Multiperson Utility

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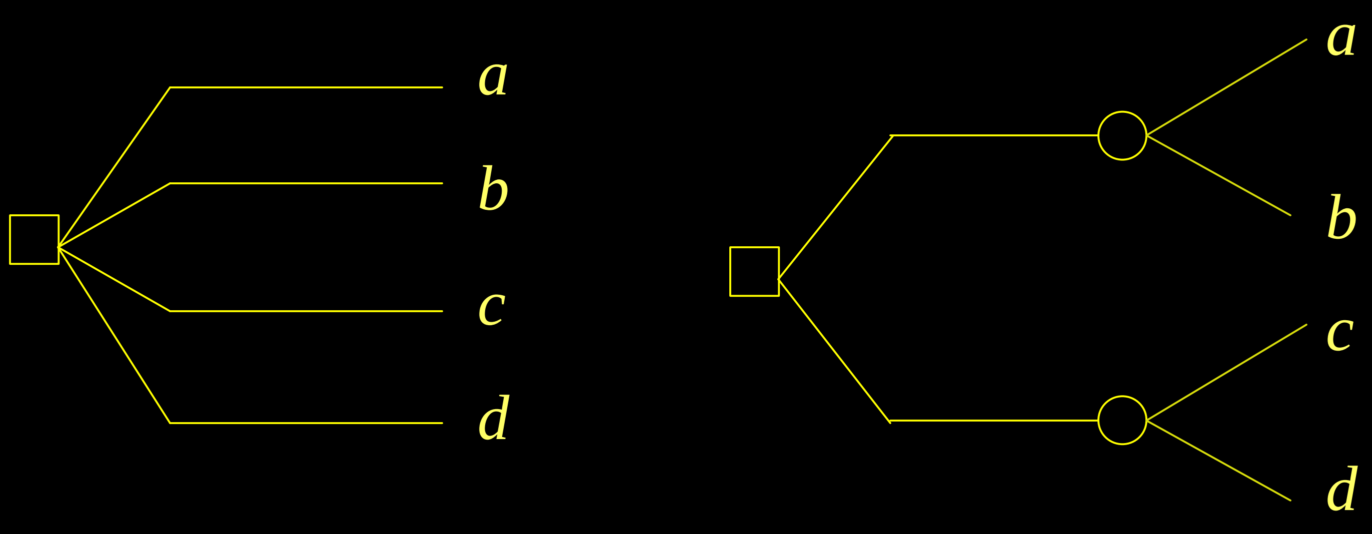
Four outcomes



Agent	*Preference*	*u(a)*	*u(b)*	*u(c)*	*u(d)*
1	*b • c • a • d*	30	100	40	0
2	*c • a • b • d*	80	50	100	0
3	*a • b • c • d*	100	75	68	0
Sum		201	225	**234**	0

Why this normalization? We could use...

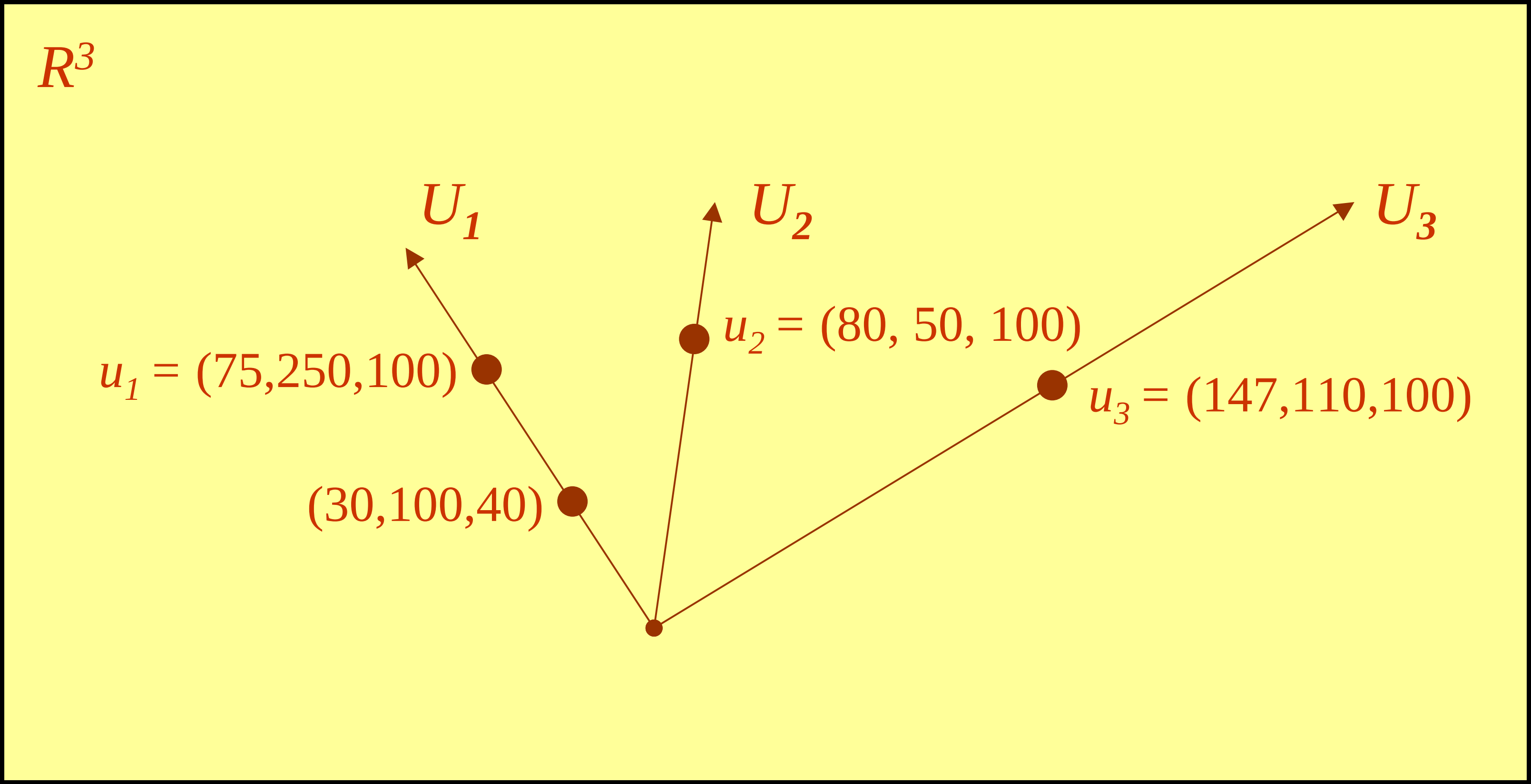
Agent	*Scaling*	*u(a)*	*u(b)*	*u(c)*	*u(d)*
1	× 1	30	100	40	0
2	× 1/2	40	25	50	0
3	× 10	1000	750	680	0
Sum		**1070**	875	770	0

Agent	*Preference*	*u(a)*	*u(b)*	*u(c)*	*u(d)*
1	*b • c • a • d*	30	100	40	0
2	*c • a • b • d*	80	50	100	0
3	*a • b • c • d*	100	75	68	0
Sum		201	225	**234**	0

Or use ...

Agent	*Scaling*	*u(a)*	*u(b)*	*u(c)*	*u(d)*
1	$\times 2.5$	75	250	100	0
2	$\times 1$	80	50	100	0
3	$\times 1.47$	147	110	100	0
Sum		302	**410**	300	0

Main result: for some choice of individual scales, the group preference will be determined by the sum of utilities.

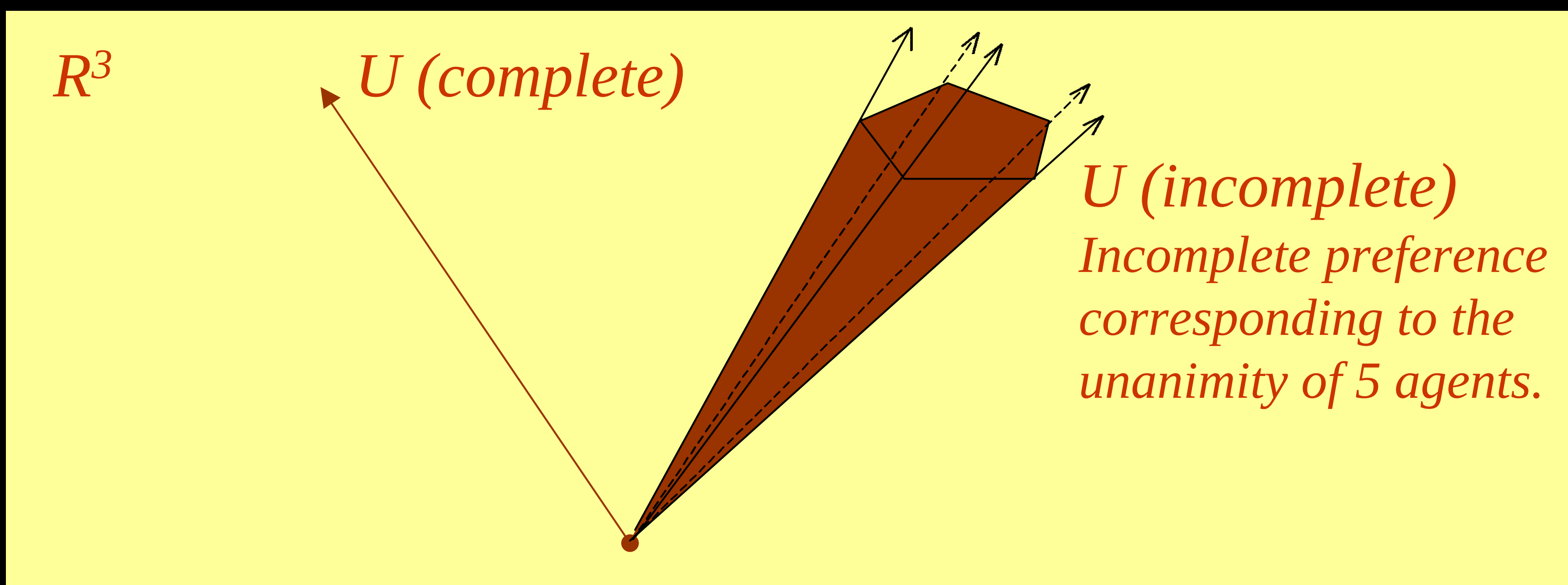


In general, we have have n rays in R^m, where n is the number of individuals and $m+1$ the number of alternatives.

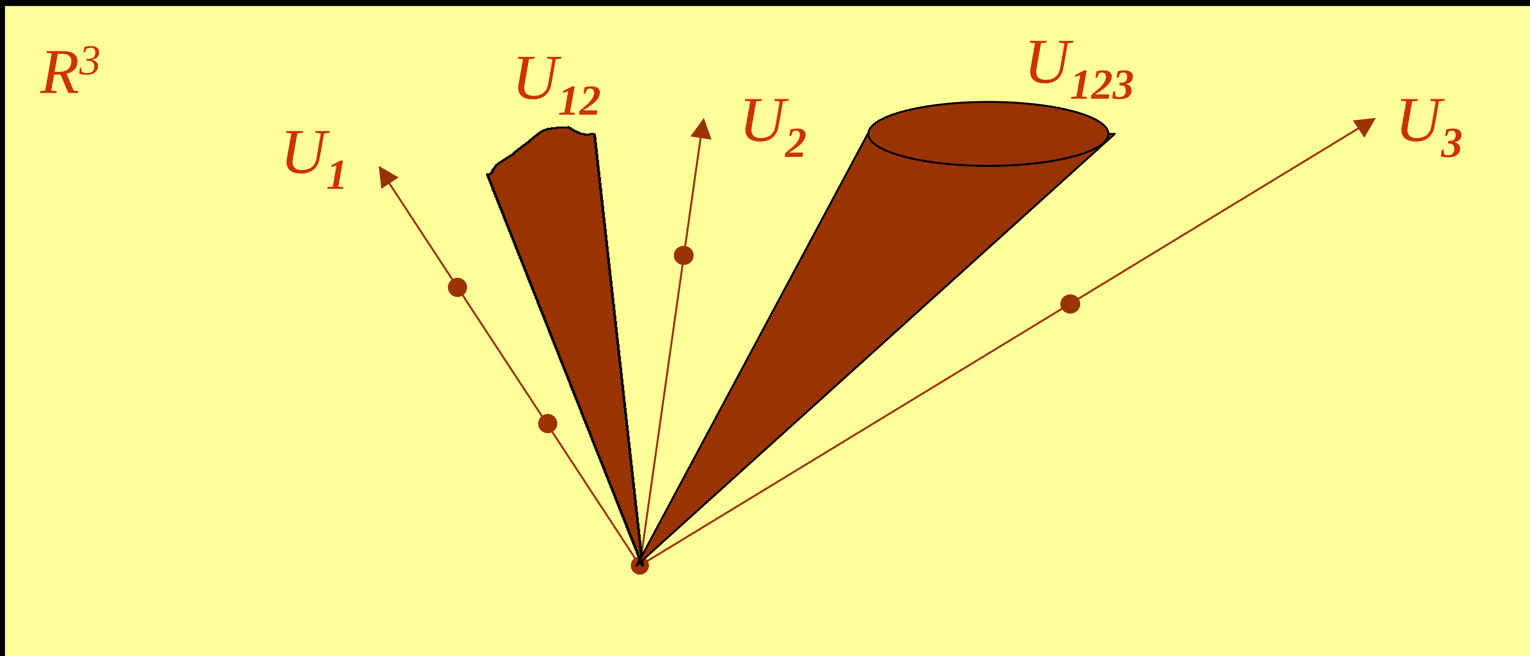
To model group preference we introduce *incomplete* cardinal preferences. The group may not be decisive, i.e., it may regard some pairs of alternatives as incomparable.

Any such incomplete preference is characterized by a convex cone *U of utility functions* such that

$$a \succsim b \quad \text{if and only if} \quad u(a) \geq u(b) \quad \text{for all } u \in U$$



We endow each coalition or subgroup S in N with an incomplete preference and its associated utility cone U_S.

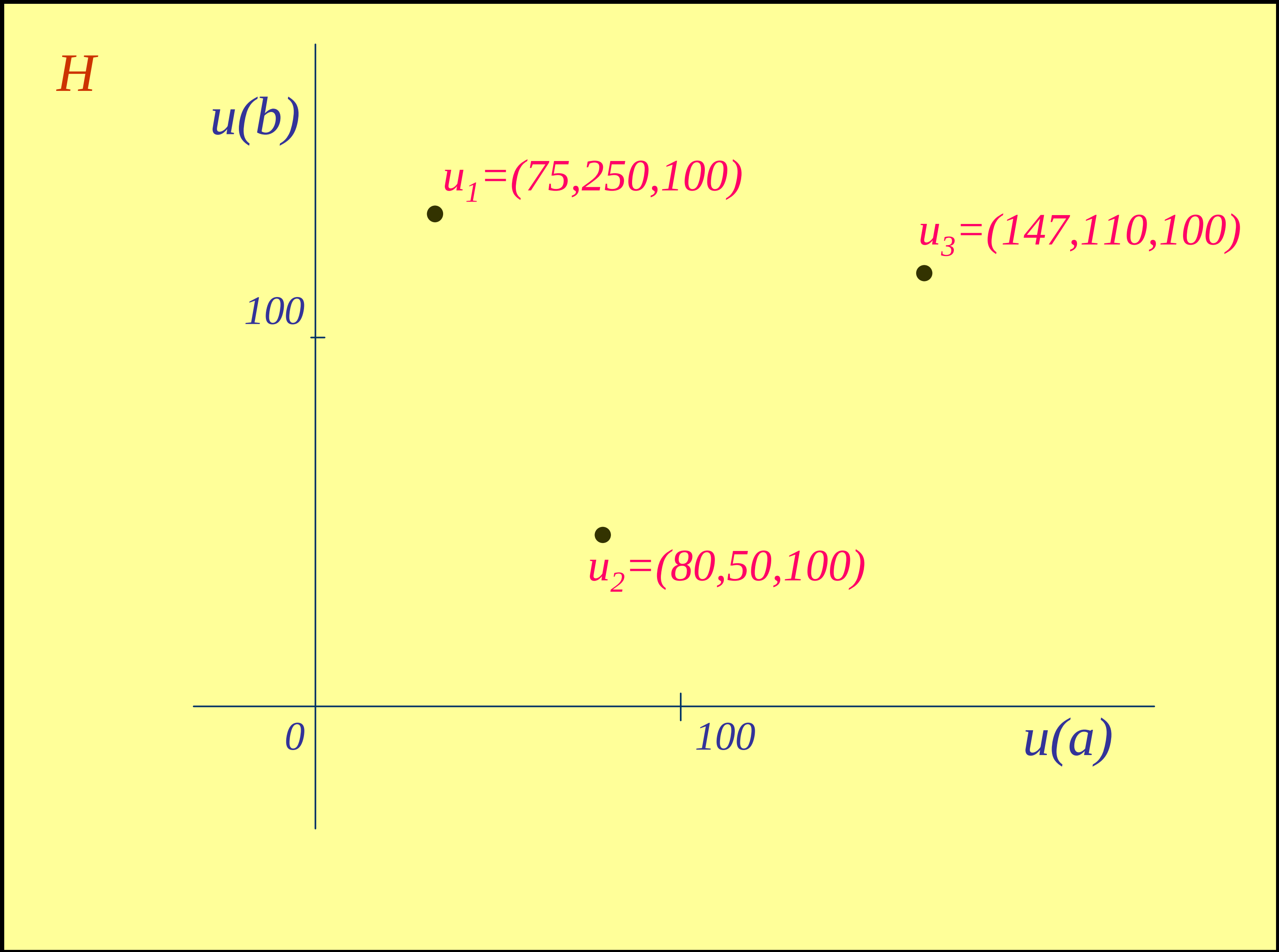


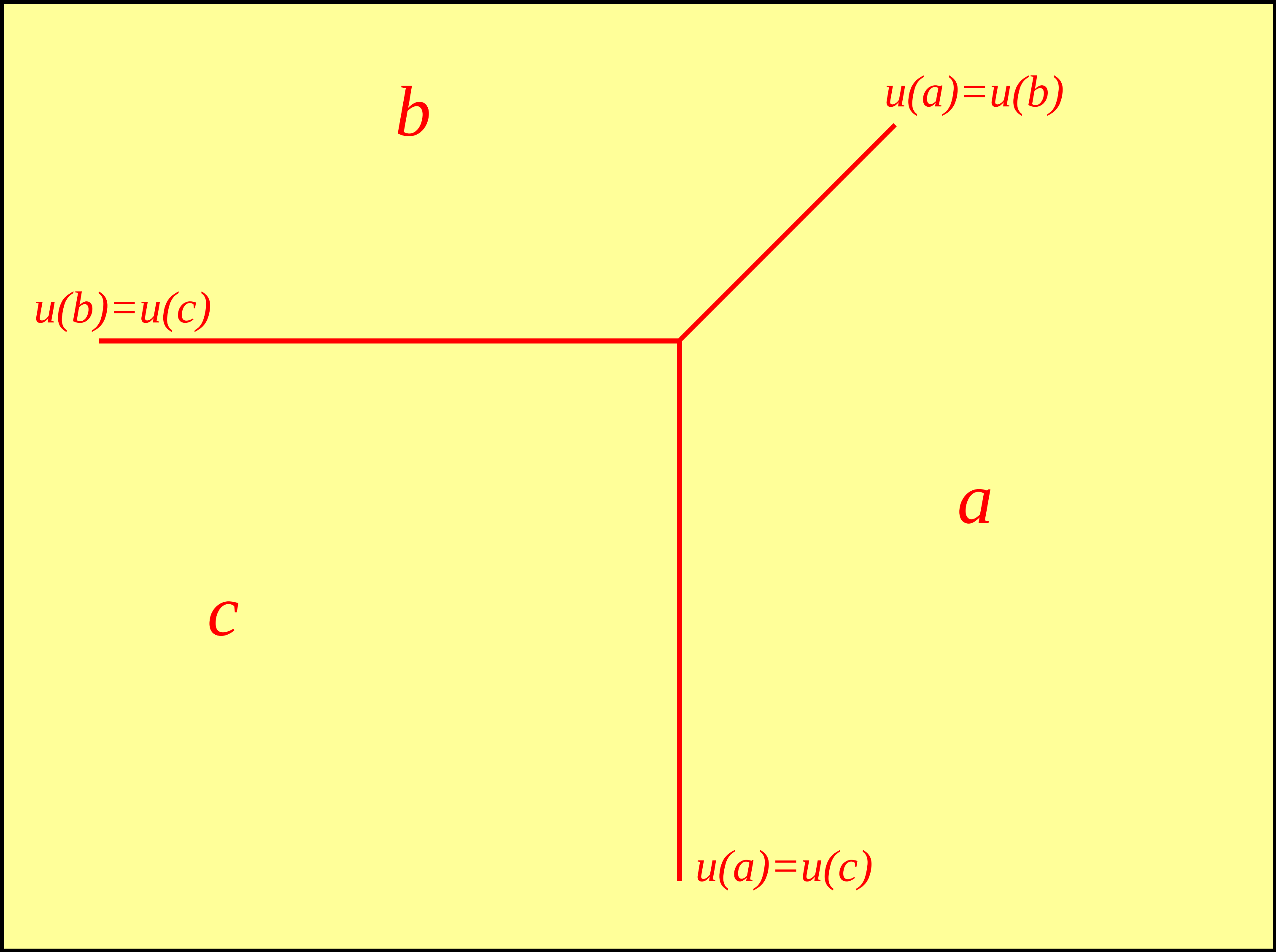
Extended Pareto Rule: for all *disjoint* coalitions S and T,

$$if \;\; a \;\; t_{S} \; b \;\; and \;\; a \;\; t_{T} \; b \; , then \;\; a \;\; t_{S \cup T} \; b$$

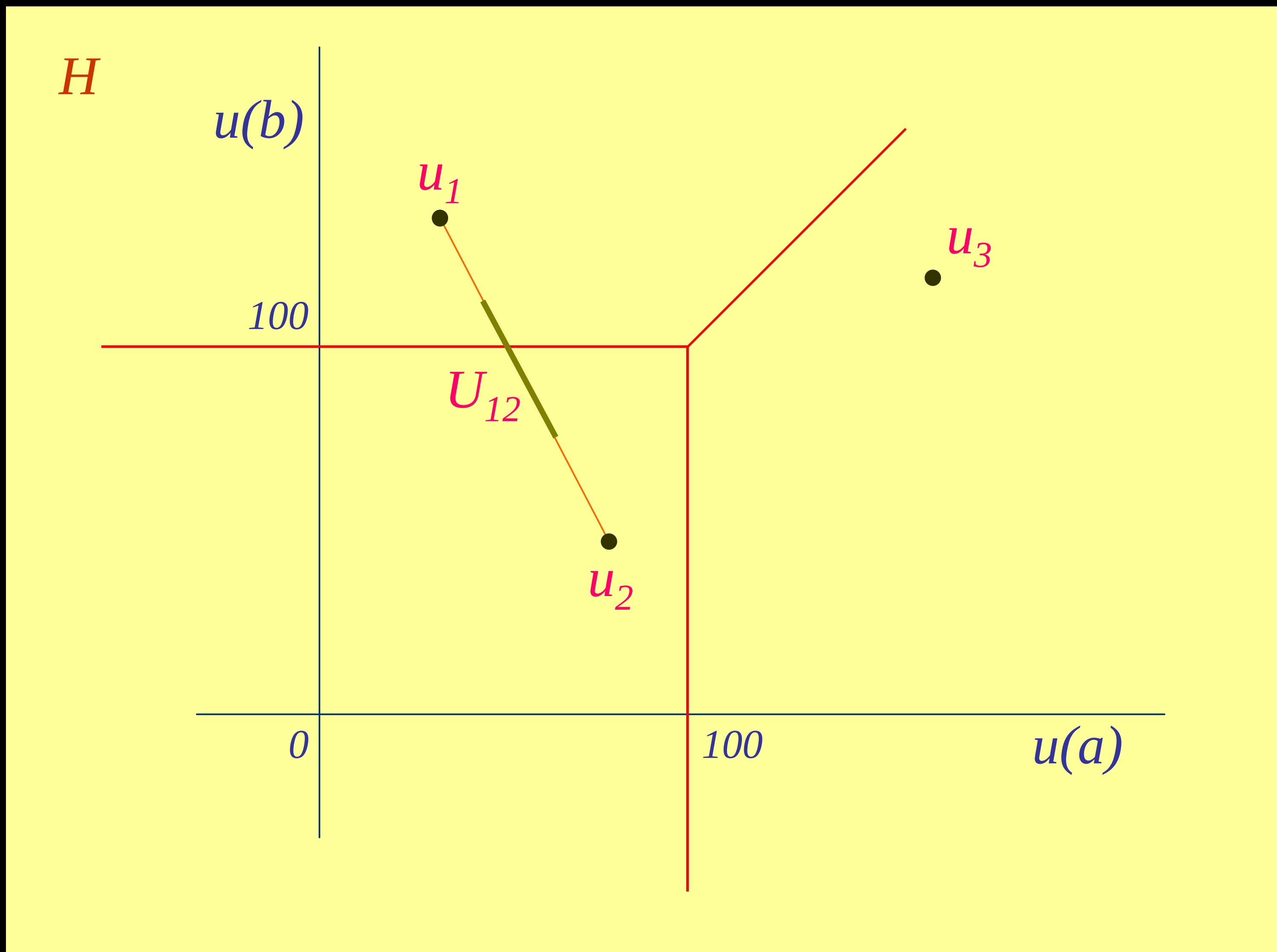
which is equivalent to $U_{S \cup T} \subseteq Co\,(\,U_S \bigcup U_T\,)$.

Let *H* be the plane containing u_1, u_2 and u_3, each normalized so that $u(c)=100$

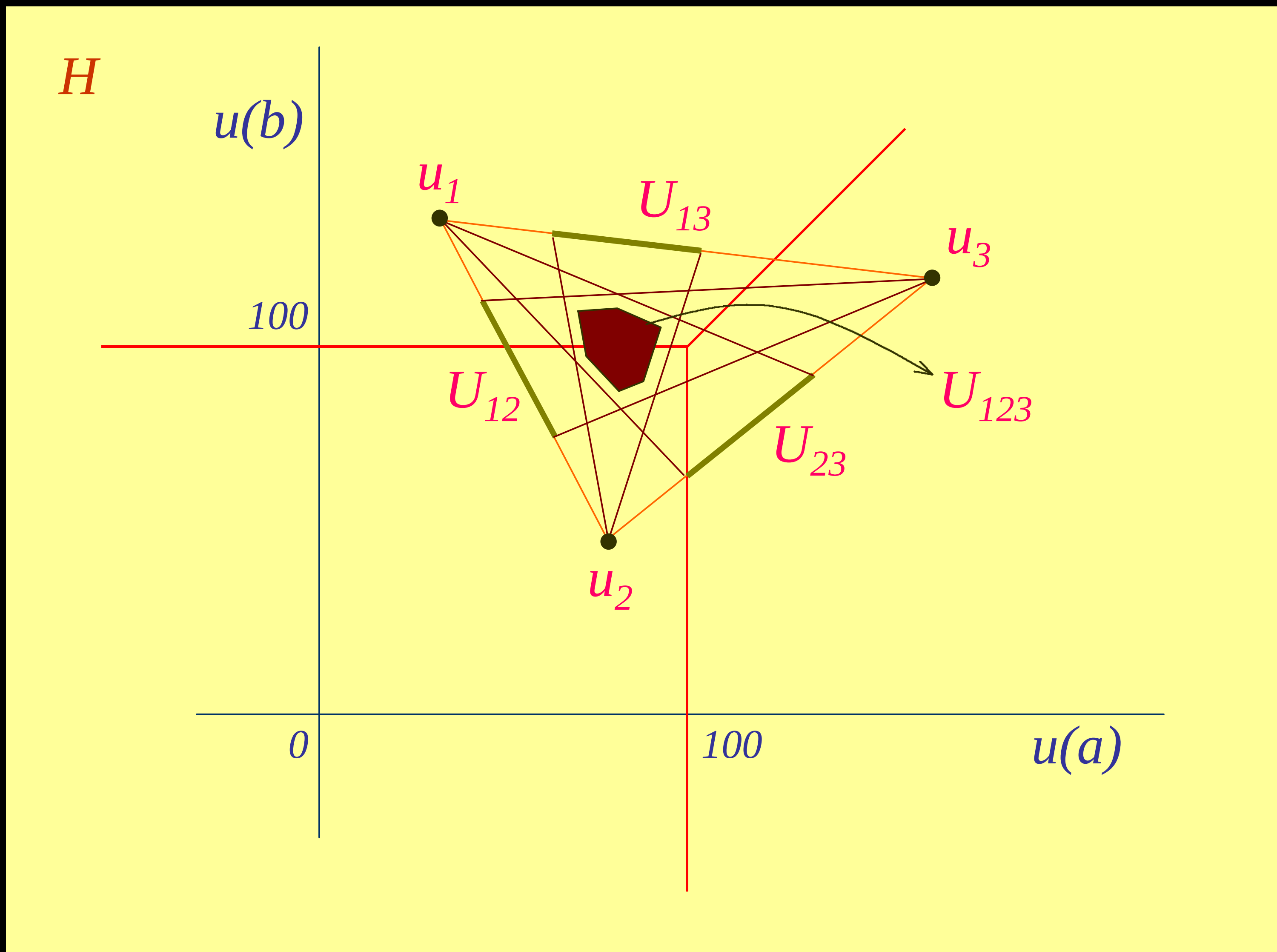



b
u(a)=u(b)
u(b)=u(c)
a
c
u(a)=u(c)

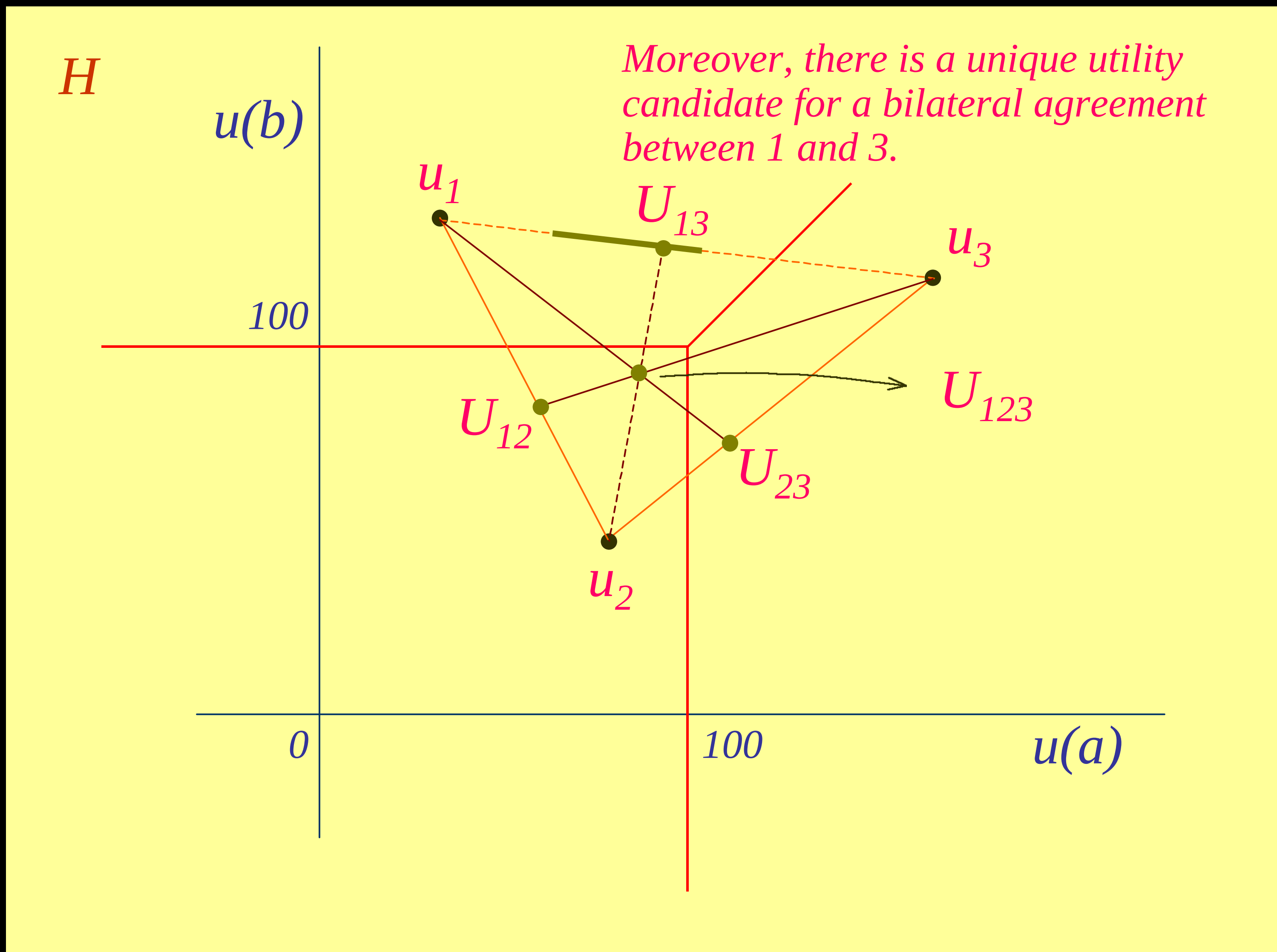
If x t $_1$ y and x t $_2$ y, then x t $_{12}$ y Û $U_{12} \subseteq Co(U_1 \cup U_2)$



If $x \; t_S \; y$ *and* $x \; t_T \; y$*, then* $x \; t_{S \cup T} \; y$ Û $U_{S \cup T} \subseteq Co(U_S \cup U_T)$

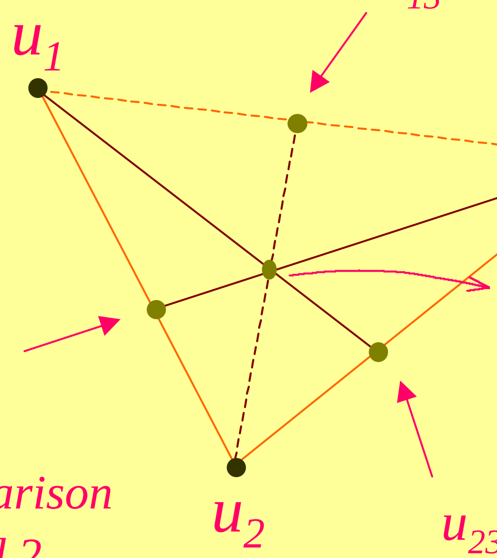


If t_{12} and t_{23} reach a complete agreement, then t_{123} is necessarily complete!



We derive a “no arbitrage” condition in the utility comparison rates.

H



$$u_{13} = \frac{u_1 + d_{13}\, u_3}{1 + d_{13}}$$

$$u_{123} = \frac{u_1 + d_{12}\, u_2 + d_{13}\, u_3}{1 + d_{12} + d_{13}}$$

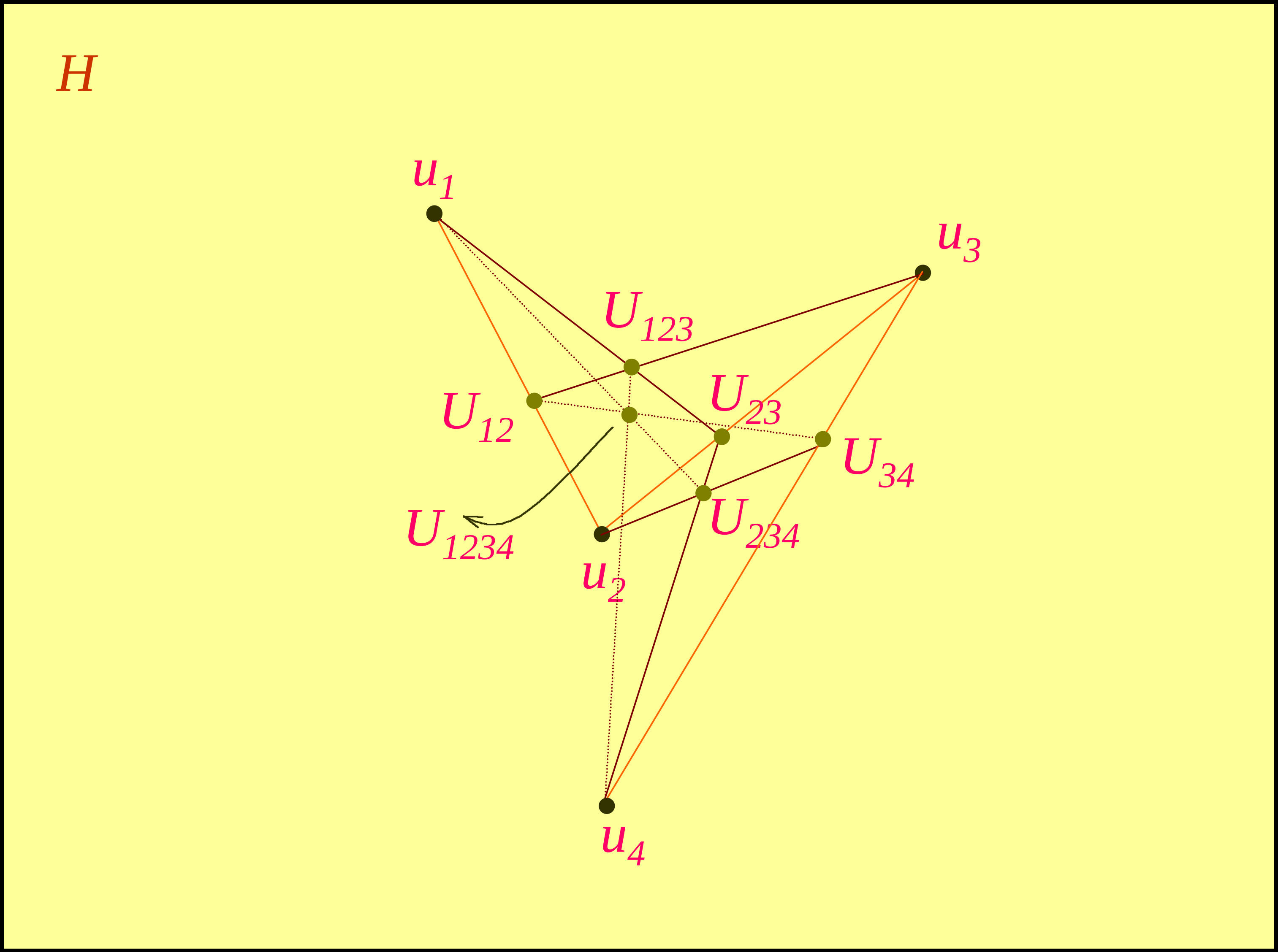
$$u_{12} = \frac{u_1 + d_{12}\, u_2}{1 + d_{12}}$$

d_{12} *is the utility comparison rate between 1 and 2*

$$u_{23} = \frac{u_2 + d_{23}\, u_3}{1 + d_{23}}$$

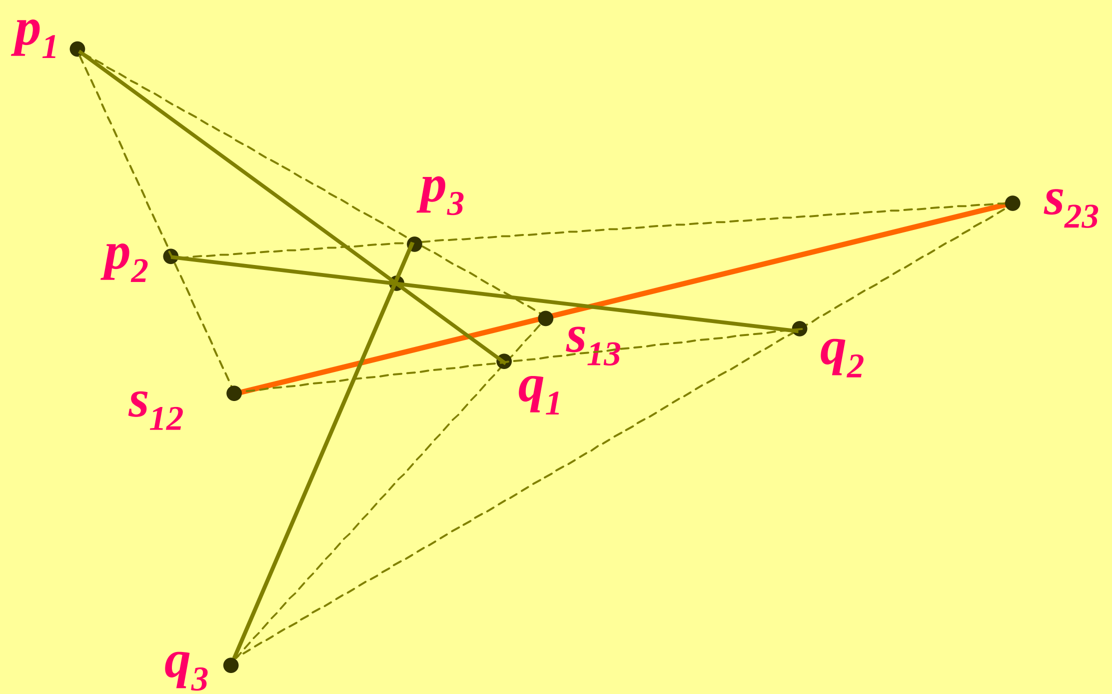
$$d_{13} = d_{12}\, d_{23}$$

The case of four agents

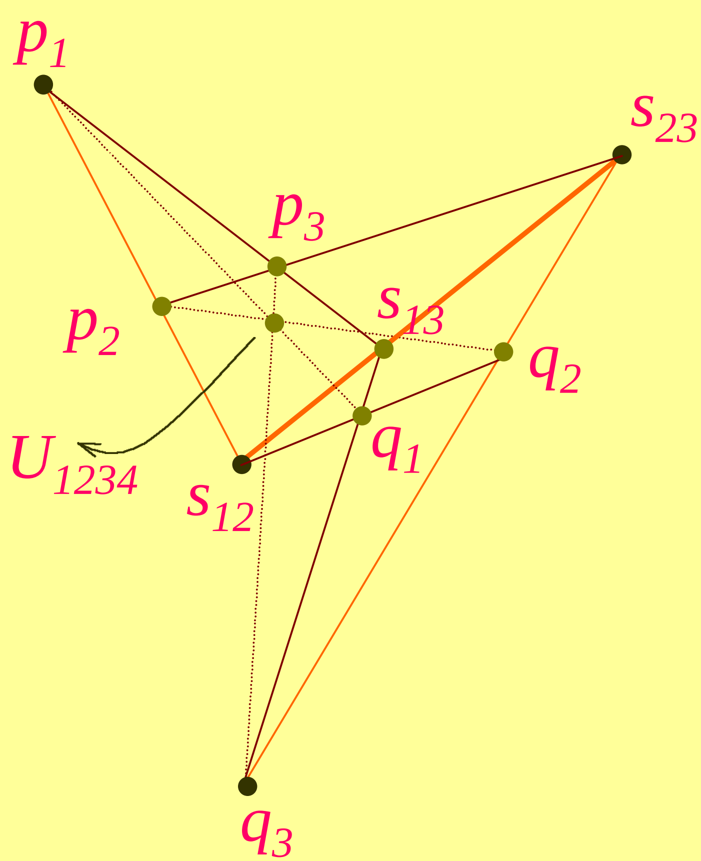


Desargues Theorem (1648)

Given p_i and q_i, i=1,2,3, let $s_{ij} = p_i p_j \cap q_i q_j$



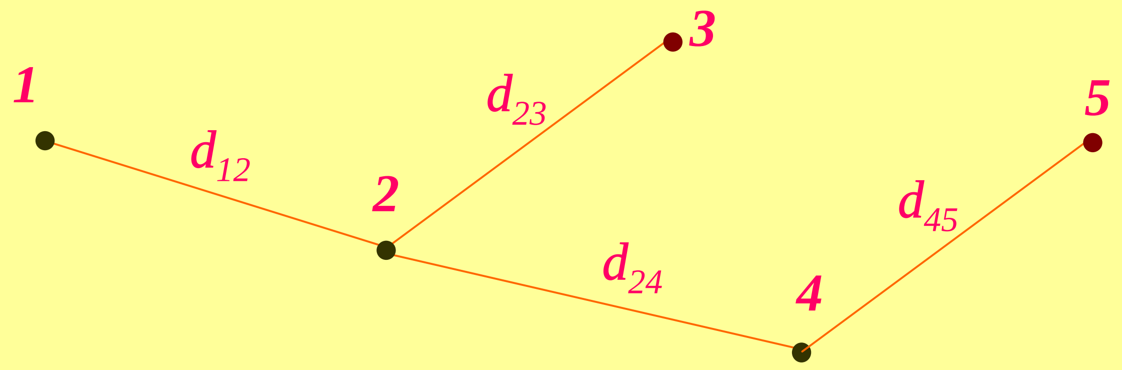
$p_i q_i$, i=1,2,3 are concurrent Û s_{ij}, 1£ i<j£ 3 are collinear.



p_1
s_{23}
p_3
p_2
s_{13}
q_2
q_1
U_{1234}
s_{12}
q_3

Theorem 1

Given an spanning tree of bilateral agreements, let λ_i be the individual weights computed as follows:



$$\lambda_1 = 1$$
$$\lambda_2 = d_{12}$$
$$\lambda_3 = d_{12}d_{23}$$
$$\lambda_4 = d_{12}d_{24}$$
$$\lambda_5 = d_{12}d_{24}d_{45}$$

If the extended Pareto Rule holds, then the group has a complete preference, with utility given by

$$u_N \equiv \sum_{i \in N} \lambda_i u_i$$

Theorem 2

Let all the pairs in *N* form bilateral agreements. Then the extended Pareto rule holds if and only if each subgroup has a complete preference, with utility

$$u_S = \sum_{i \in S} \lambda_i u_i \quad \textit{for all } S \subseteq N$$

If $x\ t_{1}\ y$ *and* $x\ t_{2}\ y$, *then* $x\ t_{12}\ y$ Û $U_{12} \subseteq Co(U_1 \cup U_2)$