Separating risk and return in the CAPM : A general utility-based model

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Abstract. The author proposes a new utility function which captures trade-offs between return and a large body of risk measures as defined by popular risk-return models in the managment science literature while exhibiting desirable properties for a financial investor. This function forms the basis for an extension to the Capital Asset Pricing Model which links general asymmetric risk measures and risk-value models with equilibrium asset pricing.

Keywords : Finance, decision theory, risk-return separation, CAPM.

1. Introduction

The management science and operations research literature has recently introduced new works in the exciting field of separating risk and return in decision-making. This includes the risk-value models of Jia and Dyer [21] and Dyer and Jia [6] and the risk-return separating utility functions of Bell ([3] and [4]). Sarin and Weber [34] call for their application to asset pricing, citing the known extensions to the Capital Asset Pricing Model (CAPM) as motivation. These extensions include the popular three-moment CAPM of Kraus and Litzenberger [22] as well as the mean-semivariance CAPM's of Bawa and Lindenberg [1] and Hogan and Warren [13] and the mean-lower partial moment CAPM of Harlow and Rao [12]. The problem with deriving extended CAPM's are mainly that in order to justify the linear pricing equation without making undesirable distributional assumptions utility must satisfy two-fund monetary separation (TFMS). This property is a much researched topic in finance and the family of utility functions with $U'(W) > 0$ and $U''(W) < 0$ for all W which display TFMS was identified in Cass and Stiglitz [5].

In this paper we present a utility function which is closely linked to popular utility functions from both the finance and management science literatures. This function is twice differentiable and allows for different risk-aversions on either sides of a pre-specified target which could be interpreted as a benchmark wealth level. By linking this function to the literature on risk-return separation we show how

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preferences depend upon a general asymmetric risk measure corresponding to the axiomatised measures of perceived risk in Fishburn [8] and Luce and Weber [24] or one of several commonly used special cases thereof. We derive an extension of the CAPM using our new utility function by obtaining TFMS without restrictive distributional assumption. Since the wealth target of the representative agent depends on the wealth distribution this is not in conflict with Cass and Stiglitz [5]. The link between axiomatised measures of perceived risk and asset pricing is then completed by considering the resulting beta's.

The paper is organised as follows : Section 2 introduces the utility function and examines its properties, putting it in relation to current developments in risk-value theory. Section 3 derives the fund separation result, aggregation and linear pricing which leads to the extended CAPM. Section 4 is reserved for our conclusions.

2. A general risk - return separating utility function

Here we introduce the utility function upon which the rest of this paper focuses and link it to the. Consider

$$U(W) = \left\{ \begin{array}{ll} \lambda W - \frac{\lambda_1}{\alpha+1}(W-\eta)^{\alpha+1} & L > W \geq \eta \\ \lambda W - \frac{\lambda_2}{\beta+1}(\eta - W)^{\beta+1} & W < \eta \end{array} \right\} \tag{1}$$

where $\lambda \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \eta \geq 0, L \leq \left(\frac{\lambda}{\lambda_1}\right)^{\alpha} + \eta$ and α and β are constants of either sign. When $\alpha > 1, \beta > 1$ and $\lambda > 0$, (1) is increasing, weakly concave and twice continuously differentiable. The upper limit restriction is not new in finance. For instance, the much used quadratic utility function has a bliss point unless the domain is shrunk appropriately. However, the wealth target η represents a point at which risk-aversion can change drastically. The existence of such a point was documented in an extensive survey by Fishburn and Kochenberger [10] and is relevant to investors such as pension funds and asset managers who are aim to beat a preset benchmark. Several functions which are related to (1) have appeared in the finance literature. Like (1) itself these can be motivated in relation to the perceived risk measures implicit in the function. To do this we turn to recent developments in risk-value theory as well as older decision rules in finance.

The risk-value models of Jia and Dyer [21] and Dyer and Jia [6] have recently been introduced in the management science and operational research literature. Previously, Bell ([3] and [4]) had derived related risk-return separation results for his one-switch utility functions, initially identified in Bell [2]. The relationships between these works are formally detailed in Pedersen and Satchell [28]. In Jia and Dyer [21] the risk-value models were derived by firstly introducing a standard risk measure

$$R(X') = -E[U(\widetilde{X} - E(\widetilde{X}))] \tag{2}$$

where $\widetilde{X}$ is a gamble with mean $E(\widetilde{X})$ and uncertainty $X' = \widetilde{X} - E(\widetilde{X})$, and then combining this with a notion of value to produce a decision framework where choice depends upon these two parameters only. A normalisation implies that risk measures and utility functions are defined in terms of returns rather than wealth. The special case of (1) when $\tau = 0$ (i.e $\eta = W_0$ so the target is the status quo) was analysed in Jia and Dyer [21]. In keeping with the original derivation we convert (1) to its returns form by $W = W_0 + \widetilde{X}$ and $\eta = W_0 + \tau$ where W is final wealth, W_0 is initial wealth and τ is a returns target and applying (2) one gets

$$\begin{aligned} R(X') &= -E\left[\lambda(\widetilde{X} - E(\widetilde{X})) - \frac{\lambda_1}{\alpha+1}(\widetilde{X} - E(\widetilde{X}))^{\alpha+1} \middle| \widetilde{X} \geq 0\right] \\ &\quad -E\left[\lambda(\widetilde{X} - E(\widetilde{X})) - \frac{\lambda_2}{\beta+1}(E(\widetilde{X}) - \widetilde{X})^{\beta+1} \middle| \widetilde{X} < 0\right] \\ &= \frac{\lambda_1}{\alpha+1}\int_0^\infty (\widetilde{X} - E(\widetilde{X}))^{\alpha+1} f(\widetilde{X}) d\widetilde{X} + \frac{\lambda_2}{\beta+1}\int_{-\infty}^0 (E(\widetilde{X}) - \widetilde{X}))^{\beta+1} f(\widetilde{X}) d\widetilde{X} \end{aligned} \tag{3}$$

This risk measure is a special case of the general axiomatisation of asymmetric risk measures in Fishburn [8] and Luce and Weber [24]. A related model results from analysing (1) using the multiplicative risk-return property of Bell [4] which is more suitable for financial applications. In the multiplicative structure gambles are rates of return rather than nominal outcomes and so $W = W_0\widetilde{X}$ and $\eta = W_0\tau$. A function is said to have the risk-return property if expected utility is a function of initial wealth, a risk measure and a return measure, where both measures are functions of the gamble only. For (1) with $\eta = W_0\tau$ expected utility simplifies to

$$\lambda W_0 E(\widetilde{X}) - \frac{\lambda_1 W_0^{\alpha+1}}{\alpha+1}\int_\tau^\infty (\widetilde{X} - \tau)^{\alpha+1} f(\widetilde{X}) d\widetilde{X} - \frac{\lambda_2 W_0^{\beta+1}}{\beta+1}\int_{-\infty}^\tau (\tau - \widetilde{X}))^{\beta+1} f(\widetilde{X}) d\widetilde{X} \tag{4}$$

which can be written as $f(r(\widetilde{X}), R_1(\widetilde{X}), R_2(\widetilde{X}), W_0)$ where $r(\widetilde{X}) = E(\widetilde{X})$,

$$R_1(\widetilde{X}) = \int_\tau^\infty (\widetilde{X} - \tau)^{\alpha+1} f(\widetilde{X}) d\widetilde{X} \text{ and } R_2(\widetilde{X}) = \int_{-\infty}^\tau (\tau - \widetilde{X}))^{\beta+1} f(\widetilde{X}) d\widetilde{X}$$

This model can be interpreted either as separating decision making into a choice over the expected value, a downside component and an upside component where the trade-offs between risk and return depend explicitly upon W_0 or a choice over the expected value and a more general asymmetric risk measure. This also allows for a general target τ which is preferred in financial applications and coincides with (3) when $\tau = E(\widetilde{X})$ which, as has been pointed out by an anonymous referee, implies that expected utility is not linear in the probabilities.

Numerous risk measures which are special cases of (3) and (4) give rise to known decision rules which are linked to the corresponding special cases of (1). When $\lambda_1 = \lambda_2$ and $\alpha = \beta = a$, both models give a symmetric risk measures

$$\int_{-\infty}^{\infty} \left|\widetilde{X} - \tau\right|^a pdf\left(\widetilde{X}\right) d\widetilde{X} \tag{5}$$

For $a = 2$ this gives mean-variance preferences and for $a = 4$ the implied risk measure can be interpreted as the fourth moment around a target or a weighted average of the first four central moments. When a is odd (5) is the absolute moment of the distribution around τ. For $a = 3$ we do not obtain skewness preferences but a measure of the cube of absolute deviations around the target.

Fishburn [9] argued for a decision rule where risk is defined by the lower partial moment

$$R_{\beta,\tau}(\widetilde{X}) = \int_{-\infty}^{\tau} (\tau - \widetilde{X})^{\beta} f(\widetilde{X}) d\widetilde{X} \tag{6}$$

where $\widetilde{X}$ is a random variable with density function f and τ is a returns target. Further special cases of this include the probability of loss and the mean absolute deviation (see Fishburn [9] for details). These form part of a survey in Pedersen and Satchell [29] where desirable measures of risk for financial investors were isolated. This corresponds to the case where $\lambda_1 = 0$ in (3) and Fishburn proves congruency between the mean-lower partial moment decision rule and a utility function obtained from putting $\lambda_1 = 0$ in (1). Asset managers in search of outperformance of a benchmark (e.g. the Dow Jones) while simultaneously reducing downside risk could use such a decision rule. Returns below the target are punished while returns above the target do not affect risk. Bawa and Lindenberg [1] derive a CAPM based on mean-lower partial moment preferences when $\beta = 1$ and η is the riskless return. This was extended by Harlow and Rao [12] to arbitrary η and Satchell [35] to all $\beta > 1$. Holthausen [14] proves a similar congruency result but uses an "upside" measure rather than expected returns. This corresponds to the case where $\lambda = 0$ in (4) and decisions are over $R_1(\widetilde{X})$ and $R_2(\widetilde{X})$ only and the congruent utility function is (1) with $\lambda = 0$. However this special case of (1) cannot be globally convex or concave (see Holthausen [14] for details) which explains why the decision rule has not been applied in financial economics.

These observations speak in favour of (1) as a representation of preferences which reflect the trade-off between risk and return while possessing the properties (monotonicity and concavity) generally considered necessary for application to financial economics thus presenting not only a generalisation of previous models but a new suitable utility representation of a decision model based on the general asymmetric risk measures of Fishburn [8] and Luce and Weber [24]. Sarin and Weber [34] have called for the application of general decision rules to asset pricing. This motivates the next section where we address the issue of extending the CAPM using (1).

3. Generalising the CAPM

We now derive an augmented version of the traditional CAPM commonly attributed to Sharpe [38], Lintner [23] and Mossin [25]. Their original work showed that when agents act as though minimising the variance of a portfolio subject to obtaining a fixed return, equilibrium dictates that

$$E[\widetilde{r}_p - r] = \left(\frac{cov\,[\widetilde{r}_p, \widetilde{r}_m]}{var\,[\widetilde{r}_m]} \right) E[\widetilde{r}_m - r] \tag{7}$$

where r is the riskless rate of return on the bond, $\widetilde{r}_p$ is the rate of return on a portfolio p, $\widetilde{r}_m$ is the rate of return on the market portfolio, defined as the sum of all individual portfolios or the portfolio of a representative agent, and E is the mathematical expectations operator. This is subject to further standard restrictions of common beliefs about distributions of returns, common time horizon, divisible assets and allowing short selling. These are maintained for the duration of this paper. The equation (7) can be rearranged to give the price of any asset or portfolio of assets in terms of market behaviour and the riskless rate. Rubinstein ([32] and [33]) derived general aggregation results which paved the way for allowing higher order moments to influence decision-making in a CAPM-setting, later utilised by Grauer [11] and Kraus and Litzenberger [22], who derive a CAPM by using cubic utility that captures preferences for positive skewness, and Sears and Wei ([36] and [37]), Homaifar and Graddy [15] and Hwang and Satchell [19] who derive fourth moment models. As mentioned in the previous section, downside risk entered the CAPM's of Bawa and Lindenberg [1], Hogan and Warren [13], Harlow and Rao [12] and Satchell [35]. In this section we extend these results by utilising (1) thus capturing all these works as special cases while extending the theory by allowing individual risk characteristics described by the asymmetric measure in (3) and (4).

As is standard in asset pricing models we will make the assumption that the following market equilibrium relationship from Huang and Litzenberger [18] (page 154) holds for our utility functions : When a representative agent exists and has utility function $U(W)$ where W is terminal wealth (7) extends to

$$E[\widetilde{r}_p - r] = \left(\frac{cov\left[\widetilde{r}_p, U'(W)\right]}{cov\left[\widetilde{r}_m, U'(W)\right]} \right) E[\widetilde{r}_m - r] \tag{8}$$

so the marginal utility of the representative agent determines the equilibrium risk measure. Our model can be considered a special case of this general approach. However the existence of a representative agent is determined by whether two-fund money separation (to be defined) obtains. By showing that (1) is sufficient for two fund money separation in the next section we in fact generalise the set of functions which are applicable to (8) without generalising the equilibrium result itself.

3.1. Fund separation and aggregation. If it can be shown that each individual optimally chooses a mixture of the riskless asset and the same portfolio (but not necessarily the same mixture) a representative agent can be solved for explicitly independent of assumptions on returns distributions and without complete markets. This is known as two-fund money separation (TFMS). TFMS can be guaranteed by assumptions on either preferences, distribution of returns, or both. Ingersoll [20], Chapter 6, discusses these options comprehensively and the results are well-known under distributional assumptions, detailed in Ross [31]. Cass and Stiglitz [5] show that within the class of functions $U(W)$ with $U'(W) > 0$ and $U''(W) < 0$ for all W, the HARA functions are necessary and sufficient for two- and three-fund monetary separation and Huang and Litzenberger [18] devote a large section of Chapter 5 to proving sufficiency results on fund separation for the CAPM-framework to hold. However the work on which we focus in Ingersoll [20], Chapter 6. He gives a short proof of sufficiency of the HARA class for TFMS. We present an extension to his proof and so generalise the adaptation by Satchell [35] who was the first to apply the procedure to piecewise functions. He showed that the mean-lower partial moment utility function of Fishburn [9] was sufficient for TFMS without distributional assumptions. Since the wealth target in this function depends on initial wealth the fund separation and aggregation result depends on the distribution of initial wealth which implies that the result is not in conflict with Cass and Stiglitz. Recall that we work with the utility function

$$U(W) = \left\{ \begin{array}{ll} \lambda W - \frac{\lambda_2}{\alpha+1}(W-\eta)^{\alpha+1} & L > W \geq \eta \\ \lambda W - \frac{\lambda_1}{\beta+1}(\eta - W)^{\beta+1} & W < \eta \end{array} \right\} \tag{9}$$

where $\lambda \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \eta \geq 0, L \leq \left(\frac{\lambda}{\lambda_1}\right)^{\alpha} + \eta, \alpha > 1$ and $\beta > 1$. We need the following Lemma whose proof is in the Appendix.

Lemma 1. *Suppose that for a twice differentiable function $U(W)$, where $W = \sum_{i=0}^{R} \alpha_i \tilde{z}_i$, $U''(W) \leq 0$ for all W, $U''(W) < 0$ for $a \leq W \leq b$ and the joint distribution of $\underline{z}$ assigns positive probability to $[a,b]$. Then the function $V(\underline{\alpha}) = E[U(W)]$ is strictly concave in $\underline{\alpha}$.*

We next turn to the main result. The proof follows the steps of Ingersoll [20] and Satchell [35] and is in the Appendix.

Theorem 1. *Suppose that there are K individuals indexed $1, 2, ...k, ..., K$. There is a riskless asset paying rate of return r, as well as R risky assets. Individual k has positive initial wealth W_{0k}, wealth target η^k and a weakly concave utility function*

$$U_k(W) = \left\{ \begin{array}{ll} \lambda^k W - \frac{\lambda_1^k}{\alpha+1}\left(W-\eta^k\right)^{\alpha+1} & L^k \geq W \geq \eta^k \\ \lambda^k W - \frac{\lambda_2^k}{\beta+1}\left(\eta^k - W\right)^{\beta+1} & W < \eta^k \end{array} \right\} \tag{10}$$

for $\alpha \geq 1$ and $\beta \geq 1$. In addition, one of the following holds :

1. $L^k \leq \eta^k + \left(\frac{\lambda^k}{\lambda_2^k}\right)^{\alpha}$,

$$\lambda_1^k = \lambda^k \left(W_{0k} f^k\right)^{-\alpha} > 0 \text{ and } \lambda_2^k = \lambda^k \left(W_{0k} f^k\right)^{-\beta} > 0 \tag{11}$$

   for all k, and the distribution of the assets assign positive probability to the event $W \neq W_{0k}$.

2. $\lambda_1^k = 0, L^k \leq \eta^k + \left(\frac{\lambda^k}{\lambda_2^k}\right)^{\alpha}$ *and*

$$\lambda_2^k = \lambda^k \left(W_{0k} f^k\right)^{-\beta} > 0 \tag{12}$$

for all k*, and the distribution of the assets assigns positive probability to the event* $W < W_{0k}$*.*

3. $\lambda_2^k = 0$ *and*

$$\lambda_1^k = \lambda^k \left(W_{0k} f^k\right)^{-\alpha} > 0 \tag{13}$$

for all k*, and the distribution of the assets assigns positive probability to the event* $W > W_{0k}$*.*

Then, if all investors hold the risky portfolio long, two fund monetary separation obtains and

$$f^k = \frac{1}{r}\left[(1+r) - \frac{\eta^k}{W_{0k}}\right] > 0 \tag{14}$$

is the fraction invested by agent k *in the risky portfolio.*

Note that in returns form the restrictions imply that we can simplify the utility functions considerably. Consider Case 1 and a gamble $\widetilde{X}$. We get

$$\begin{aligned} U_k(\widetilde{X}) &= \left\{ \begin{array}{ll} \lambda^k W_{0k}\widetilde{X} - \frac{\lambda_1^k}{\alpha+1}\left(W_{0k}\widetilde{X} - W_{0k}\tau^k\right)^{\alpha+1} & l^k \geq W \geq \tau^k \\ \lambda^k W_{0k}\widetilde{X} - \frac{\lambda_2^k}{\beta+1}\left(W_{0k}\tau^k - W_{0k}\widetilde{X}\right)^{\beta+1} & W < \tau^k \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \lambda^k W_{0k}\widetilde{X} - \frac{\lambda_1^k W_{0k}^{\alpha+1}}{\alpha+1}\left(\widetilde{X} - \tau^k\right)^{\alpha+1} & l^k \geq W \geq \tau^k \\ \lambda^k W_{0k}\widetilde{X} - \frac{\lambda_2^k W_{0k}^{\beta+1}}{\beta+1}\left(\tau^k - \widetilde{X}\right)^{\beta+1} & W < \tau^k \end{array} \right\} \end{aligned}$$

which by using (11) and dividing by $\lambda^k W_{0k}$ gives

$$U_k(\widetilde{X}) = \left\{ \begin{array}{ll} \widetilde{X} - \frac{(f^k)^{-\alpha}}{\alpha+1}\left(\widetilde{X} - \tau^k\right)^{\alpha+1} & l^k \geq W \geq \tau^k \\ \widetilde{X} - \frac{(f^k)^{-\beta}}{\beta+1}\left(\tau^k - \widetilde{X}\right)^{\beta+1} & W < \tau^k \end{array} \right\}$$

so the expected utility function will exhibit risk-return separation where trade-offs depend on the initial wealth, the risk-free rate and the target. The associated risk measures are those given in (4). The fraction allocated to the risky portfolio, f^k, increases if and only if the trade-offs decrease. Observe that f^k is increasing in W_{0k} and decreasing in η^k so that as initial wealth increases agents diversify away from the bond, consistent with the property of decreasing absolute risk aversion, and investors with relatively high target rebalance in favour of the safe asset to reduce the chance of a shortfall. The restriction that f^k be positive implies that agents can not short the market. However, w_0, the weight of the bond in the market, can be used to control, the risk of the overall portfolio. Furthermore, it appears that the absence of distributional assumptions and the nature of (9) imply that fund separation requires the parameters of the utility function explicitly depend upon initial wealth, consistent with the observations of Satchell [35] and a further abstraction from Cass and Stiglitz [5].

Having obtained the demands of agents as fractions of their wealth allocated between the riskless asset and a fixed risky portfolio $\underline{w}$, aggregate demand is easily calculated. We thus construct a representative agent. A good discussion of the relationship between fund separation and representative agents can be found in either Ingersoll [20], Huang and Litzenberger [18] (Chapter 4) or Muellbauer [26]. The following theorem establishes the existence of a representative agent. Its proof is also relegated to the Appendix.

Theorem 2. *The aggregate demand in an economy where all individuals have utility as described in Case* 1 *in Theorem* 1 *(i.e. (10) with (11) satisfied) is identical to that of a single representative consumer with utility function*

$$U(W) = \left\{ \begin{array}{ll} \lambda W - \frac{\lambda_1}{\alpha+1}(W-\eta)^{\alpha+1} & L \geq W \geq \eta \\ \lambda W - \frac{\lambda_2}{\beta+1}(\eta-W)^{\beta+1} & W < \eta \end{array} \right\} \tag{15}$$

where $L \leq \eta + \left(\frac{\lambda}{\lambda_2}\right)^{\alpha}, \lambda_1 = \lambda (W_0 f)^{-\alpha} > 0, \lambda_2 = \lambda (W_0 f)^{-\beta} > 0, f = \frac{1}{r}\left[(1+r) - \frac{\eta}{W_0}\right]$, *initial wealth* $W_0 = \sum\limits_k W_{0k}$ *and target* $\eta = \sum\limits_k \eta^k$. *The aggregate demand in an economy where all individuals have utility as described in Case* 2 *in Theorem* 1 *(i.e. (10) with (12) satisfied) is identical to that of a single representative consumer with utility function*

$$U(W) = \left\{ \begin{array}{ll} \lambda W - \frac{\lambda_1}{\alpha+1}(W-\eta)^{\alpha+1} & L \geq W \geq \eta \\ \lambda W & W < \eta \end{array} \right\} \tag{16}$$

where $L \leq \eta + \left(\frac{\lambda}{\lambda_2}\right)^{\alpha}, \lambda_1 = \lambda (W_0 f)^{-\alpha} > 0, f = \frac{1}{r}\left[(1+r) - \frac{\eta}{W_0}\right]$, *initial wealth* $W_0 = \sum\limits_k W_{0k}$ *and target* $\eta = \sum\limits_k \eta^k$. *The aggregate demand in an economy where all individuals have utility as described in Case* 3 *in Theorem* 1 *(i.e. (10) with (11) satisfied) is identical to that of a single representative consumer with utility function*

$$U(W) = \left\{ \begin{array}{ll} \lambda W & L \geq W \geq \eta \\ \lambda W - \frac{\lambda_2}{\beta+1}(\eta-W)^{\beta+1} & W < \eta \end{array} \right\} \tag{17}$$

where $\lambda_2 = \lambda (W_0 f)^{-\beta} > 0, f = \frac{1}{r}\left[(1+r) - \frac{\eta}{W_0}\right]$, *initial wealth* $W_0 = \sum\limits_k W_{0k}$ *and target* $\eta = \sum\limits_k \eta^k$.

Hence even when individuals have different targets and initial wealths demands aggregate in a straightforward manner if the utility functions satisfy the appropriate restrictions. The representative agents utility function has very similar structure to those of the underlying individuals. In addition her wealth target is defined as the sum of the individuals wealth targets and the fraction of wealth she places in the risky portfolio is a weighted average of the corresponding individual fractions.

3.2. Linear pricing and beta. We are now in a position to derive equilibrium CAPM's since the fund separation property validates the use of (8) without distributional assumptions. By differentiating (9) we get

$$U^{'}(\widetilde{W}) = \Delta\left[\lambda + \lambda_1\left(\widetilde{W} - \eta\right)^{\alpha}\right] + (1-\Delta)\left[\lambda + \lambda_2\left(\eta - \widetilde{W}\right)^{\beta}\right] \tag{18}$$

where $\Delta = 0$ when $\widetilde{W} < \eta$ and $\Delta = 1$ otherwise. We need to get everything in terms of rates of return rather than wealth. If W_m is total initial wealth of the economy, all of which is invested, and $\widetilde{r}_m$ is the return on the market total final wealth, $\widetilde{W}$, satisfies

$$\widetilde{W} = W_m(1 + \widetilde{r}_m) \tag{19}$$

The wealth target, η, is related to a corresponding aggregate target rate of return, τ, thus

$$\eta = W_m(1 + \tau) \tag{20}$$

Substituting these into (18) and simplifying using (11) yields

$$U^{'}(\widetilde{r}_m) = \Delta f^{\alpha}\left[1 + (\widetilde{r}_m - \tau)^{\alpha}\right] + (1-\Delta) f^{\beta}\left[1 + (\tau - \widetilde{r}_m)^{\beta}\right] \tag{21}$$

where $\Delta = 1$ when $\widetilde{r}_m > \tau$ and $\Delta = 0$ otherwise. When $\Delta = 1$, the second term is zero and the constant f^{α} cancels. We can cancel f^{β} similarly when $\Delta = 0$. Substituting (21) into (8) and simplifying the extended pricing equation becomes

$$E[\widetilde{r}_p - r] = \Pi E[\widetilde{r}_m - r]$$

where

$$\begin{aligned} \Pi &= \frac{f^{\alpha} cov\left[\widetilde{r}_p, \Delta\left(\widetilde{r}_m - \tau\right)^{\alpha}\right] + f^{\beta} cov\left[\widetilde{r}_p, (1-\Delta)\left(\tau - \widetilde{r}_m\right)^{\beta}\right]}{f^{\alpha} cov\left[\widetilde{r}_m, \Delta\left(\widetilde{r}_m - \tau\right)^{\alpha}\right] + f^{\beta} cov\left[\widetilde{r}_m, (1-\Delta)\left(\tau - \widetilde{r}_m\right)^{\beta}\right]} \\ &= \left[\left.\frac{cov\left[\widetilde{r}_p, (\widetilde{r}_m - \tau)^{\alpha}\right]}{cov\left[\widetilde{r}_m, (\widetilde{r}_m - \tau)^{\alpha}\right]}\right| \Delta = 1\right] + \left[\left.\frac{cov\left[\widetilde{r}_p, (\tau - \widetilde{r}_m)^{\beta}\right]}{cov\left[\widetilde{r}_m, (\tau - \widetilde{r}_m)^{\beta}\right]}\right| \Delta = 0\right] \end{aligned} \tag{22}$$

is the beta. It is trivial to verify that if $U(\widetilde{r}_m)$ is quadratic (i.e. $\alpha = 1$ and $\Delta = 1$ in (22)) the expression reduces to the traditional CAPM-beta (7). General power utility is obtained when $\Delta = 1$ for all returns which can be guaranteed by making the target redundant (i.e. $\tau \leq -1$). The cubic case further requires $\alpha = 2$ so our result contains the extensions in Grauer [11], Kraus and Litzenberger [22], Sears and Wei ([36] and [37]), Homaifar and Graddy [15] and Hwang and Satchell [19] made possible by Rubinsteins work in [32] and [33]. When all individuals satisfy case 2 in Theorem 1 we get

$$\Pi = \frac{cov\left[\widetilde{r}_p, \Delta\left(\widetilde{r}_m - \tau\right)^\alpha\right]}{cov\left[\widetilde{r}_m, \Delta\left(\widetilde{r}_m - \tau\right)^\alpha\right]}$$

This seems somewhat non-sensical as the risk measure is now a function only of returns above the threshold and so one would hedge against profits, not losses. Under these restrictions individuals utility functions separate risk and return where risk is measured purely on returns above the target ! Case 3 in Theorem 1 yields the CAPM derived in Satchell [35]

$$\Pi = \frac{cov\left[\widetilde{r}_p, (1-\Delta)\left(\tau - \widetilde{r}_m\right)^\beta\right]}{cov\left[\widetilde{r}_m, (1-\Delta)\left(\tau - \widetilde{r}_m\right)^\beta\right]} \tag{23}$$

Investors in this example have preferences separated into the expected value and lower partial moments of gambles but they may have different risk-return trade-offs. By setting $\beta = 2$ one gets the mean-lower partial moment beta of Harlow and Rao [12] and by further restricting $\tau = r_f$, the return on a safe asset, the semivariance beta of Bawa and Lindenberg [1] and Hogan and Warren [13] is obtained.

The general CAPM pricing equation (22) corresponds to case 1 in Theorem 1 when all investors have utility given by (1). The asymmetric contributions of "upside" and "downside" returns to equilibrium risk when $\alpha \neq \beta$ allows us to weigh differently responses from upturns and downturns in the market while not relying on downside contributions only. This would appeal to most investors in general as equilibrium risk is measured as a mixture of overall dispersion and possibly more heavily weighted contributions from returns below a certain level, reflecting the possible asymmetries in the associated individual risk measures (3) and (4). Note also that the beta's are independent of initial wealth despite both marginal utility and risk-return trade-offs at both the individual and aggregate level having wealth-dependent parameters.

4. CONCLUSION

Motivated by recent models of risk and return separation in the management science and operational research literature (Bell ([3] and [4]), Jia and Dyer [21] and Dyer and Jia [6]) a new utility function has been proposed which can be expressed in terms of risk and return for a large number of risk measures including the general asymmetric risk measures axiomatised by Fishburn [8] and Luce and Weber [24]. It is also related to previous utility functions in the finance literature which arise from decision rules over downside risk and return. Sarin and Weber [34] have called for an application of such models in asset pricing. We have derived an extension of the Capital Asset Pricing Model (CAPM) by using an approach of Ingersoll [20] and Satchell [35] to establish two-fund money separation and consequently applied a general result in Huang and Litzenberger [18]. The new CAPM contains as special cases most other popular CAPM extensions including Kraus and Litzenberger [22], Hogan and Warren

[13], Bawa and Lindenberg [1], Harlow and Rao [12], Satchell [35], Sears and Wei ([36] and [37]), Homaifar and Graddy [15] and Hwang and Satchell [19].

Several works have tested for differences in asset pricing models based on different utility functions (Harlow and Rao [12], Homaifar and Graddy ([17] and [16]), Price, Price and Nantell [30], Eftekhari and Satchell [7] and Pedersen [27] amongst others) and our general linear model is convenient for nesting one model in another (see Pedersen [27] for details). Such differences have important implications for asset pricing, optimal asset allocations and performance measurement in empirical finance. Moreover by the explicit link to the underlying theoretical frameworks presented in this paper such tests ultimately provide information about the relative merits of different decision models in modeling preferences of economic agents and financial investors.

5. APPENDIX

Proof of Lemma 1

Let $V(\underline{\alpha}) = E[U(W)] = E\left[U\left(W_0[(1-\underline{\alpha}^T\underline{e})r + \underline{\alpha}^T\underline{z}\right)\right]$, where $\underline{e}$ is a vector of ones. Since $U(W)$ is twice differentiable and the expectations operator is a positive linear mapping, $V(\underline{\alpha})$ is also twice differentiable. We get

$$V'(\underline{\alpha}) = E\left[U'(W)(\underline{z} - r\underline{e})\right]$$

and

$$V''(\underline{\alpha}) = E\left[U''(W)(\underline{z} - r\underline{e})^T(\underline{z} - r\underline{e})\right]$$

Since the support of $\underline{z}$ was supposed to cover $[a, b]$, there is positive probability of obtaining W for which $U''(W) < 0$. We know $U''(W) \leq 0$ elsewhere. Thus, since $E[(\underline{z} - r\underline{e})^T(\underline{z} - r\underline{e})]$ is positive definite, $V''(\underline{\alpha})$ is negative definite whenever $\underline{\alpha}^* \neq 0$. Since $U''(W)$ is continuous, the second partial derivatives are continuous and Theorem 4.4.10 on page 152 of Stoer and Witzgall [39] gives the result. In fact, the result may hold directly for $\underline{\alpha}^* = 0$ as well. In this case,

$$V''(\underline{\alpha}) = U''(W_0(1+r))E[\underline{z}^T\underline{z}]$$

This is negative definite if

$$U''(W_0(1+r)) < 0$$

i.e. if $W_0(1+r) \in [a, b]$. Then the result immediately holds since $V''(\underline{\alpha})$ would be negative definite everywhere. ■

Proof of Theorem 1

Consider individual k's maximisation problem when choosing an optimal portfolio. Denote by asset 0 the riskless bond paying $(1+r)$. If she assigns weights w_i^k to risky asset i with random variable rate of return $\widetilde{z}_i$,terminal wealth is

$$\widetilde{W}_k = W_{0k}\sum_{i=0}^{R} w_i^k(1+\widetilde{z}_i)$$

Define

$$P^k = \Pr\left(\widetilde{W}_k < \eta^k\right) = \Pr\left(\sum_{i=0}^{R} w_i^k\widetilde{z}_i < \frac{\eta^k}{W_{0k}}\right) \tag{24}$$

the probability that final wealth is below the target. Call this event E_1^k. Similarly define $(1- P^k)$ and event E_2^k. Lemma 1 applies due to the distributional assumption, and so her first order conditions are necessary and sufficient for a maximum. These are

$$E\left[(\widetilde{z}_i - r)U_k'\left(W_{0k}\sum_{i=0}^{R} w_i^k(1+\widetilde{z}_i)\right)\right] = 0 \tag{25}$$

which (in case 1) become

$$P_1^k E\left[(\widetilde{z}_i - r)\left[\lambda^k + \lambda_1^k\left(W_{0k}\sum_{i=0}^{R} w_i^k(1+\widetilde{z}_i) - \eta^k\right)^{\alpha}\right]/E_1^k\right]$$

$$+(1-P_1^k)E\left[(\widetilde{z}_i-r)\left[\lambda^k+\lambda_2^k\left(\eta^k-W_{0k}\sum_{i=0}^{R}w_i^k(1+\widetilde{z}_i)\right)^{\beta}\right]/E_2^k\right]=0 \tag{26}$$

where E is an expectations operator conditional on the respective events. The proof now runs in two stages. Suppose initially that $\eta^k=W_{0k}$. Clearly, (24) becomes

$$P_1^k=\Pr\left(\sum_{i=0}^{R}w_i^k(1+\widetilde{z}_i)<1\right) \tag{27}$$

which depends on k only in the choice variable and so the superscript can be dropped. A similar argument holds for the events. The first order conditions simplify to

$$P_1E\left[(\widetilde{z}_i-r)\left[\lambda^k+\lambda_1^k(W_{0k})^{\alpha}\left(\sum_{i=0}^{R}w_i^k(1+\widetilde{z}_i)-1\right)^{\alpha}\right]/E_1\right]$$

$$+(1-P_1)E\left[(\widetilde{z}_i-r)\left[\lambda^k+\lambda_2^k(W_{0k})^{\beta}\left(1-\sum_{i=0}^{R}w_i^k(1+\widetilde{z}_i)\right)^{\beta}\right]/E_2\right]=0 \tag{28}$$

Using (12) and simplifying gives

$$P_1E\left[(\widetilde{z}_i-r)\left[1+\left(1-\sum_{i=0}^{R}w_i^k(1+\widetilde{z}_i)\right)^{c_1}\right]/E_1\right]$$

$$+(1-P_1)E\left[(\widetilde{z}_i-r)\left[\left(\sum_{i=0}^{R}w_i^k(1+\widetilde{z}_i)-1\right)^{c_2}\right]/E_2\right]=0 \tag{29}$$

By Lemma 1, the function is concave and so the solution to this equation will be a unique portfolio containing both the riskless asset and the R gambles, which is independent of k. Call this solution $\underline{w}=(w_0,w_1,....w_R)$. This will form the risky portfolio of the two-fund separation.

To see this, consider the general case where $\eta^k\neq W_{0k}$. Suppose the agent initially puts a fraction $(1-f^k)$ of initial wealth into the safe asset and then optimally invests the other f^k in a portfolio $\underline{v}=(v_0,v_1,....v_R)$. Since the riskless asset is included in $\underline{v}$, this is without loss of generality. The optimal weights are now $\underline{x}=(x_0,x_1,....x_R)$, where

$$\begin{array}{ll} x_0^k=(1-f^k)+f^kv_0 & \\ x_i^k=f^kv_i & i\neq 0 \end{array} \tag{30}$$

and observe also that E_1^k can be written as

$$\sum_{i=0}^{R}x_i^k(1+\widetilde{z}_i)<\frac{\eta^k}{W_{0k}} \tag{31}$$

which by substitution of (30) yields

$$(1-f^k)(1+r)+f^k\sum_{i=0}^{R}v_i(1+\tilde{z}_i)<\frac{\eta^k}{W_{0k}} \tag{32}$$

Simplifying using (14), noting that $f^k>0$ by assumption gives

$$\sum_{i=0}^{R}v_i(1+\tilde{z}_i)<1 \tag{33}$$

Similarly, (24) becomes

$$P_1^k=\Pr\left(\sum_{i=0}^{R}v_i(1+\tilde{z}_i)<1\right) \tag{34}$$

so both events and probabilities are again independent of k, and the superscripts are dropped. Compare this to (27). Substituting $\underline{x}$ using (30) into the first-order conditions (26) gives

$$P_1E\left[(\tilde{z}_i-r)\left[\lambda^k+\lambda_1^k\left(W_{0k}\left[(1-f^k)(1+r)+f^k\sum_{i=0}^{R}v_i(1+\tilde{z}_i)\right]-\eta^k\right)^{\alpha}\right]/E_1\right]$$
$$+(1-P_1)E\left[(\tilde{z}_i-r)\left[\lambda^k+\lambda_2^k\left(\eta^k-W_{0k}\left[(1-f^k)(1+r)+f^k\sum_{i=0}^{R}v_i(1+\tilde{z}_i)\right]\right)^{\beta}\right]/E_2\right]=0 \tag{35}$$

Substituting for f^k and simplifying gives

$$P_1E\left[(\tilde{z}_i-r)\left[\lambda^k+\lambda_1^k(W_{0k}f^k)^{\alpha}\left(\sum_{i=0}^{R}v_i(1+\tilde{z}_i)-1\right)^{\alpha}\right]/E_1\right]$$
$$+(1-P_1)E\left[(\tilde{z}_i-r)\left[\lambda^k+\lambda_2^k(W_{0k}f^k)^{\beta}\left(1-\sum_{i=0}^{R}v_i(1+\tilde{z}_i)\right)^{\beta}\right]/E_2\right]=0 \tag{36}$$

Applying (12) and the definition of f^k, we can rewrite (36) as

$$P_1E\left[(\tilde{z}_i-r)\left[1+\left(\sum_{i=0}^{R}v_i(1+\tilde{z}_i)-1\right)^{\alpha}\right]/E_1\right]$$
$$+(1-P_1)E\left[(\tilde{z}_i-r)\left[1+\left(1-\sum_{i=0}^{R}v_i(1+\tilde{z}_i)\right)^{\beta}\right]/E_2\right]=0$$

which is identical to (29). Hence, by the uniqueness of the solution to (29), $\underline{v}=\underline{w}$ and 2-fund separation obtains. All investors choose a mix of the riskless asset and $\underline{w}$, the investors for whom $\eta^k=W_{0k}$ allocating all to the latter.

The proofs for cases (2) and (3) follow an identical route ■

Proof of Theorem 2

By Theorem 1, the representative agent will exhibit two-fund separation and she will place a fraction f in the risky portfolio $\underline{w}$. Note that since

$$f^k = \frac{1}{r}\left((1+r) - \frac{\eta^k}{W_{0k}}\right) \forall k$$

we have

$$\eta^k = (1+r)W_{0k} - rW_{0k}f^k \qquad \forall k$$

Summing over k gives

$$\sum_k \eta^k = (1+r)\sum_k W_{0k} - r\sum_k W_{0k}f^k$$

Dividing by $\sum\limits_k W_{0k}$ and rearranging gives

$$\frac{\sum\limits_k W_{0k}f^k}{\sum\limits_k W_{0k}} = \frac{1}{r}\left((1+r) - \frac{\sum\limits_k \eta^k}{\sum\limits_k W_{0k}}\right) = f$$

which means that

$$f = \frac{1}{r}\left((1+r) - \frac{\eta}{W_0}\right) = \frac{\sum\limits_k W_{0k}f^k}{\sum\limits_k W_{0k}} > 0 \tag{37}$$

so we can apply Theorem 1. Consequently, her demand for the riskless bond will be

$$d_0^R = W_0\left[(1-f) + fw_0\right] \tag{38}$$

i.e. the fractions of initial wealth put straight into the bond plus the fraction of the bond in the risky portfolio held. Also, she will demand

$$d_i^R = W_0 f w_i \tag{39}$$

Similarly, individual k demands

$$d_0^k = W_0^k\left[(1-f^k) + f^k w_0\right] \tag{40}$$

of the bond. Summing over k gives

$$\sum_k d_0^k = \sum_k W_{0k} - \sum_k W_{0k}f^k + w_0\sum_k W_{0k}f^k$$

In terms of the aggregate parameters this becomes

$$\sum_k d_0^k = W_0 - W_0 f + w_0 W_0 f$$

$$= W_0\left[(1-f) + fw_0\right] = d_0^R$$

Total demand for risky asset i is likewise

$$\sum_k d_i^k = \sum_k W_{0k}f^k w_i = w_i\sum_k W_{0k}f^k = W_0 f w_i = d_i^R$$

■

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