)	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	**0, 1**
(nt, –, –, –, d)	0, 3	0, 3	0, 3	0, 1	0, 1	0, 1	2, 0	2, 0	2, 0	0, 2
(nt, –, –, –, nd)	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1	**0, 1**
kappa values	0	1	3	1	1	1	5	3	2	

Oil Wildcatter's Problem

- Decision Tree Solution (using rollback):



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Summary and Future Work

- Have defined a qualitative linear utility theory for Spohn's epistemic belief calculus.
- It's "linear" since U = EU
- It's qualitative since U: $\mathscr{L} \rightarrow B_0$
- The utility theory can be used for solving decision problems in which uncertainty is characterized by Spohn's calculus.
- Copy of paper available from my homepage at http://lark.cc.ukans.edu/~pshenoy:

  Giang, P. H. & P. P. Shenoy (2000), "A Linear Qualitative Utility Theory for Spohn's Theory of Epistemic Beliefs," in C. Boutilier and M. Goldszmidt (eds.), *Uncertainty in Artificial Intelligence: Proceedings of the Sixteenth Conference (UAI-2000)*, 220-229, Morgan Kaufmann, San Francisco, CA.

- Future Work
  - Risk attitudes and their characterizations
  - Decomposition of utility functions
  - Computational Issues