INFORMS San Antonio Meeting

November 5, 2000

A Measure of Relevance

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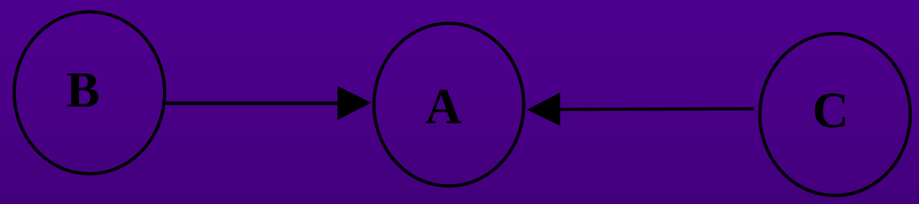
The Issues of this Talk

1. **Relevance is a very important element in decision analysis modeling.**
2. **But relevance is practically never quantified.**
3. **Similarity of probability distributions as a basis for measuring relevance.**
4. **Definition of a measure of relevance at the state level and relevance at the event level.**
5. **Applications of the measure of relevance.**

Modeling Relevance in Decision Analysis

- **Relevance, or probabilistic dependence, is a key modeling element in decision analysis.**
- **However, relevance is represented in a rough way. Only existence or inexistence of relevance is modeled. No mention is made of degrees of relevance.**



- **This happens in part because we do not have an intuitive and widely used measure of relevance.**

Proposed Measures of Probabilistic Dependence Include:

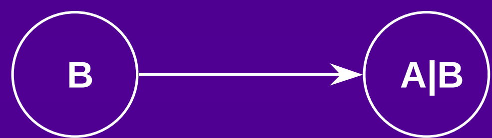
- **Mutual Information based on Shannon's measure of information.**
  $$MI(E,F) = \sum_i \sum_j \{E_i,F_j|\&\} \ln(\{E_i,F_j|\&\} / \{E_i|\&\}\{F_j|\&\}) .$$
- **Bayes factor for hypothesis testing given a set of independent sampled data, D.** $B_{1,0} = \{D|H_1\} / \{D|H_0\}$.
- **Correlation coefficient for some types of probability functions.**

However, none of those measures seems to be intuitive and general enough to be of common use in DA.

◊ **This work proposes a new measure of relevance which is simple to understand and compute, and can enhance DA models.**

Relevance ⇔ Difference between prior and conditioned probabilities

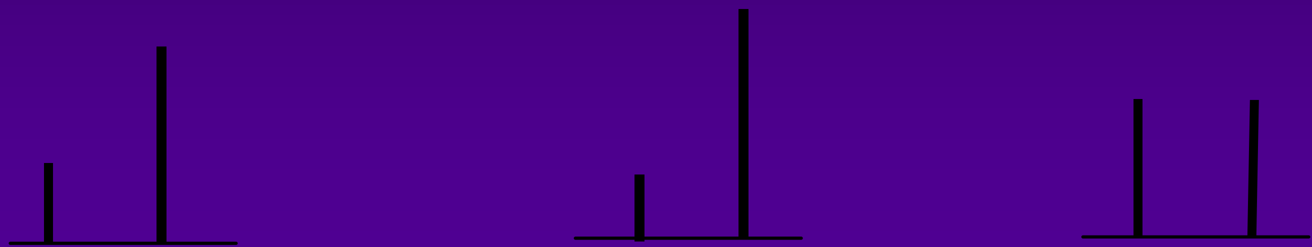
We can identify relevance by looking at the <u>difference</u> between the <u>prior probability</u> distribution and the <u>conditioned probability</u> distributions (conditioned on each outcome).



	prior probability	**conditioned probabilities**		
	{A|&)	**{A|B$_1$&)**	**{A|B$_2$&)**	**{A|B$_3$&)**
A_1	**.20**	**.90**	**.30**	**.10**
A_2	**.50**	**.10**	**.40**	**.20**
A_3	**.30**	**.00**	**.30**	**.70**

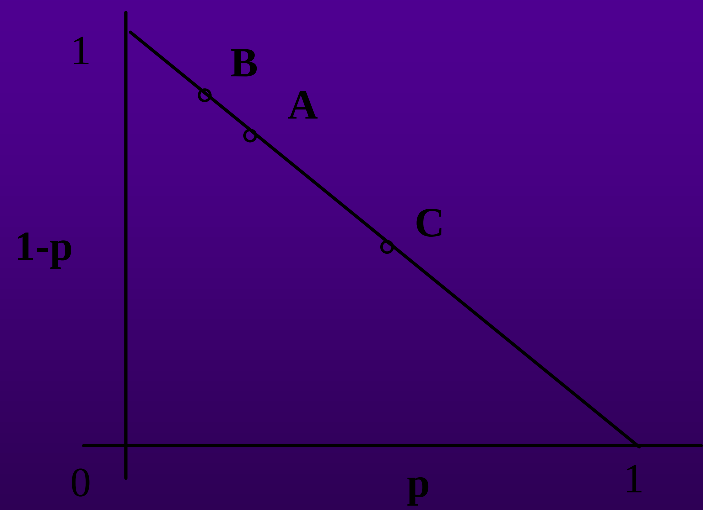
Similarity between probability distributions can be expressed as geometric distance.

Three different two-state probability distributions

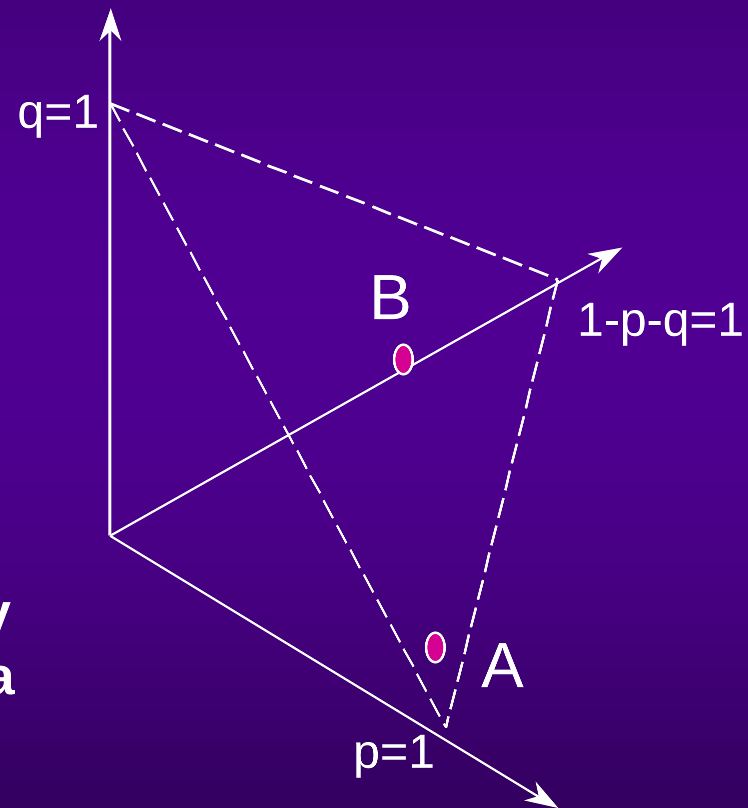


{A|&} {B|&} {C|&}

Representing an n-outcome probability distribution as a point in an n-dimensional space.



Representing Three-Outcome Probability Distributions



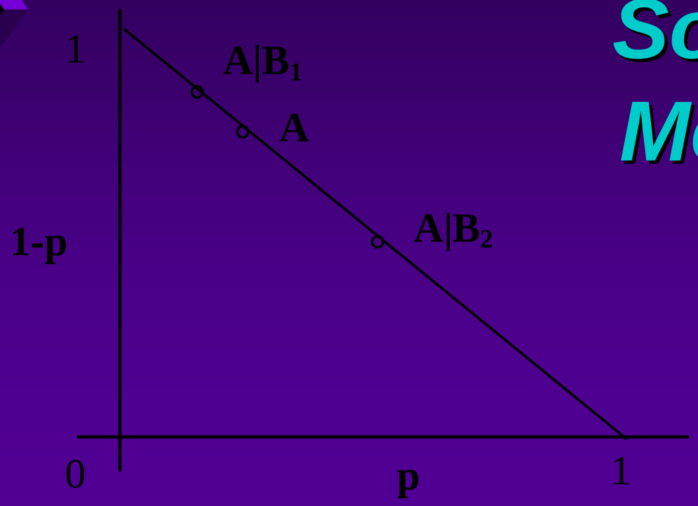
Points representing three-state probability distributions lie only on the marked triangular area (a two-dimensional hyperplane).

Defining relevance of one state B_i to an event A: $R(A,B_i)$

Let us define the *relevance of state B_i to event A* as the geometric distance between two points, one representing the prior probability distribution of *A*, {*A*|&}, with values $p_1, p_2, ..., p_n$, and the other representing the posterior probability distribution of *A* given the outcome B_i, {$A|B_i$&}, with values $q_1, q_2, ..., q_n$.

$$R(A,B_i) = [\,(p_1-q_1)^2 + (p_2-q_2)^2 + ... + (p_n-q_n)^2\,]^{1/2}$$

- $R(A,B_i)$ measures the difference in shape between the prior distribution and the posterior distribution given a particular state.

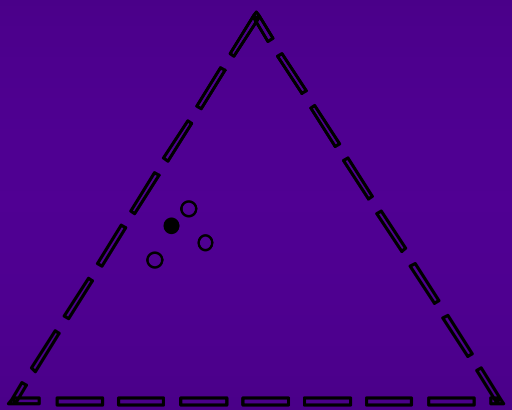


Some Properties of the Measure of Relevance, $R(A,B_i)$

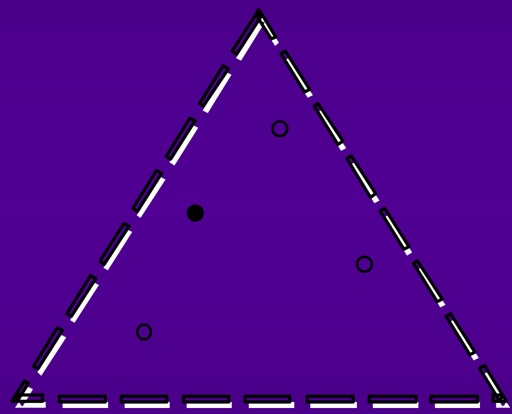
1. Relevance is defined for any pair of events with discrete probability distributions and is continuous in that space.
2. Irrelevant states have $R(A,B_i) = 0$ and relevant states have $R(A,B_i) > 0$.
3. Relevance has a finite range of values (between 0 and $\sqrt{2}$).
4. $R(A,B_i)$ can be very different from $R(B,A_i)$. This property reflects the asymmetry of conditional probabilities.

Looking at the relevance of the <u>states</u> of two events (B and C) to an event A

States of event B are less relevant to event A than states of event C.



Clear dots represent posterior probabilities of A conditioned on states $\mathbf{B_1}$, $\mathbf{B_2}$ and $\mathbf{B_3}$.



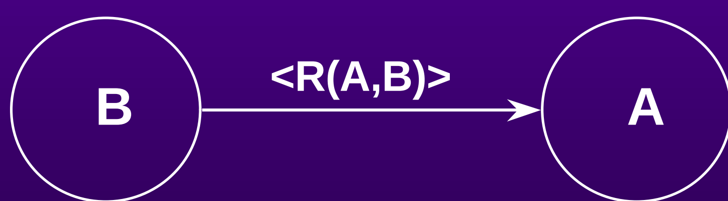
Clear dots represent posterior probabilities of A conditioned on states $\mathbf{C_1}$, $\mathbf{C_2}$ and $\mathbf{C_3}$.

A "front view" of the triangular area where three-state probability distributions lie .

Relevance of one event to another: Weighted Relevance <R(A,B)>

The expected value of relevance for the states of the conditioning variable is a measure of the relevance between events. It is called **weighted relevance** and is computed as:

$$<R(A,B)> = \sum_{i=1}^{n} \{B_i|\&\} R(A,B_i)$$



Some Properties of Weighted Relevance

1. If events A and B have the same number of states, <R(A,B)> **=** <R(B,A)>. This shows an underlying symmetry of relevance.

2. If event A has more states than event B, <R(B,A)> **>** <R(A,B)>. The event with more states is more informative.

3. Weighted relevance does not change as a result of interchanging the columns of the conditioned probabilities matrix.

Computing Relevance and Weighted Relevance

Events A and B have three states each.

$\{B|\&\} = [1/6, 1/3, 1/2]$

	$\{A\|\&\}$	$\{A\|B_1\&\}$	$\{A\|B_2\&\}$	$\{A\|B_3\&\}$
A_1	0.25	0.70	0.10	0.20
A_2	0.40	0.20	0.20	0.60
A_3	0.35	0.10	0.70	0.20

Relevance of each state of B:

$R(A,B_1) = 0.552$ $\quad$ $R(A,B_2) = 0.430$ $\quad$ $R(A,B_3) = 0.255$

Weighted Relevance of B to A is:

$\langle R(A,B)\rangle = 1/6\ (0.552) + 1/3\ (0.430) + 1/2\ (0.255) = \underline{0.363}$

In the other direction we have: **$\langle R(B,A)\rangle = \underline{0.363}$** (same value) with

$R(B,A_1) = 0.374$ $\quad$ $R(B,A_2) = 0.312$ $\quad$ $R(B,A_3) = 0.414$

Weighted relevance between events with different number of states

$E \rightarrow F$

$$\begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array} \begin{pmatrix} .35 \\ .41 \\ .24 \end{pmatrix} \qquad \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \end{array} \begin{array}{cc} E_1 & E_2 \\ \begin{pmatrix} .80 \\ .15 \\ .05 \end{pmatrix} & \begin{pmatrix} .20 \\ .50 \\ .30 \end{pmatrix} \end{array}$$

$R(F,E_1) = .554$

$R(F,E_2) = .185$

Weighted Relevance <R(F,E) > = .25(.554) + .75(.185) = <u>.277</u>

$F \rightarrow E$

$$\begin{array}{c} E_1 \\ E_2 \end{array} \begin{pmatrix} .25 \\ .75 \end{pmatrix} \qquad \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{ccc} F_1 & F_2 & F_3 \\ \begin{pmatrix} .57 \\ .43 \end{pmatrix} & \begin{pmatrix} .09 \\ .91 \end{pmatrix} & \begin{pmatrix} .05 \\ .95 \end{pmatrix} \end{array}$$

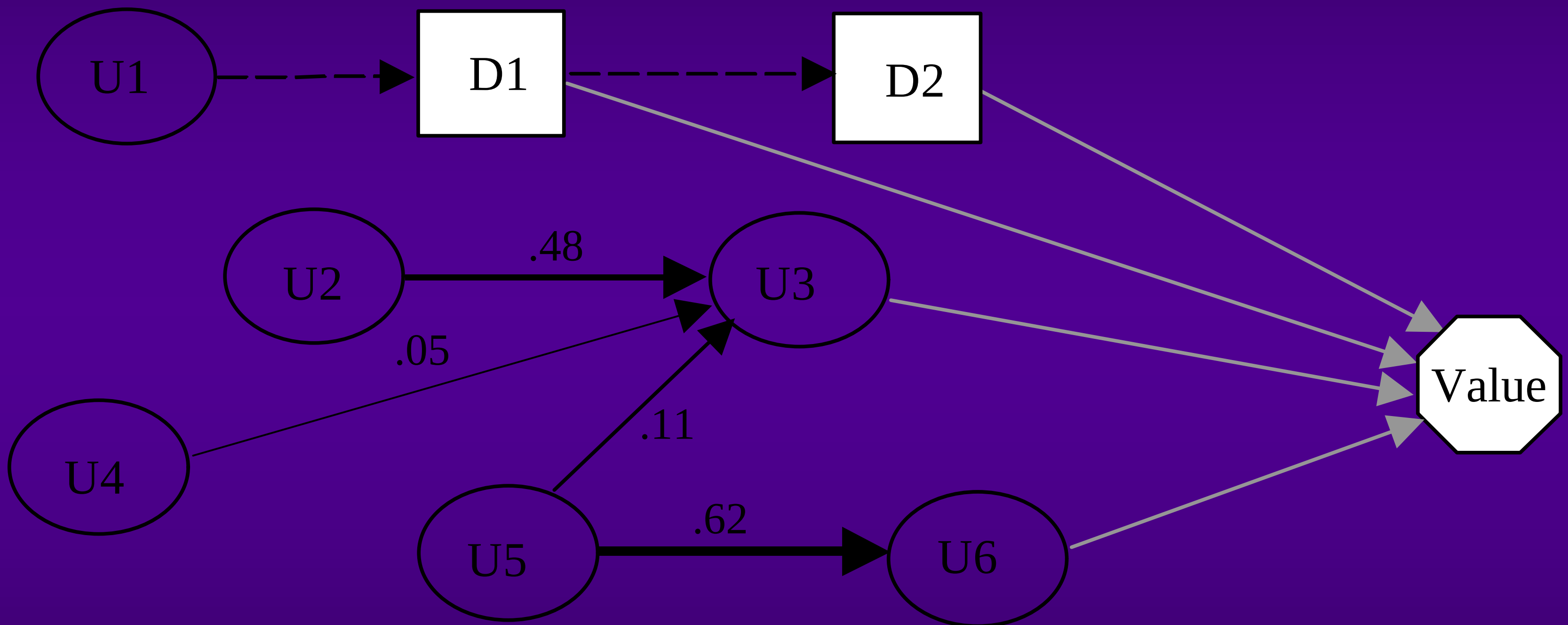
$R(E,F_1) = .455$

$R(E,F_2) = .225$

$R(E,F_3) = .279$

Weighted Relevance <R(E,F) > = .35(.455) + .41(.225) + .24(.279) = <u>.318</u>

Enriching Relevance Representation in Influence Diagrams



Weighted Relevance values on the arrows and thicker lines for more relevant links show at a glance the intensity of relevance.

These enhancements could be features of DA software.

Measuring Influence (Relevance of Alternatives)

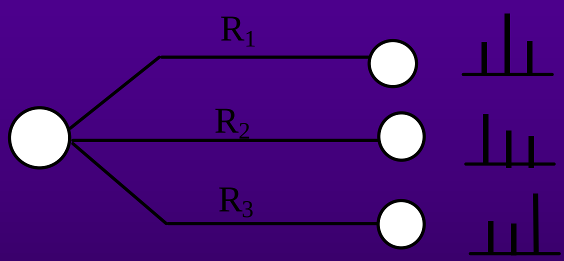
- **A decision influences a probabilistic node when the alternative selection changes the probability distribution of the node.**
- **Relevance can be computed for individual alternatives that influence probabilistic events.**
- **Weighted relevance is not defined for influence, but <u>maximum influence</u> can be used if a single indicator is desired.**

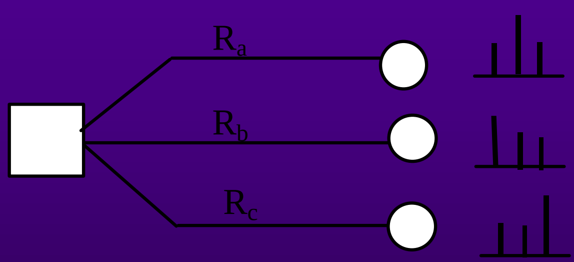
Indicating Relevance and Influence in Decision Trees

We can make decision trees more informative by quantifying relevance of states and alternatives.

<u>Relevance of states</u> can be written on the branches from probabilistic nodes to probabilistic nodes.

<u>Influence of alternatives</u> can be written on the branches from decision nodes to probabilistic nodes.

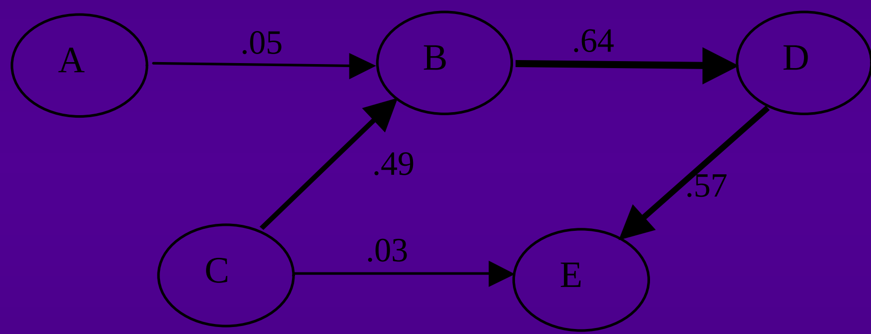




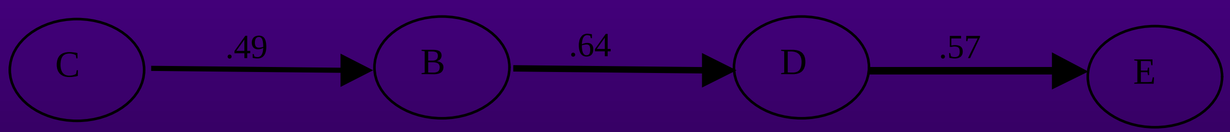
DA software could also draw thicker branches for expressing higher relevance.

Potential Use as a Tool for Pruning Influence Diagrams

Arrows with low weighted relevance can be good candidates for elimination or at least for exploring its elimination.



This diagram can be simplified as follows by dropping the low relevance links:



Summary

1. **Similarity between discrete probability distributions with *n* states can be represented by the distance between points in an *n*-dimensional space.**

2. **Relevance of a state can be measured by the similarity between the prior probability distribution and the corresponding conditioned probability distribution.**

3. **Weighted Relevance measures the overall importance of the probabilistic link between two events.**

4. **We can compute influence of alternatives on probabilistic events.**

5. **This measure can enhance the representation capability of influence diagrams and improve communication between DA analysts and their clients.**

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