On the characterisation of investor preferences by changes in wealth

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Abstract. Traditional risk concepts have created links between specific partial orderings of gambles and the shape of investors utility functions, independently of initial wealth. However, when we want to consider complete orderings of gambles with respect to a specific utility function, wealth cannot be ignored. Bell (1988, 1995a and 1995b) has addressed this difficult question and characterised the specific functional form of utility functions which allow a finite number of switches between two arbitrary gambles over the entire range of initial wealth. Extending this analysis, the authors characterise a large set of utility functions with respect to their switching characteristics.

Keywords : Stochastic dominance, utility, switching between gambles.

In the seminal work of Pratt (1964) and Arrow (1971), decision making under uncertainty depends jointly upon three things; the utility functions, wealth and the distribution of gambles. In this triplet of considerable complexity, stochastic dominance (Hadar and Russell (1969) and (1971), Fishburn (1974), Bawa et.al (1985) and many others) has been a central idea in the assessment of the relative riskiness of two gambles, not least because the relationships, being independent of the level of initial wealth, greatly simplifies the analysis. For whole classes of utility functions and all wealth levels, stochastic dominance enables one to characterise the distributions which are preferred in some pairwise comparison. For example, if all investors with increasing and concave utility functions prefer gamble A to gamble B, then A is said to stochastically dominate B monotonically of the second degree. Hadar and Russell (1969) derive the set of distributions ordered by such a dominance criterion and show that it is only a partial ordering of gambles.

When we move to the situation where one investment does not stochastically dominate another, the role of initial wealth, hitherto irrelevant, becomes important. Although questions of general portfolio weight determination via the behaviour of notions of absolute and relative risk aversion have been examined, the question of which asset should be preferred in a complete ordering with respect to a specific utility function has not been explored a great deal in the economics literature. Bell (1988, 1995a and 1995b) has identified the specific functional form of utility functions with the property that, as initial wealth increases, preferences switch a finite number of times (typically once or not at all) between two arbitrary gambles. By coupling this with standard assumptions such as decreasing absolute risk aversion, he isolates a small set of desirable utility functions (see for example Bell (1988), Proposition 3). The purpose of our paper is to show that these results, however attractive they may be in reducing the number of 'desirable' utility functions to a small family, hinge on two features, namely additive gambles and an assumption we refer to as 'global' solutions to differential equations. When we allow for more general types of gambles and/or local solutions we generate a rather large class of utility functions compatible with switching. This is not to belittle Bell's contributions; we find the identification of utility functions by their gamble and switching characteristics most interesting.

The paper is organised as follows : Section 2 reviews the necessary concepts and discusses the desirable features of an investors utility function. Section 3 contains the main switching results and a discussion of the relationship to the rest of the literature while section 4 is reserved for conclusions.

1. Desirable properties of investor's utility

In this section, we review the basic concepts of investor preferences. Arrow (1971) and Pratt (1964) initially wrote of what were the desirable features of an investor's utility function. In particular, they argued that a utility function $U(W)$ defined on wealth, W, should be increasing and continuous. Furthermore, they introduced the Arrow-Pratt coefficient of absolute risk aversion

$$r_A(W) = -\frac{U''(W)}{U'(W)} \tag{1}$$

This is positive for increasing, concave $U(W)$, which would imply that you prefer a safe bet to a risky one with the same expected value. In addition, they proposed that $r_A(W)$ should be decreasing in W, so that your aversion to risk decreases as you get wealthier. The relative risk aversion coefficient $r_R(W) = Wr_A(W)$, which measures aversion to a multiplicative gamble, was introduced by Arrow. This should logically also be positive for all W, but whether it should be increasing or decreasing in W is not entirely obvious. Eeckhoudt and Gollier (1995) give a good discussion of these points. While these ideas have remained as fundamental concepts in the theory of risk and utility, it is noteworthy that no specific functional forms for the utility functions which satisfy any combination of the above properties is readily available.

For utility functions which are risk-averse everywhere we can say something further about attitude to gambles. If all investors with such increasing and concave utility functions prefer a gamble A to gamble B, then we say that A stochastically dominates B monotonically of the second order, which we denote by $A_{SSD}B$. First -, third -, and higher-order stochastic dominance are also concepts used in finance (see Huang and Litzenberger (1988), Chapter 2 for details). However, second order is most commonly used because of its tie to increasing, risk-averse utility and so is that upon which we focus our attention. Following Huang and Litzenberger (1988), Chapter 2, the necessary and sufficient conditions for $A_{SSD}B$ is that $E[A] \geq E[B]$ and $S(y) = \int_0^y [F_A(z) - F_B(z)]\, dz \leq 0$ for all y, where $F_I(z)$ is the distribution function of I. This links the distribution functions of gambles to desirable overall characteristics of utility functions. Rothschild and Stiglitz (1970) also derive properties on distribution functions which should lead to a natural ranking of gambles for risk-averse investors. In particular, they show that B differs from A by a series of Mean-Preserving-Spreads (i.e. weight is shifted to the tails without affecting the mean) if and only if $B = A + \epsilon$, where ϵ is a zero-mean noise uncorrelated with A if and only if all investors with concave utility functions prefer A to B. However, these characterisations require all gambles intended for comparison to have identical means and again, the role of initial wealth is ignored. As discussed earlier, if we wish to construct a complete ordering of arbitrary gambles for a given utility function, initial wealth plays an important role.

We next present a series of papers from the management science literature which addresses precisely this issue.

2. Preference switching

Bell (1988) defined the notion of an **n−switch utility function** as a function which has the property that, as initial wealth increases, we switch preferences between any two additive gambles at most n times. We denote by **n-switch preferences** the preferences underlying an n-switch utility function. Pedersen and Satchell (1997) provide a thorough examination of his theorems. There are at least three important applications of the switching property. One is that of discounted cash flows. Here the notion of switching is simply that of impatience where wealth becomes analogous to the timing of satisfaction. This is detailed in Bell (1988), page 1421-1423. Switching is also linked to issues of contextual uncertainty, i.e. the possibility of removing one of two uncertainties that affect your initial wealth prior to decision making. It is shown that there is a close relationship between contextual uncertainties and one-switching in Bell (1995b), Theorem 1.

Perhaps the most important property of one-switching, however, is the link to risk-return separation. In Bell (1995a), it is shown that utility functions are one-switch if and only if their expectations are a function of a risk measure, a return measure and initial wealth. This in turn identifies the set of risk measures which can be represented in this way by a suitable utility function. This approach was applied to multiplicative switching (to be defined) in Bell (1996), and the procedure was extended to take into account downside risk measures in Pedersen (1998). Jia and Dyer (1996 and 1997) derive risk-value models which allow a similar separation, though for risk measures derived from more theoretical considerations. In Pedersen and Satchell (1997), it is shown that this is equal to zero-switching between identical gambles and in current work, Pedersen (1998) links higher order switching to more general risk-return models.

The class of switching functions have only been completely classified in the case where gambles are defined additively and utility functions are infinitely differentiable and are global solutions to particular differential equations in a sense to be defined. We briefly introduce the results of Bell and extensions for the additive case before arguing the case for our extensions. By additive gambles, we mean that wealth is defined additively with respect to gambles, that is, final wealth $W = W_0 + \widetilde{X}$, where W_0 is initial wealth and $\widetilde{X}$ is the gamble. The proofs of the following two theorems can be found in the original text.

Theorem 1. ***(Bell (1988), Prop. 1)***

The only zero-switch utility functions for additive gambles are the exponential and the linear functions.

Theorem 2. ***(Bell (1988), Prop. 2)***

The only one-switch utility functions for additive gambles are

$$
\begin{aligned}
(a) \qquad U(W) &= aW^2 + bW + c \\
(b) \qquad U(W) &= ae^{bW} + ce^{dW}
\end{aligned}
$$

$$(c) \qquad U(W) = aW + be^{cW}$$

$$(d) \qquad U(W) = (aW + b)e^{cW} \tag{2}$$

for arbitrary constants a, b, c *and* d.

The utility functions (2) are identical to the utility functions found by Farquhar and Nakamura (1987) to satisfy their augmented constant exchange risk condition. Bell additionally shows that (b) and (c) are increasing and have absolute risk-aversion and decreasing absolute risk-aversion, so that they not only display the one-switch characteristics for arbitrary gambles but are consistent with second order stochastic dominance for gambles of identical mean. Although these have received most attention in terms of applications (see Bell (1995a and 1995b)), the result is extended to $n-$ switch additive functions (1988, Proposition 9), which are of the form

$$U(W) = f_0(W) + \sum_{i=0}^{K} e^{c_i W} f_i(W) \tag{3}$$

where $f_i(.)$ is a polynomial of order n_i such that

$$\sum_{i=0}^{K} n_i \leq n + 1 - K$$

and c_i is a constant for all i.

In many investment decisions, particularly when W cannot be negative, gambles are more meaningfully defined as fractions of initial wealth; that is, if you invest W_0 in a gamble $\widetilde{X}$ the return on $\widetilde{X}$ is a percentage, so that $W = W_0\widetilde{X}$. The set of switching functions for this alternative gamble representation is now presented.

2.1. Multiplicative gambles. The analysis of multiplicative shocks as opposed to additive shocks with respect to measuring risk and return was studied in detail in Bell (1996) and Jia and Dyer (1997). In this section we provide a full characterisation of what functions admit switching when risks are multiplicative rather than additive. We give precise mathematical conditions under which our results hold. We then examine the properties of the one-switch functions in detail with respect to monotonicity, risk aversion and decreasing risk aversion and provide an intuitive link to the additive case.

We start by introducing some further notation and definitions. For multiplicative gambles, zero-switching occurs whenever $\widetilde{X} \succeq \widetilde{Y}$ (or $EU(W_0\widetilde{X}) \geq EU(W_0\widetilde{Y})$) for all W_0. To follow Bell, we restrict our attention to utility functions in the class $D_{\infty,}$ i.e. the functions for which all derivatives exist. We also require the validity of the Taylor series

$$EU(W_0\widetilde{X}) = \sum_{i=0}^{\infty} \frac{W_0^i U^i(W_0) m_i}{i!} \tag{4}$$

where U^i is the ith derivative of $U(.)$ and $m_i = E[(\widetilde{X}-1)^i]$, the ith central moment of $\widetilde{X}$. The conditions for the existence of this expansion are given in the following Lemma, whose proof in the Appendix.

Lemma 1. *If we assume that*

$$\sum_{n=0}^{\infty} \frac{W_0^n U^n(W_0)(\widetilde{X}-1)^n}{n!} = U(W_0\widetilde{X}) \tag{5}$$

for all W_0 *and* $\widetilde{X}$ *almost everywhere and*

$$\sum_{i=0}^{\infty} \frac{W_0^n \left|U^n(W_0)\right| E\left|(\widetilde{X}-1)^n\right|}{n!} < \infty \tag{6}$$

then (4) is valid.

Although Bell's proof relies upon the existence of a similar series, he does not derive explicitly the necessary mathematical conditions. Note also that no explicit relationship between W_0 and $\widetilde{X}$ is required. It might be thought that these mathematical requirements are of little or no consequence. However, the conditions required for validity of (4) are quite stringent. As a counter-example to these conditions, consider constant absolute risk aversion (i.e. exponential utility) and gambles defined as log-normal multiplicative shocks. In this case the left hand side of (4) becomes $E[e^{-W_0\widetilde{X}}]$ where $\widetilde{X}$ is log-normal, i.e. the moment-generating function of a log-normal random variable. It is a known result in probability theory that this cannot be represented by an infinite Taylor series over its moments (see Durrett (1991), page 89-93 for details). Since log-normality and exponential utility are basic building blocks in finance, this counter-example is far from pathological.

We now present theorems for multiplicative gambles analogous to those in Bell (1988). The proofs are all relegated to the Appendix. Our new theorems start with the multiplicative zero-switch utility functions.

Theorem 3. *In the class of infinitely differentiable utility functions defined upon multiplicative gambles, with finite moments about* 1 *such that (4) holds, the only* *zero-switch utility functions are* $\ln W$ *and* bW^{α}, *when* $\widetilde{X}$ *is a multiplicative gamble such that* $E[\ln(\widetilde{X})]$ *and* $E[\widetilde{X}^{\alpha}]$ *exists and* b *and* α *are constants.*

Note that these are the additive zero-switch functions where $\ln W$ replaces W. We will encounter this point again later, but now turn to the one-switch functions.

Theorem 4. *Under equivalent conditions to theorem* 3, *and assuming all relevant expectations exist, the one-switch multiplicative risk utility functions are*

$$
\begin{aligned}
(a) \qquad U(W) &= a(\ln W)^2 + b\ln W + c \\
(b) \qquad U(W) &= aW^b + cW^d \\
(c) \qquad U(W) &= a\ln W + bW^c \\
(d) \qquad U(W) &= (a\ln W + b)W^c
\end{aligned} \tag{7}
$$

where a, b, c *and* d *are all constants.*

We thus have a class of one-switch utility functions with respect to multiplicative gambles. Under the continued assumption of infinite differentiability and global solutions, the result can however be generalised to $n-$switch utility functions by employing the same transformation as that utilised in the previous theorem, yielding

$$U(W) = f_0(\ln W) + \sum_{i=0}^{k} W^{c_i} f_i(\ln W) \tag{8}$$

where $f_i(.)$ is a polynomial of order n_i such that

$$\sum_{i=0}^{k} n_i \leq n + 1 - k$$

and c_i is a constant for all i. A proof of this is found in Pedersen and Satchell (1997). These are effectively sums of products of power functions and polynomials in $\ln W$. Not many of these have been used in the existing literature. However, given the reasons why the multiplicative risk framework is preferred in finance, it would appear that these could be given more consideration. We briefly turn to the isolation of desirable one-switch utility functions with the multiplicative gamble structure. There is no reason, however, why the following analysis could not be extended to functions displaying higher order switching. We get the following result, whose proof is in the Appendix.

Theorem 5. *If a decision maker*

(a) *prefers more to less*

(b) *wishes to obey the axioms of expected utility*

(c) *is risk averse at all wealth levels*

(d) *has decreasing absolute risk aversion at all wealth levels*

(e) *wishes to obey the multiplicative one-switch rule*

then the only feasible utility functions which are infinitely differentiable everywhere, are

$$U(W) = a \ln W - bW^{-c}$$

and

$$U(W) = -aW^{-b} - cW^{-d}$$

for positive numbers a, b, c and d.

These two functions appear in Bell (1996) as the only two infinitely differentiable, increasing, risk-averse and decreasingly risk averse utility functions which separate risk and return for multiplicative risks. Indeed, the conditions for separation are the same as one-switching and so the above result can be looked at in parallel to Bell (1996), Theorem 1. We now have two further utility functions which display globally desirable properties with respect to risk aversion and the ranking of gambles. Before extending our analysis to a third type of gambles, we examine the possibility of allowing local rather than global solutions.

2.2. Global versus local solutions. Bell's proofs in all of the above theorems assume that $U(W)$ is infinitely differentiable everywhere, which justifies the use of Taylor expansions, and that the solutions derived at are global solutions. While economists rarely require popular functions to be more than twice differentiable for analytical purposes, almost all functions used are in fact infinitely differentiable, so this may not seem unreasonable. Indeed, in his most recent work on this topic (1996), Bell argues in support of this assumption, that a utility function should be smooth and not contain target levels of wealth. In addition, Bell (1988) contains a section which claims to exclude trick solution (see the elegant Lemma in (1988), Section 6, pages 1423-1424), which results in a claim that no other utility functions, however obscurely constructed, may be a one-switch function.

However, a closer examination of the proof of Bell's main theorems reveals that switching in fact is a local property. In particular, the results so far presented do not require $U(W)$ defined on $[0,\infty)$ to have the same functional form over the whole range. Therefore, we could restrict the range of our utility functions to subsets of $[0,\infty)$, impose different functional forms, and then piece these together to generate new functions. A restriction we need to impose in some cases is that final wealth cannot jump to a different range of the utility functions than where initial wealth currently resides. To see this, we will demonstrate a couple of examples of the latter here before introducing a new class of affine gambles in the next section to expand upon the former.

Consider the utility function

$$U(W) = \left\{ \begin{array}{cc} \lambda W & W > \eta \\ \lambda W - (\eta - W)^{\alpha} & W \leq \eta \end{array} \right\} \tag{9}$$

for $\alpha > 2, \lambda > 0$ and where $\eta = W_0 + \tau$ for some returns target τ, is an additive zero-switch utility function. This utility function is the mean-lower partial moment utility function, introduced in Fishburn (1977). It is continuous, twice continuously differentiable, locally risk averse and decreasingly locally risk averse, but the third derivative is not defined at $W = \eta$ although at all other levels of wealth it is infinitely differentiable. The mean-lower partial moment decision rule congruent to (9) was related to second degree stochastic dominance in (1977). Note that it is of the form linear minus power function where the power function is defined over a restricted domain. The linear function is an additive zero-switch and the restricted power a zero-switch for an affine gamble (to be defined later). Pedersen (1998) shows that

(9) is an additive zero-switch function. This has interesting effects for the risk-return applications of switching in Bell (1995a) in that (9) will introduce downside risk measures in such a framework. If we use multiplicative gambles, we can show that (9) is one-switch. The proof of the following theorem is in the Appendix.

Theorem 6. *With multiplicative risks so that* $W = W_0\widetilde{X}$ *and* $\eta = W_0\tau$, *(9) is a one-switch utility function.*

These extensions also hold for other piecewise functions. The next section repeats the switching theorems by allowing the gambles to be affine functions of initial wealth. This will yield solutions which are global solutions but only defined over a restricted range.

2.3. Affine risk exposures. Rather than arguing for a straight multiplicative gamble, as was done in the previous sections, we will assume that an agent with initial wealth W_0 to invest, elects (or is forced) to hold a pre-determined fixed quantity $\overline{W} < W_0$ in a riskless asset paying interest rate r but takes a multiplicative gamble $\widetilde{X}$ with the remaining $(W_0 - \overline{W})$. This makes our strategy look very similar to what is known in finance as portfolio insurance, since at the end of an investment period, we have insured against wealth falling below $\overline{W}$. The above we call a **Lower Bound Affine Risk Exposure (LBARE)** gamble**.** Algebraically, our final wealth is given by

$$W = (W_0 - \overline{W})\widetilde{X} + \overline{W} \tag{10}$$

where $\overline{W} < W_0$. An LBARE gamble may well be an example of a gamble by an institution subject to external restrictions. We will set $r = 0$ for convenience without loss of generality. In this case, $\overline{W}$ is a benchmark wealth level which results if the gamble returns a zero contribution to total wealth. The following results rely upon a similar mathematical condition to that proved in Lemma 4, found by replacing W_0 by $W_0 - \overline{W}$. The proofs are in the Appendix.

Theorem 7. *For the class of LBARE gambles given by (10), the zero-switch infinitely differentiable functions are*

$$\begin{aligned} (a) \qquad U(W) &= \ln(W - \overline{W}) \\ (b) \qquad U(W) &= (W - \overline{W})^b \end{aligned} \tag{11}$$

for $W > \overline{W}$ and real number b.

As may have been observed by the reader, the function $(W - \overline{W})^b$ is one of the popular HARA functions (to be defined). A later section will illustrate how the HARA functions can be linked to different switching rules, some using these LBARE gambles. Note that the functions can be arbitrarily defined below the target, which implies that the set of zero-switch functions is uncountably infinite. Next we give the one-switch functions.

Theorem 8. *For the class of LBARE gambles given by (10), the infinitely differentiable one-switch utility functions are*

$$(a) \qquad U(W) = a\ln(W - \overline{W})^2 + b\ln(W - \overline{W}) + c$$

$$\begin{aligned}(b) \quad U(W) &= a(W-\overline{W})^b + c(W-\overline{W})^d \\ (c) \quad U(W) &= a\ln(W-\overline{W}) + b(W-\overline{W})^c \\ (d) \quad U(W) &= (a\ln(W-\overline{W}) + b)(W-\overline{W}) \qquad (12)\end{aligned}$$

for $W > \overline{W}$ and constants a, b and c.

These are yet another set of functions which display one-switching for a type of gamble which is directly applicable to finance, but nevertheless have not appeared in the literature. The properties of these functions are the same as those of the multiplicative functions and Theorem 5 can be used to classify them according to risk aversion and/or decreasing risk aversion. Replacing W by $(W-\overline{W})$ gives the relevant conditions. We choose not to investigate this further here, but note that the extension to the $n-$ switch case, is given by

$$U(W) = f_0[\ln(W-\overline{W})] + \sum_{i=0}^{k}(W-\overline{W})^{c_i} f_i[\ln(W-\overline{W})]$$

where $W > \overline{W}$ and $f_i(.)$ is a polynomial of order n_i such that

$$\sum_{i=0}^{k} n_i \leq n+1-k$$

The proof of this can be found in Pedersen and Satchell (1997).

We now define a different exposure, which is symmetric to the Lower Bound Affine Risk Exposures just discussed. Consider the **Upper Bound Affine Risk Exposures (UBARE)**

$$W = (\overline{W} - W_0)\widetilde{X} + (2W_0 - \overline{W}) \qquad (13)$$

for $W_0 < \overline{W} < 2W_0$, which ensures that wealth stays below $(2W_0 - \overline{W})$. This rule implies that we take on less risk as our initial wealth gets closer to the upper target. In other words, agents facing the choice of a riskless asset and a risky asset have a bliss point and as they approach it elect to shift out of the risky asset and into the riskless asset. By symmetry to the lower bound affine investment rule, we get the following corollary. The proof of this is exactly the same as for the previous theorems, replacing $(W - \overline{W})$ by $(\overline{W} - W)$, and consequently is omitted.

Corollary 1. *For the class of UBARE gambles in (13), the zero-switch utility functions are*

$$\begin{array}{lrcl} (a) & U(W) & = & \ln(\overline{W} - W) \\ (b) & U(W) & = & (\overline{W} - W)^b \end{array} \tag{14}$$

the one-switch

$$\begin{array}{lrcl} (a) & U(W) & = & a(\ln(\overline{W} - W))^2 + b\ln(\overline{W} - W) + c \\ (b) & U(W) & = & a(\overline{W} - W)^b + c(\overline{W} - W)^d \\ (c) & U(W) & = & a\ln(\overline{W} - W) + b(\overline{W} - W)^c \\ (d) & U(W) & = & (a\ln(\overline{W} - W) + b)(\overline{W} - W) \end{array} \tag{15}$$

and the $n-$ switch

$$f_0[\ln(\overline{W} - W)] + \sum_{i=0}^{k} (\overline{W} - W)^{c_i} f_i[\ln(\overline{W} - W)] \tag{16}$$

where $W < \overline{W}$ and $f_i(.)$ is a polynomial of order n_i such that $\sum_{i=0}^{k} n_i \leq n + 1 - k$. The parameters $a, b, c, d,$ and c_i for all i are all constants in the above.

These are the same as those in Theorem 9 only defined on the lower part of the distribution. The feature of moving out of the risky asset and into the riskless asset which characterise gambles from the UBARE class helps us to understand the well-known criticism of the quadratic utility that as you increase wealth your dollar position in the risky asset falls. In the zero-switch case (b) above, the results hold for $U(W) = -(\overline{W} - W)^b$ and $b = 2$ will then give the quadratic. The functions in Corollary 1 also help to cast light on a close relationship between switching and a popular general class of utility functions, which we now introduce.

2.4. Switching and the HARA functions. In order to get a better overview of the large number of functions we encountered so far, it will be convenient to introduce the **Hyperbolic Absolute Risk Aversion (HARA)** functions, defined by

$$U_{HARA}(W) = \frac{1-\gamma}{\gamma}\left[\frac{aW}{1-\gamma} + b\right]^{\gamma} \tag{17}$$

where $W > \frac{b(\gamma-1)}{a}$. This class is presented and analysed in depth in Ingersoll (1987) and Eeckhoudt and Gollier (1995). A large body of commonly used functions are special cases of the HARA class. In particular, (17) contains the constant absolute risk aversion functions

$$U_{CARA}(W) = -\left(\frac{1}{\alpha}\right)e^{-\alpha W} \tag{18}$$

derived from (17) by setting $\frac{a}{b} = \alpha$ and letting $\gamma \to \infty$, and the constant relative risk aversion functions

$$U_{CRRA}(W) = \left\{ \begin{array}{ll} \frac{W^{1-\alpha}}{1-\alpha} & \alpha > 0, \alpha \neq 1 \\ \ln W & \alpha = 1 \end{array} \right\} \tag{19}$$

which we get from (17) by setting $a = 1-\alpha, b = 0$ and $\gamma = \alpha$ (letting $\gamma \to 1$ gives the logarithm). The zero-switch additive functions (see Theorem 1) are the constant absolute risk aversion HARA functions (18), while the zero-switch multiplicative utility functions (Theorem 4) are the constant relative risk aversion HARA functions (19). Apart from the limit functions mentioned above, the HARA class contains several polynomial functions, including the quadratic, which was shown to be an additive one-switch (see (2)).

Additionally, the other one-switch additive functions are all defined as sums of, or sums of products of, separate HARA functions. Such functions arise when we consider representative agent utility functions where a social planner has allocated positive weights to utility functions and all agents have been given the same wealth. It should be clear that simple manipulation of (17) leads to $U(W) = \frac{(1-\gamma)^{1-\gamma}}{\gamma}[aW + b(1-\gamma)]^{\gamma}$ and by setting $a = 1$ and $b(1-\gamma) = -\overline{W}$ (or $a = -1$ and $b(1-\gamma) = \overline{W}$) we recover the zero-switch functions corresponding to the risk exposures (10) (or (13)) from the previous section. Note that the logarithmic functions are again obtained in the limit. It has thus been shown that all the HARA functions can be identified as either zero-switch multiplicative functions, $n-$ switch additive functions for some non-negative integer n or zero-switch affine risk exposure functions. The following table sums up this discussion - ARA stands for Absolute Risk Aversion and DARA for Decreasing Absolute Risk Aversion. Parameters refer to (17). The double (x, y) under Rule denotes gamble type x (either additive (A), multiplicative (M) or affine risk exposure (ARE)), and whether it is a zero- or one-switch, i.e. $y = 0$ or 1.

Table 1. The HARA functions and their switching rules

Function	Parameters	ARA	DARA	Rule
Log	$a = 1, b = 0, \gamma = 0$	$\frac{1}{W}$	Always	(M,0)
Power	$a = 1 - \beta, b = 0, \gamma = \beta$	$\frac{(1-\beta)}{W}$	$\beta < 1$	(M,0)
Quadratic	$a = 2\beta, b = 1, \gamma = 2$	$\frac{2\beta}{(1-2\beta W)}$	Never	(A,1)
Exponential	$\frac{a}{b} = \beta, \gamma = -\infty$	-β	Never	(A,0)
$(W - \overline{W})^{\beta}$	$a = 1, b(1-\gamma) = \overline{W}, \gamma = \beta$	$\frac{(1-\beta)}{(W-\overline{W})}$	$\beta < 1$	(LBARE,0)
$(\overline{W} - W)^{\beta}$	$a = -1, -b(1-\gamma) = \overline{W}, \gamma = \beta$	$\frac{(1-\beta)}{(\overline{W}-W)}$	$\beta < 1$	(UBARE,0)

(20)

In addition, the $n-$ switch additive gamble functions which are not in (17) can be modelled by the appropriate sum or product of HARA functions. As the HARA's are central to decision making under uncertainty, this would add support to the use of preference switching as an aid to understanding desirable utility functions. However, it simultaneously weakens the switching property since it is clear that the choice of gamble influences the set of switching functions quite substantially..

3. CONCLUSION

Our paper has demonstrated that the switching gamble characterisation can be used as a taxonomic device for utility which, unlike notions of stochastic dominance, gives an explicit role to the wealth level. It is not unique but helpful in analysing the complexity of preferences over different types of gambles and levels of wealth. It does not lead to a small number of desirable utility functions as Bell (1988) has claimed.

Indeed, by relaxing the assumptions of Bell and allowing for local solutions and/or alternative types of gambles, we have identified a large number of utility functions which allow any finite number of switches for specific types of gambles. In particular, we show that the HARA functions can be reconstructed from a series of switching theorems. Our analysis is by no means exhaustive. There are gambles that one could consider different from our multiplicative, LBARE and UBARE definitions for example. Furthermore, switching may not be an important behavioral characteristic to focus analysis around. Some other aspect of the interrelationship between wealth preference and gambles may work much better.

4. APPENDIX

Proof of Lemma 1

Note first that (6) gives

$$\begin{aligned} \infty &> \sum_{i=0}^{\infty} \frac{W_0^n \left|U^n(W_0)\right| E\left|(\widetilde{X}-1)^n\right|}{n!} \\ &= \sum_{i=0}^{\infty} \frac{\left|W_0^n U^n(W_0)\right| \left|\int \left|(\widetilde{X}-1)^n pdf(\widetilde{X})\right|\right|}{n!} \end{aligned}$$

since $E\left|(\widetilde{X}-1)^n\right|$ and $pdf(\widetilde{X})$ are both positive. This in turn implies that

$$\begin{aligned} \infty &> \sum_{i=0}^{\infty} \left| \frac{W_0^n U^n(W_0) \int (\widetilde{X}-1)^n pdf(\widetilde{X})}{n!} \right| \\ &= \sum_{i=0}^{\infty} \left| \int \frac{W_0^n U^n(W_0)(\widetilde{X}-1)^n pdf(\widetilde{X})}{n!} \right| \end{aligned} \tag{21}$$

We apply the Beppo-Levi theorem[1] in Asplund and Bungart (1966) (page 64-66) directly. Letting $f_n(\widetilde{X}) = \frac{U^n(W_0)(\widetilde{X}-1)^n}{n!} pdf(\widetilde{X})$ and

$$f(\widetilde{X}) = U(W_0\widetilde{X}) pdf(\widetilde{X})$$

[1] The Beppo-Levi theorem is not commonly used in economics. Formally, it says that if $\sum_{n=1}^{\infty} f_n$ is a series of integrable functions such that

$$\sum_{n=1}^{\infty} \int \mid f_n \mid dx < \infty$$

then $\sum_{n=1}^{\infty} f_n(x)$ converges for almost all x, and if additionally f is a function that equals $\sum_{n=1}^{\infty} f_n$ almost everywhere, then f is integrable and

$$\int f = \sum_{n=1}^{\infty} \int f_n dx$$

It is closely related to the convergence theorems of Lebesque (see, for instance, Grimmett and Stirgazer [14], page 160).

then (5) and (21) imply that the conditions of the Beppo-Levi theorem are satisfied. As a result, we get that $U(W_0\widetilde{X})pdf(\widetilde{X})$ is integrable and

$$\int U(W_0\widetilde{X})pdf = \sum_{n=0}^{\infty} \int \frac{U^n(W_0)(\widetilde{X}-1)^n}{n!} pdf(\widetilde{X})$$

which in turn implies that

$$E[U(W_0\widetilde{X})] = \sum_{n=0}^{\infty} \frac{U^n(W_0) \int (\widetilde{X}-1)^n pdf(\widetilde{X})}{n!} = \sum_{i=0}^{\infty} \frac{W_0^i U^i(W_0) m_i}{i!}$$

where U^i is the ith derivative of $U(.)$ and $m_i = E[(\widetilde{X}-1)^i]$, the ith central moment of $\widetilde{X}$ about 1. ■

Proof of Theorem 3

(Sufficiency) Consider any gamble $\widetilde{X}$. Clearly,

$$E[\ln(W_0\widetilde{X})] = \ln W_0 + E[\ln(\widetilde{X})]$$

and

$$E[b(W_0\widetilde{X})^\alpha] = bW_0^\alpha E[\widetilde{X}^\alpha]$$

so that, for all W_0,

$$E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] = E[\ln(\widetilde{X}) - \ln(\widetilde{Y})]$$

and

$$E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] = bW_0^\alpha\{E[\ln(\widetilde{X})] - E[\ln(\widetilde{Y})]\}$$

respectively. These are always positive or always negative, and so no preference switching can occur.

(Necessity)

Suppose that $\widetilde{X}$ and $\widetilde{Y}$ are gambles[2] and, without loss of generality, $\widetilde{X} \succeq \widetilde{Y}$. Clearly, for any wealth level W_0

$$
\begin{aligned}
& E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] \\
= & \sum_{i=1}^{\infty} U^i(W_0)W_0^i q_i = a(W_0) \qquad (22)
\end{aligned}
$$

where $q_i = \frac{E[(\widetilde{X}-1)^i]-E[(\widetilde{Y}-1)^i]}{i!}$. If $U(.)$ exhibits zero-switching, it must be that for any pair of wealth levels, W_0 and W_1, $a(W_0)a(W_1) \geq 0$. In particular, the restricted system

$$\sum_{i=1}^{2} U^i(W_0)W_0^i q_i = a(W_0) \qquad (23)$$

cannot have a solution for two different wealth levels W_0 and W_1 unless $a(W_0)a(W_1) \geq 0$. This implies that the system can not have a general solution, which requires the existence of a real non-zero solution θ to the differential equation

$$U''(W)W^2 + \theta U'(W)W = 0 \qquad (24)$$

for all W, since otherwise we could use (22) to find solutions to (23) for $a(W_0) < 0$ and $a(W_1) > 0$ which would imply a switch in preferences. Solving (24), we get

$$U(W) = \begin{cases} a \ln W & \theta = -1 \\ bW^{\theta-1} & \theta \neq -1 \end{cases} \qquad (25)$$

for constants a, b and θ.

[2]Notice that unlike Bell [5] we do not need the gamble to be 'suitably small'. Instead, we need only have that gambles cannot take a non-positive value.

Proof of Theorem 4

(Sufficiency)

To show that the four functions all satisfy the one-switch rule, we again consider each one in turn. It suffices to show that $E[U(W_0\widetilde{X})]-E[U(W_0\widetilde{X})]=0$ has at most one real solution for W_0.

(a) $E[U(W_0\widetilde{X})]-E[U(W_0\widetilde{X})]$

$$\begin{aligned}
&= E\{a(\ln(W_0\widetilde{X}))^2+b\ln(W_0\widetilde{X})+c-a(\ln(W_0\tilde{Y}))^2+b\ln(W_0\tilde{Y})-c\}\\
&= (2a\ln W_0+b)(E(\ln\widetilde{X})-E(\ln\tilde{Y}))+a(E(\ln\widetilde{X})^2-E(\ln\tilde{Y})^2)\\
&= 2a(E(\ln\widetilde{X})-E(\ln\tilde{Y}))\ln W_0+b(E(\ln\widetilde{X})-E(\ln\tilde{Y}))+a(E(\ln\widetilde{X})^2-E(\ln\tilde{Y})^2)\\
&= 0 \Leftrightarrow W_0=e^{\frac{-b(E(\ln\widetilde{X})-E(\ln\tilde{Y}))-a(E(\ln\widetilde{X})^2-E(\ln\tilde{Y})^2)}{2a(E(\ln\widetilde{X})-E(\ln\tilde{Y}))}}
\end{aligned}$$

(b) $E[U(W_0\widetilde{X})]-E[U(W_0\tilde{Y})]$

$$\begin{aligned}
&= aE(W_0\widetilde{X})^b+cE(W_0\widetilde{X})^d-aE(W_0\tilde{Y})^b-cE(W_0\tilde{Y})\\
&= aW_0^b(E(\widetilde{X}^b)-E(\tilde{Y}^b))+cW_0^d(E(\widetilde{X}^d)-E(\tilde{Y}^d))\\
&= 0 \Leftrightarrow W_0=\left(\frac{-c(E(\widetilde{X}^d)-E(\tilde{Y}^d))}{a(E(\widetilde{X}^b)-E(\tilde{Y}^b))}\right)^{\frac{1}{b-d}}
\end{aligned}$$

(c) $E[U(W_0\widetilde{X})]-E[U(W_0\tilde{Y})]$

$$\begin{aligned}
&= aE(\ln(W_0\widetilde{X}))+cE(W_0\widetilde{X})^d-aE(\ln(W_0\tilde{Y}))-cE(W_0\tilde{Y})^d\\
&= a(E(\ln\widetilde{X})-E(\ln\tilde{Y}))+cW_0^d(E(\widetilde{X}^d)-E(\tilde{Y}^d))\\
&= 0 \Leftrightarrow W_0=\left(\frac{-a(E(\ln\widetilde{X})-E(\ln\tilde{Y}))}{c(E(\widetilde{X}^d)-E(\tilde{Y}^d))}\right)^{\frac{1}{d}}
\end{aligned}$$

and finally,

(d) $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})]$

$$= E\{(a\ln(W_0\widetilde{X})) + b)(W_0\widetilde{X})^c\} - \{E(a\ln(W_0\widetilde{Y})) + b)(W_0\widetilde{Y})^c\}$$

$$= aW_0^c(\ln W_0)E(\widetilde{X}^c) + bW_0^cE(\widetilde{X}^c) + aW_0^cE(\widetilde{X}^c\ln\widetilde{X})$$

$$-aW_0^c(\ln W_0)E(\widetilde{Y}^c) - bW_0^cE(\widetilde{Y}^c) - aW_0^cE(\widetilde{Y}^c\ln\widetilde{Y})$$

$$= W_0^c\{(a\ln W_0 + b)(E(\widetilde{X}^c) - E(\widetilde{Y}^c)) + a(E(\widetilde{X}^c\ln\widetilde{X}) - E(\widetilde{Y}^c\ln\widetilde{Y}))\}$$

$$= 0 \Leftrightarrow W_0 = e^{\frac{-a(E(\widetilde{X}^c\ln\widetilde{X})-E(\widetilde{Y}^c\ln\widetilde{Y}))-b(E(\widetilde{X}^c)-E(\widetilde{Y}^c))}{a(E(\widetilde{X}^c)-E(\widetilde{Y}^c))}}$$

(Necessity)

Suppose that $H(V)$ is a one-switch utility function. Since (4) holds, this implies that for three distinct wealth levels, $V_1 > V_2 > V_3$, the corresponding numbers $a(V_1), a(V_2)$ and $a(V_3)$ (as defined by (22)), cannot satisfy $a(V_1) > 0, a(V_2) < 0$ and $a(V_3) > 0$ or $a(V_1) < 0, a(V_2) > 0$ and $a(V_3) < 0$. A necessary condition for this is that the $H'(V), H''(V)$ and $H'''(V)$ are collinear (since if not, one could solve the sub-system of equations (22) ($i = 1, 2, 3$) for ANY values of $a(V_i)$ by simple matrix invertion. This means we can solve

$$\theta_1 H'(V)V + \theta_2 H''(V)V^2 + H'''(V)V^3 = 0 \tag{26}$$

for θ_1 and θ_2. Note the simple extension to (24) offered by (26). To solve this, we use a simple transformation motivated by the observation that the four sufficient functions we seek are in fact equal to Bell's one-switch functions when V is replaced by ln$V$. By letting $W = \ln V$ and $H(V) = U(\ln V)$, we get

$$H'(V) = U'(W) * \frac{dW}{dV} = U'(W) * \frac{1}{V} \tag{27}$$

Thus,

$$H''(V) = U''(W) * \frac{dW}{dV} * \frac{1}{V} - \frac{1}{V^2} * U'(W)$$

so that

$$H''(V) = \frac{1}{V^2}[U''(W) - U'(W)] \tag{28}$$

Finally, it can be shown by a similar argument that

$$H'''(V) = \frac{1}{V^3}[U'''(W) - 3U''(W) + U'(W)] \tag{29}$$

Substituting (27), (28) and (29) into (26) and simplifying then gives

$$U'(W)[2 - \theta_1 - \theta_2] + U''(W)[\theta_2 - 3] + U'''(W) = 0$$

Setting $\phi_1 = 2 - \theta_1 - \theta_2$ and $\phi_2 = \theta_1 - 3$, which is a one-to-one mapping, this can be simplified to

$$\phi_1 U'(W) + \phi_2 U''(W) + U'''(W) = 0$$

as required. The real solutions to this equation are given by (2) and setting $W = \ln W$ yields the result. ■

Proof of Theorem 5

Proof. By Theorem 4, only utility functions in (7) satisfy the one-switch rule. All of these obey the axioms of expected utility. Hence (b) and (e) are satisfied. Now consider the properties $(a), (c)$ and (d) for each function in turn. For $U(W) = a(\ln W)^2 + b\ln W + c$, $U'(W) = \frac{2a\ln W + b}{W}, U''(W) = \frac{2a(1-\ln W)-b}{W^2}, r(W) = \frac{1}{W}\left[1 - \frac{2a}{(2a\ln W+b)}\right]$ and $r'(W) = \frac{1}{W^2}\left[\frac{4a^2}{(2a\ln W+b)^2} + \frac{2a}{(2a\ln W+b)} - 1\right]$. $U(W)$ will be increasing and risk-averse for $W > e^{1-\frac{b}{2a}}$. However, this would imply $r'(W) > 0$.

Hence, this function can not be both risk-averse and have decreasing risk aversion. For $U(W) = aW^b + cW^d$, we have $U'(W) = abW^{b-1} + cdW^{d-1}$, $U''(W) = ab^2W^{b-2} + cd^2W^{d-2}, r(W) = \frac{-ab^2W^{b-2} - cd^2W^{d-2}}{abW^{b-1} + cdW^{d-1}}$ and $r'_M(W) = \frac{a^2b^3W^{2b-4} + c^2d^3W^{2d-4} + abcd(b+d)W^{b+d-4}}{(abW^{b-1} + cdW^{d-1})}$ Hence, we can get $U'(W) > 0, r_M(W) > 0$ and $r'_M(W) < 0$ for all $W \geq 0$ by selecting a, b, c, d all negative. When $U(W) = a \ln W + bW^c, U'(W) = \frac{a}{W} + bcW^{c-1}, U''(W) = \frac{-a}{W^2} + bc(c-1)W^{c-2}, r(W) = \frac{1}{W}\left[\frac{a - bc(c-1)W^c}{a + bcW^c}\right] = \frac{1}{W}\left[1 - \frac{bc^2W^c}{a + bcW^c}\right]$ and $r'(W) = \frac{1}{W}\left[-\frac{1}{W} + \frac{bc^2W^c}{(a + bcW^c)} - \frac{abc^3W^c}{(a + bcW^c)^2}\right]$. This means we again can ensure $U'(W) > 0, r_M(W) > 0$ and $r'_M(W) < 0$ for all $W \geq 0$, this time by selecting $a > 0, b < 0$ and $c < 0$. The fourth and final case is $U(W) = (a \ln W + b)W^c$, for which $U'(W) = W^{c-1}[a(1 + \ln W) + b], U''(W) = W^{c-2}[a(2 + \ln W) + b], r(W) = -\frac{[a(2+\ln W)+b]}{W[a(1+\ln W)+b]} = \frac{1}{W}\left[1 - \frac{a}{[a(2+\ln W)+b]}\right]$ and $r'(W) = \frac{1}{W^2}\left[\left(\frac{a}{a(2+\ln W)+b}\right)^2 + \left(1 - \frac{a}{[a(2+\ln W)+b]}\right)\right]$. Hence $U(W)$ is increasing and risk averse only when $W > e^{-\frac{b}{a}}$. However, this is exactly the condition which makes the second bracketed term in $r'(W)$ positive. Hence, this function cannot be increasing, risk-averse and decreasingly risk averse over any wealth region. Thus, only the two utility functions mentioned in the theorem satisfies all the axioms ■

Proof of Theorem 6

Recall (9), but now assume $\eta = W_0\tau$, (τ now being a multiplicative returns target) and consider two gambles $\widetilde{X}$ and $\widetilde{Y}$. We get

$$EU(W_0\widetilde{X}) > EU(W_0\widetilde{Y})$$

$$\Leftrightarrow \Pr\{W_0\widetilde{X} > \eta\}E[\lambda(W_0\widetilde{X}) \mid W_0\widetilde{X} > \eta]$$

$$+\Pr\{W_0\widetilde{X} < \eta\}(E[\lambda(W_0\widetilde{X}) - (\eta - W)^\alpha \mid W_0\widetilde{X} < \eta])$$

$$> \Pr\{W_0\widetilde{Y} > \eta\}E[\lambda(W_0\widetilde{Y}) \mid W_0\widetilde{Y} > \eta]$$

$$+\Pr\{W_0\widetilde{Y} < \eta\}(E[\lambda(W_0\widetilde{Y}) - (\eta - W)^\alpha \mid W_0\widetilde{Y} < \eta])$$

$$\Leftrightarrow \lambda E[\widetilde{X}] - W_0^\alpha \Pr\{\widetilde{X} < \tau\}E[(\tau - \widetilde{X})^\alpha \mid \widetilde{X} < \tau]$$

$$> \lambda E[\widetilde{Y}] - W_0^\alpha \Pr\{\widetilde{Y} < \tau\}E[(\tau - \widetilde{Y})^\alpha \mid \widetilde{Y} < \tau]$$

$$\Leftrightarrow W_0^\alpha > \frac{\lambda(E[\widetilde{Y}] - E[\widetilde{X}])}{\Pr\{\widetilde{Y}<\tau\}E[(\tau-\widetilde{Y})^\alpha|\widetilde{Y}<\tau] - \Pr\{\widetilde{X}<\tau\}E[(\tau-\widetilde{X})^\alpha|\widetilde{X}<\tau]}$$

so that preference between $\widetilde{X}$ and $\widetilde{Y}$ switch at most once.

Proof of Theorem 7

We prove this theorem directly, although one can employ the multiplicative zero-switch functions for an alternative approach.

(Sufficiency)

For (a), $E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})]$

$$\begin{aligned} &= E[\ln[(W_0 - \overline{W})\widetilde{X}][-E[\ln[(W_0 - \overline{W})\widetilde{Y}]] \\ &= \ln(W_0 - \overline{W}) + \ln(\widetilde{X}) - \ln(W_0 - \overline{W}) - \ln(\widetilde{Y}) \\ &= \ln(\widetilde{X}) - \ln(\widetilde{Y}) \end{aligned}$$

which is independent of W_0. Likewise for (b), we get $E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})]$

$$= E[[(W_0 - \overline{W})\widetilde{X}]^b] - E[[(W_0 - \overline{W})\widetilde{Y}]^b]$$

$$= \quad (W_0 - \overline{W})^b (E[\widetilde{X}^b] - E[\widetilde{Y}^b])$$

which will not change sign as W_0 increases, since $\overline{W} < W_0$.

(Necessity)

Following Theorem 3, expanding using a Taylor series around $\widetilde{X} = 1$ gives

$$\begin{aligned} & E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{Y} + \overline{W})] \\ = & \sum_{i=1}^{\infty} U^i (W_0 - \overline{W})(W_0 - \overline{W})^i m_i \end{aligned} \tag{30}$$

where $m_i = \frac{E[(\widetilde{X}-1)^i] - E[(\widetilde{Y}-1)^i]}{i!}$. A necessary condition for $U(.)$ to be zero-switch is that the system

$$\begin{aligned} \sum_{i=1}^{\infty} U^i (W_0 - \overline{W})(W_0 - \overline{W})^i m_i &= a(W_0) \\ \sum_{i=1}^{\infty} U^i (W_1 - \overline{W})(W_1 - \overline{W})^i m_i &= a(W_1) \end{aligned}$$

does not have a solution where $a(W_0)a(W_1) < 0$. Obviously this will then also have to hold for the smaller set

$$\begin{aligned} \sum_{i=1}^{2} U^i (W_0 - \overline{W})(W_0 - \overline{W})^i m_i &= a(W_0) \\ \sum_{i=1}^{2} U^i (W_1 - \overline{W})(W_1 - \overline{W})^i m_i &= a(W_1) \end{aligned}$$

which in turn requires that we cannot find a general solution to this sub-system, i.e. that the first two derivatives of $U(.)$ are collinear. This means that

$$U''(W - \overline{W}) + \theta U'(W - \overline{W}) = 0$$

which has as solution the two functions in (11) ■

Proof of Theorem 8

(Sufficiency)

To show sufficiency, we substitute for wealth as defined in (10) and show there is at most one switch between two different gambles $\widetilde{X}$ and $\tilde{Y}$. To simplify proceedings, we note the following relationship : If $U(W)$ is a multiplicative function, then $U(W) = U(W_0\widetilde{X})$. If $U(W)$ is a function defined on mixed gambles where final wealth is one of the above, then by definition $U[W - \overline{W}] = U[(W_0 - \overline{W})\widetilde{X}]$. Hence, we can simplify our analysis by referring to the sufficiency conditions in Theorem 4 and replace W_0 by $(W_0 - \overline{W})$. Since $W_0 > \overline{W}$ all logs and fractional powers are still defined. We now solve

$$(a)\quad E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\tilde{Y} + \overline{W})]$$

$$= (2a\ln(W_0 - \overline{W}) + b)(E(\ln\widetilde{X}) - E(\ln\tilde{Y})) + a(E(\ln\widetilde{X})^2 - E(\ln\tilde{Y})^2)$$

$$(b)\quad E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\tilde{Y} + \overline{W})]$$

$$= a(W_0 - \overline{W})^b(E(\widetilde{X}^b) - E(\tilde{Y}^b)) + c(W_0 - \overline{W})^d(E(\widetilde{X}^d) - E(\tilde{Y}^d))$$

$$(c)\quad E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\tilde{Y} + \overline{W})]$$

$$= a(E(\ln\widetilde{X}) - E(\ln\tilde{Y})) + c(W_0 - \overline{W})^d(E(\widetilde{X}^d) - E(\tilde{Y}^d))$$

$$(d)\quad E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\tilde{Y} + \overline{W})]$$

$$= (W_0 - \overline{W})^c\{(a\ln(W_0 - \overline{W}) + b)(E(\widetilde{X}^c) - E(\tilde{Y}^c)) + a(E(\widetilde{X}^c\ln\widetilde{X}) - E(\tilde{Y}^c\ln\tilde{Y}))\}$$

By the arguments in Theorem 4, all of these will switch sign at most once as W_0 increases.

(Necessity)

We again appeal to a connection to Theorem 4. When final wealth is defined as

$W = (W_0 - \overline{W})\widetilde{X} + \overline{W}$, consider

$$\begin{aligned} & E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{Y} + \overline{W})] \\ = & \sum_{i=1}^{\infty} U^i(W_0 - \overline{W})(W_0 - \overline{W})^i m_i \end{aligned} \tag{31}$$

by the Taylor series around $\widetilde{X} = 1$. A necessary condition for a utility function of this type to be one-switch is that the system

$$\begin{aligned} \sum_{i=1}^{\infty} U^i(W_0 - \overline{W})(W_0 - \overline{W})^i m_i &= a(W_0) \\ \sum_{i=1}^{\infty} U^i(W_1 - \overline{W})(W_1 - \overline{W})^i m_i &= a(W_1) \\ \sum_{i=1}^{\infty} U^i(W_2 - \overline{W})(W_2 - \overline{W})^i m_i &= a(W_2) \end{aligned}$$

has no solution for a restricted range of the $a(W_i)-$ functions and different wealth levels W_0, W_1 and W_2. Following the proof of Theorem 5, we need to ensure the subsystem consisting of the lowest derivative terms (replace $\sum_{i=1}^{\infty}$ by $\sum_{i=1}^{3}$) cannot have a general solution, which in turn requires us to have the following linearity condition

$$\phi_1 U'(W - \overline{W})(W - \overline{W}) + \phi_2 U''(W - \overline{W})(W - \overline{W})^2 + U'''(W - \overline{W})(W - \overline{W})^3 = 0 \tag{32}$$

Now let $V = W - \overline{W}$ and $H(V) = U(W - \overline{W})$. Note that $H'(V) = U'(W - \overline{W})\frac{d(W_I - \overline{W})}{dV} = U'(W - \overline{W})$, $H''(V) = U''(W - \overline{W})\frac{d(W - \overline{W})}{dV} = U''(W - \overline{W})$ and $H'''(V) = U'''(W - \overline{W})\frac{d(W - \overline{W})}{dV} = U'''(W - \overline{W})$, since $\frac{d(W - \overline{W})}{dV} = 1$. Hence, (32) becomes

$$\phi_1 H'(V)V + \phi_2 H''(V)V^2 + H'''(V)V^3 = 0$$

which is identical to (26). We know the real solutions to this equation are the functions in (7). Transforming these back in terms of $U(W - \overline{W}) = H(V)$, and using (10), we get the solution as $U(W - \overline{W})$ as required. The functions which give one-switching when final wealth is defined as in (10) are those in (12) ■

References

[1] Arrow, K. (1971), *Essays in the theory of risk bearing.* Markham, Chicago.

[2] Asplund, K. and Bungert, L. (1996), *A first course in integration.* Holt, Rinehart and Winston

[3] Bawa, V., Bodurtha, J., Rao, M. and Suri, H. (1985), On determination of stochastic dominance optimal sets. *The Journal of Finance*, 40(2):417–431.

[4] Bawa, V. and Lindenberg, E. (1977), Capital market equilibrium in a mean-lower partial moment framework. *Journal of Financial Economics 5*, pages 189–200.

[5] Bell, D. (1988), One-switch utility functions and a measure of risk. *Management Science*, 34(12):1416–1424.

[6] Bell, D. (1995a), A contextual uncertainty condition for behaviour under risk. *Management Science*, pages 1145–1150.

[7] Bell, D. (1995b), Risk, return and utility. *Management Science*, 41(1):23–30.

[8] Bell, D. (1996), Measuring risk and return of portfolios. In *Wise Choices.* HBS Publishing.

[9] Durrett, R. (1991), *Probability : Theory and examples.* Wadsworth Publishiing Company.

[10] Eeckhoudt, L. and Gollier, C. (1995), *Risk : Evaluation, management and sharing.* Prentice Hall/Harvester Wheatleaf.

[11] Farquhar, P. and Nakamura, Y. (1987), Constant exchange risk properties. *Operations Research*, pages 206–214.

[12] Fishburn, P. (1974), Convex stochastic dominance with continuous distribution functions. *Journal of Economic Theory,* 7:143-158.

[13] Fishburn, P. (1977), Mean-risk analysis with risk associated with below-target returns. *The American Economic Review*, 67(2):116–126, March.

[14] Grimmett, G. and Stirgazer, D. (1992), *Probability and Random Processes.* Clarendon Press, Oxford, 2nd edition.

[15] Hadar, J. and Russell, W. (1969), Rules for ordering uncertain prospects. *American Economic Review*, 59:25–34.

[16] Hadar, J. and Russell, W. (1971), Stochastic dominance and diversification. *Journal of Economic Theory*, 3:288–305.

[17] Huang, C. and Litzenberger, R. (1988), *Foundations for financial economics.* Elsevier Science Publishing Co., Inc.

[18] Ingersoll, J. (1987), *Theory of financial decision making.* Rowman

[19] Jia, J. and Dyer, J. (1996), A standard measure of risk and risk-value models. *Management Science*, 42(12):1691–1705.

[20] Jia, J. and Dyer, J. (1997), Relative risk-value models. *European Journal of Operational Research.*

[21] Pedersen, C. (1998), *Topics in risk in finance.* PhD thesis, Trinity College, University of Cambridge, England, (To be submitted November 1998).

[22] Pedersen, C. and Satchell, S. (1997). Risk, utility and switching between gambles. *DAE Working Paper, Amalgamated Series No. 9735, University of Cambridge.*

[23] Pratt, J. (1964), Risk aversion in the small and the large. *Econometrica*, 32(1-2):122–136.

[24] Rothschild, M. and Stiglitz, J. (1970), Increasing risk : I. a definition. *Journal of Economic Theory*, 2(3):225–243.