UTILITY FUNCTIONS WHOSE PARAMETERS DEPEND ON INITIAL WEALTH

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Abstract. Conventional one-period utility functions in Economics assume that initial wealth only enter preferences through the definition of final wealth. Consequently, those utility functions most utilised (i.e. exponential and quadratic) have implausible risk characteristics. The authors characterise a new class of utility function whose parameters depend upon initial wealth. Several desirable results are obtained. In particular, investors with quadratic and exponential utility functions can have decreasing risk aversion and will increase their share of the risky asset as they get wealthier. The authors conclude with a number of applications demonstrating the efficacy of this new class of utility functions.

1. Introduction

The purpose of this paper is to present a new class of utility function which has the feature that the parameters that constitute part of their specification depend explicitly upon initial wealth. That utility functions could have such a property was

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recognised as early as 1951 by Mosteller and Nogee [27], who comment upon their experimental investigation into choice under uncertainty: *"One possible criticism (of their experiments) could be that the subject changes his utility curve with these changes in capital, so that each decision he makes depends on the amount of money he has on hand at that particular moment"*. Their results provided the motivation for the Consistency Axiom used by Pfanzagl [30] in his derivation of a theory of measurable utility, and led to the literature on wealth-dependent preference switching, originating with Bell [3].

A motivation for this class is the well-known fact that the few tractable utility functions available to economists typically have undesirable risk characteristics - for example, the exponential utility function has constant absolute risk aversion whilst quadratic utility has increasing absolute risk aversion. Arrow [2] has shown that in an optimal choice over two assets, one riskless and the other risky, investors with such risk characteristics would keep constant or decrease their holdings in the risky asset as wealth increases. This has long been seen as unrealistic and could easily contaminate the many applications of these utility functions to solving decision problems.

Our suggested utility functions eliminate this problem by specifying extra relationships between initial wealth and the coefficients that measure risk without departing from the expected utility framework. Although more general theories of choice under uncertainty exist, this particular approach has not, to the best of our knowledge, been explicitly formulated before. Indeed, within a more general theoretical framework, it is hard to place the antecedents of these ideas. One may relate the concept of initial-wealth dependent utility to recent developments in economic theory, which consider aggregation issues by discussing the shape of the wealth distribution across agents and model jointly utility and wealth (see for instance Grandmont [12] and Hildenbrand [15]). However, our own approach in this paper is not to focus on equilibrium issues. In section 2, we present details of where these ideas have appeared in the literature, define our class of utility function and give a definition of the associated measure of risk-aversion. In section 3, we specialise to the HARA class and present conditions under which exponential and quadratic utility have decreasing absolute risk aversion. We show that the Cass and Stiglitz [4] necessity result on fund separation no longer holds and investigate the effect on comparative statics of optimal portfolio problems. Section 4 illustrates the flexibility of this approach by considering as an example an extension to a simple principal-agent model which enable us to account for stylized facts on the relationship between asset managers and pension funds. Section 5 is reserved for our conclusions.

2. UTILITY FUNCTIONS WHOSE PARAMETERS DEPEND UPON INITIAL WEALTH

In conventional theory, utility $U(\cdot)$ is defined exclusively in terms of terminal wealth, $\widetilde{W}$, which in turn is defined as

$$\widetilde{W} = W_0 \left(1 + \widetilde{X}\right) \tag{1}$$

for initial wealth W_0 and the rate of return, $\widetilde{X}$, on a portfolio or gamble. Choice under uncertainty is then analysed by maximising the expectation of $U(\widetilde{W})$ subject to the appropriate budget constraints. In such a framework, W_0 only enters the utility functions through $\widetilde{W}$ although there is no underlying theoretical reason why it should not be allowed to enter separately. We thus present our general class of utility function.

Definition 1. *The class of utility functions whose parameters depend on initial wealth,* W_0*, is given by*

$$U\left(W_0, \widetilde{W}\right) \tag{2}$$

where the first component refers to when initial wealth enters the utility function independently of $\widetilde{W}$*.*

To interpret such a utility function, we find the following ideas helpful. Expected utility theory defines preferences over a set of gambles for a given wealth level W_0. It is clearly possible to conceptualise what ones preferences over the same set of gambles would be if wealth were changed. If we now vary wealth over some index set, then $U(W_0, \widetilde{W})$ represents the family of utility functions defined over that set. Comparative statics that involve changing attitudes to risk as W_0 vary must presumably have such a model in mind - our definition adds no complication except to make the functional dependence explicit through other channels than final wealth.

Expected utility theory tells us that, in a context where initial wealth is fixed, it is only final wealth that our preferences are based on. To the authors knowledge, it has not been established anywhere in the literature that the underlying utility function which forms the basis for the expected utility function has to be independent of endowments, since this preserves linearity in probabilities. Indeed, Mas-Colell et al. [24], page 185 (top), remark: *"Although the axioms (of expected utility) yield the expected utility representation they place no restrictions whatsoever on the form of the utility function".*

Utility functions of the form given by (2) have already appeared in both the Economics and Finance literature although the authors are not aware of any formal treatment of such until the present. For instance, Grauer [13] defines a generalised logarithmic utility function $U(W_0, \widetilde{W}) = \ln\left(\widetilde{W} - \frac{W_0}{2}\right)$ which forms the basis for a general asset pricing model. A further case in Cho, Edison and West [5] - who use a generalised quadratic utility function, is discussed later. Functions in (2) also arise naturally when working with target-based preferences, where utility functions defined on returns take different forms above and below a preset threshold (see for instance Fishburn [11] and Holthausen [16]). In Satchell [36] and Pedersen [28], such piecewise utility functions are taken from the management science literature and applied to asset pricing. This involves transforming utility functions defined on returns using (1), implying that constant returns targets are transformed to wealth targets that vary with initial wealth, which locates the resulting utility function defined on wealth in the general class (2). This notion of transforming from utility of return to utility of wealth is discussed in more detail in Ingersoll [19], pages $216 - 218$.

Another area of possible interest might be to portfolio turnpike theory (see Ross [35] and Huberman and Ross [18]). Portfolio turnpike theory is the study of multi-period investment strategies that maximise final period expected utility of wealth and where the strategy is simplified so that the impact of changing wealth through the investment periods is not considered; in Ross [35] the strategies are defined by constant portfolio weights for the multiple holding periods. It may well be possible to extend existing results to look at certain turnpike problems where the wealth variation changes the risk characteristics of the utility function.

Functions in (2) have also occurred in analysis by financial practitioners, where the expected utility function does not necessarily satisfy the Von Neumann - Morgernstern axioms. Consider the standard mean-variance utility function

$$V\left(\mu, \sigma^2\right) = E\left[U(\widetilde{W})\right] = \mu - \lambda\sigma^2 \tag{3}$$

where μ is the expected return, σ the standard deviation of returns and λ a constant trade-off. This was introduced in Markowitz [23] and forms a basis for the Capital Asset Pricing Model derived independently by Sharpe [38], Lintner [22] and Mossin [26]. The mean-variance utility function is justified by assuming either underlying quadratic utility or multivariate elliptical returns (see Chamberlain [6]). Recently, Van Eaton and Conover [9] generalised (3) to

$$V\left(\mu, \sigma^\beta\right) = E\left[U(\widetilde{W})\right] = \mu - \lambda\sigma^\beta \tag{4}$$

As an example of the usefulness of our approach, we will demonstrate how a utility function such as (4) might arise. Firstly, risk-value models of the form

$$E\left[U(\mu)\right] - \lambda(\mu)E\left[U(\widetilde{X} - \mu)\right] \tag{5}$$

were introduced in Jia and Dyer ([20] and [21]) and Dyer and Jia [8]. When $U(\widetilde{W}) = \widetilde{W}^\beta, \lambda(\mu) = \lambda$ and $\widetilde{X}$ is a member of the location-scale family, we obtain (4) from (5). Alternatively, we could consider an underlying utility function in the class (2) of the form

$$U\left(W_0, \widetilde{W}\right) = \widetilde{W} - \phi(W_0)\left|\widetilde{W} - E(\widetilde{W})\right|^\beta \tag{6}$$

Then clearly

$$\begin{aligned} U\left(W_0, \widetilde{W}\right) &= W_0\left(1 + \mu\right) - \phi(W_0)W_0^\beta E\left(\left|\widetilde{r} - \mu\right|^\beta\right) \\ &= \mu - \phi(W_0)W_0^{\beta-1}E\left(\left|\widetilde{r} - \mu\right|^\beta\right) \end{aligned} \tag{7}$$

where we assume that the utility function is unique up to affine transformations although this may not necessarily be consistent with the non-linear probability structure in (6). If $\widetilde{r} \sim N\left(\mu, \sigma^2\right)$ and we let $\widetilde{z} = \widetilde{r} - \mu$, then $\widetilde{z} \sim N\left(0, \sigma^2\right)$ and

$$E\left(\left|\widetilde{r} - \mu\right|^{\beta}\right) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} z^{\beta} e^{-\frac{z^2}{2\sigma^2}} dz \tag{8}$$

Consider the substitution $v = \frac{z^2}{2\sigma^2}$. This implies that $z = \sigma\sqrt{2v}$ and $dz = \frac{\sigma}{\sqrt{2v}} dv$ so (8) becomes

$$\begin{aligned} E\left(\left|\widetilde{r} - \mu\right|^{\beta}\right) &= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} \left(\sigma\sqrt{2v}\right)^{\beta} e^{-v} \frac{\sigma}{\sqrt{2v}} dv \\ &= \frac{2^{\frac{\beta}{2}}\sigma^{\beta}}{\sqrt{\pi}} \int_0^{\infty} v^{\frac{\beta-1}{2}} e^{-v} dv \\ &= \frac{2^{\frac{\beta}{2}}\sigma^{\beta}}{\sqrt{\pi}} \Gamma\left(\frac{\beta+1}{2}\right) \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function (see, for instance, DeGroot [7], page 286). We then obtain (4) from (7) by setting

$$\phi(W_0) = \frac{\sqrt{\pi}}{2^{\frac{\beta}{2}} \Gamma\left(\frac{\beta+1}{2}\right) W_0^{\beta-1}} \tag{9}$$

Thus we are able to support the choice of expected utility function by giving precise sufficient conditions on the underlying preferences, endowments and the distribution of returns when abstracting to the more general framework.

We do not claim that the cited works are the only ones in the literature which use members of our new class of utility functions. It is obvious that where used, the purpose has been one of capturing effects not customarily associated with traditional utility functions, often to make the analysis wealth-independent. We now turn our attention to the how the added initial wealth effect implicit in (2) affects some fundamental results in Economics. We will start by discussing how the definition of absolute risk aversion should be amended in this more general framework.

2.1. Definition of absolute risk aversion.

For our utility function in (2) we will refer to

$$R_A\left(W_0\right)=-\frac{U_{0,2}\left(W_0,W_0\right)}{U_{0,1}\left(W_0,W_0\right)} \tag{10}$$

where $U_{i,,j}$ denotes differentiating i times with respect to the first argument and j times with respect to the second argument, as the *Conventional Arrow-Pratt coefficient of Absolute Risk Aversion (CARA).* This is invariant to positive affine transformations of $U(W_0,\widetilde{W})$. However it is no longer interpretable as a quantity proportional to the risk premium, $\pi(W_0,\widetilde{X})$, the amount of wealth an individual is willing to forego to avoid a mean-zero additive gamble $\widetilde{X}$, for utility functions in (2). In this case, one solves for the risk premium from the equation

$$U\left(W_0-\pi\left(W_0,\widetilde{X}\right),W_0-\pi\left(W_0,\widetilde{X}\right)\right)=E\left[U\left(W_0,W_0+\widetilde{X}\right)\right] \tag{11}$$

where E denotes the expectations operator. In (11), the risk premium enters the left hand side twice as we forego initial wealth while only once in the right hand side since the gamble taken only affects terminal wealth and not initial wealth. We thus get the following theorem, proved in the Appendix.

Theorem 1. *With utility given as in (2), the risk premium for a small gamble* $\widetilde{X}\sim\left(0,\sigma^2\right)$ *is given by*

$$\pi\left(W_0,\widetilde{X}\right)=-\frac{1}{2}*\sigma^2*\frac{U_{0,2}\left(W_0,W_0\right)}{U_{1,0}\left(W_0,W_0\right)+U_{0,1}\left(W_0,W_0\right)} \tag{12}$$

It is immediately obvious that when the first component is zero or, more generally, W_0 does not affect utility independently of $\widetilde{W}$, equation (12) reduces to equation (10) as $U_{1,0}\left(W_0, W_0\right) = 0$. From this it is immediate that the expression corresponding to the (CARA) for (2) is

$$R_A^G(W_0) = -\frac{U_{0,2}\left(W_0, W_0\right)}{U_{1,0}\left(W_0, W_0\right) + U_{0,1}\left(W_0, W_0\right)} \tag{13}$$

which we dub the *Generalised Arrow-Pratt measure of Absolute Risk Aversion (GARA).* The following lemma, which is proved in the Appendix, shows that this generalised measure of risk aversion is also invariant to affine transformations of the utility function.

Lemma 1. *The generalised Arrow-Pratt measure of absolute risk aversion (13) is invariant to affine transformations of the utility function.*

We make further assumptions about $U(W_0, \widetilde{W})$ starting with the following standard properties.

Assumption 1 : $U_{1,0}\left(W_0, W_0\right) + U_{0,1}\left(W_0, W_0\right) > 0$

Assumption 2 : $U_{0,1}\left(W_0, W_0\right) > 0$

Assumption 3 : $U_{0,2}\left(W_0, W_0\right) < 0$

The first two assumptions imply that an increase in initial wealth increases utility and that the contribution to this increase through final wealth is strictly positive. It is an interesting consideration that an initial wealth increase could in fact decrease the components which do not include final wealth (i.e. we could have $U_{1,0}\left(W_0, W_0\right) < 0$). The third assumption is the standard concavity assumption which, combined with the second assumption, simply asserts that conventional absolute risk aversion holds.

We can further use (11) to find exact conditions under which the risk premium is decreasing in initial wealth. We get the following result, also proved in the Appendix.

Theorem 2. *When utility is given by (2) and* ***Assumption*** 1 *holds, the risk premium decreases in initial wealth if and only if*

$$\begin{aligned} &E\left(U_{0,1}\left(W_0, W_0+\widetilde{X}\right)+U_{1,0}\left(W_0, W_0+\widetilde{X}\right)\right) \\ &>U_{1,0}\left(W_0-\pi\left(W_0, \widetilde{X}\right), W_0-\pi\left(W_0, \widetilde{X}\right)\right) \\ &+U_{0,1}\left(W_0-\pi\left(W_0, \widetilde{X}\right), W_0-\pi\left(W_0, \widetilde{X}\right)\right) \end{aligned} \tag{14}$$

Note that in the case $U_{1,0}(.,.) = 0$ the result collapses to the traditional characterisation in the conventional framework. Also, for utility functions which are three times differentiable in both arguments, (14) can be interpreted as imposing certain conditions on the third partial derivatives. In conventional theory, decreasing risk aversion implies that the third derivative is positive, as can be seen easily from (10). The changes in the general approximation (12) with changes in wealth will similarly imply restrictions on the third partial derivatives of our general utility function (2). We now turn to more specific functions in order to illustrate some of the advantages of our new framework with respect to results in the theory of Financial Economics.

3. HARA UTILITY, FUND SEPARATION AND TWO-ASSET PORTFOLIOS

In this section, we first specialise to the HARA utility functions, which are described in detail in Eeckhoudt and Gollier [10], and then select the exponential and quadratic functions for more detailed analysis. The HARA class are most often used in Economics because they include most commonly-used utility functions (exponential, quadratic, power and logarithmic). The first issue we address concerns this class as a whole. Cass and Stiglitz [4] proved that necessary and sufficient conditions for two fund money separation without distributional assumptions or joint assumptions on preferences and endowments are that utility satisfies

$$U_k'(\widetilde{W}) = \left(a_k + b_k\widetilde{W}\right)^c \tag{15}$$

for constants a_k and b_k or an exponential function, which is one way to characterise the HARA's. While a_k and b_k can vary across individuals, who are indexed by k, they do not depend explicitly upon the endowments (i.e. initial wealth) of the agents. To see how this makes a difference, we consider the following simple example taken from Pedersen [29]: suppose that individuals have a utility function which satisfies

$$U_k'(\widetilde{W}) = \lambda_k + \left(b_k\widetilde{W}\right)^c \tag{16}$$

which does not satisfy (15) and so does not give separation. Indeed, we will not get separation even if $\lambda_k = \lambda$ for all k and $b_k = b$ for all k. However, if we let $\lambda_k = b_k^c W_{0k}^c$, then

$$U_k'(\widetilde{W}) = b_k^c W_{0k}^c + \left(b_k\widetilde{W}\right)^c$$

so that

$$\begin{aligned} U_k(\widetilde{W}) &= b_k^c W_{0k}^c \widetilde{W} + \frac{b_k^c}{c+1}\widetilde{W}^{c+1} \\ &= b_k^c W_{0k}^{c+1}\left(1 + \widetilde{X} + \frac{1}{c+1}\left(1+\widetilde{X}\right)^{c+1}\right) \\ &= \left(1 + \widetilde{X} + \frac{1}{c+1}\left(1+\widetilde{X}\right)^{c+1}\right) \end{aligned}$$

where the last equality follows from the fact that utility is unique up to positive affine transformations. Thus, all individuals have identical preferences and choose the same fund. This is trivially a special case of two-fund separation; consequently, when making parameters depend on wealth, the necessity result of Cass and Stiglitz [4] does not apply, and the class of utility functions which suffice for fund separation is enlarged.

Next, we specialise further and introduce the general quadratic and exponential utility functions.

Definition 2. *The general quadratic is given by*

$$U\left(W_0, \widetilde{W}\right) = \widetilde{W} - b\left(W_0\right)\widetilde{W}^2 \tag{17}$$

and the general exponential is given by

$$U\left(W_0, \widetilde{W}\right) = -e^{-b(W_0)\widetilde{W}} \tag{18}$$

where $b(W_0)$ is a function of initial wealth satisfying $b(W_0) \geq 0$ for all W_0. Further, $b(W_0)$ is infinitely differentiable - that is, $b^r(W_0)$ exists, where $b^r(W_0)$ denotes the rth derivative of $b(W_0)$.

Cho, Edison and West [5], page 36, use *"....quadratic utility with constant absolute risk aversion equal to one"*. This would imply that

$$\frac{2b(W_0)}{1 - 2b(W_0)W_0} = 1$$

in our notation, which solves for

$$b(W_0) = \frac{1}{2(W_0 + 1)}$$

so that, implicitly, b is a function of W_0, and their utility function is thus located in our general class (2).

Note that although we move to a more general framework the quadratic is still consistent with mean-variance preferences. One major criticism of the quadratic utility function in the conventional framework is that it displays increasing risk aversion; similarly the exponential utility function has constant absolute risk aversion. This problem of asset allocations in a two-asset world is addressed in the next section. First we show that the generalised exponential and quadratic functions can give both decreasing general absolute risk aversion (DGARA) and decreasing conventional absolute risk aversion (DCARA). The proof of the following theorem is in the Appendix.

Theorem 3. *The general quadratic (17) has DGARA iff*

$$2W_0b(W_0)b'(W_0)+W_0^2b(W_0)b''(W_0)+W_0b'(W_0)-W_0^2\left(b'(W_0)\right)^2+b'(W_0)-b(W_0)+2\left(b(W_0)\right)^2<0 \tag{19}$$

and DCARA iff

$$b'(W_0)+2\left(b(W_0)\right)^2<0 \tag{20}$$

The general exponential (18) has DGARA iff

$$2b'(W_0)W_0+2\left(b'(W_0)\right)^2-b(W_0)-b(W_0)b''(W_0)<0 \tag{21}$$

and DCARA iff

$$b'(W_0)<0 \tag{22}$$

The interpretation of this result for the quadratic case is that an increase in initial wealth needs to decrease the risk-return trade-off sufficiently to off-set the tendencies of agents with quadratic utility to prefer riskless assets as we approach the bliss point $\widehat{W}=\frac{1}{2b(W_0)}$. Another interpretation is that an increase in initial wealth moves the bliss point by decreasing $b(W_0)$; moving the bliss point further away makes risky assets more attractive, which implies a decreasing aversion to risk. Taken together, these considerations make a convincing case for $b'(W_0) < 0$.

We next consider - as a specialisation of Theorem 3 - a power function representation for $b(W_0)$ in both the extended exponential and extended quadratic functions. This will allow us to derive more specific conditions in which desirable risk characteristics are attainable, and explore how these characteristics vary in wealth. We state the following corollary without proof.

Corollary 1. *Suppose that*

$$b(W_0) = W_0^{\alpha} \tag{23}$$

Then (19) is satisfied if $\alpha < 0$ *and*

$$(1-\alpha)W_0 - \alpha > (\alpha+2)W_0^{\alpha+1}$$

while (20) is satisfied if $\alpha < 0$ *and*

$$W_0 > \left(-\frac{2}{\alpha}\right)^{-\left(\frac{1}{\alpha+1}\right)}$$

Also (21) is satisfied if $\alpha < 0$ *and*

$$W_0 > \left(\frac{\alpha(\alpha-1)}{1-2\alpha}\right)^{\frac{1}{2-\alpha}}$$

while (22) is satisfied if $\alpha < 0$.

Hence the selection of a power representation for $b(W_0)$ allows decreasing risk aversion - in both the new and conventional sense and in both exponential and quadratic utility - without forcing an overly parametric relationship between wealth and utility.

As a natural extension of this, we now turn to optimal asset allocations in a world with a riskless and a risky asset.

3.1. One riskless and one risky asset.

There are only a small number of utility functions which admit closed form solutions for optimal asset allocations when making non-trivial distributional assumptions, and even fewer allowing for such for all distributions. In this section we examine how our new general exponential and quadratic functions ((17) and (18) respectively) can produce more appropriate results and comparative statics with respect to initial wealth. We consider an individual with utility function $U(W_0, \widetilde{W})$ facing an investment decision where she needs to allocate an optimal amount, a, of initial wealth, W_0, to a risky asset with rate of return, $\widetilde{r}$, and the remaining amount, $W_0 - a$, is invested in a safe asset with fixed rate of return, denoted by r_f. Clearly, final wealth will then be given by

$$\widetilde{W} = W_0(1 + r_f) + a(\widetilde{r} - r_f)$$

and the standard optimisation problem is then

$$\max_a U\left(W_0, W_0(1 + r_f) + a(\widetilde{r} - r_f)\right) \tag{24}$$

Before moving to specific cases we present a general result which, for our framework, corresponds to the standard result in portfolio theory - proved in Arrow [2] and Huang and Litzenberger [17], Chapter 2 - that with decreasing risk aversion, the fraction allocated to the risky asset decreases in initial wealth. The following result is proved in the Appendix.

Theorem 4. *If the general utility function (2) has DCARA holds*

$$E\left(U_{1,1}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)\right) > 0$$

then

$$\frac{\partial a}{\partial W_0} > 0$$

If (2) has ICARA and

$$E\left(U_{1,1}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)\right) < 0$$

then

$$\frac{\partial a}{\partial W_0} < 0$$

Once more conditions reduce to the conventional result in Arrow [2] when we remove the independent initial wealth term. We now use this result to demonstrate how asset allocations with the general quadratic and exponential functions differ from the conventional set-up. The next two theorems are proved in the Appendix.

Theorem 5. *For the general quadratic utility function in (17), the optimal allocation in equity is*

$$a = \frac{\mu(1 - 2cb(W_0))}{2b(W_0)\left(\mu^2 + \sigma^2\right)} \tag{25}$$

where $c = W_0(1 + r_f)$, $\mu = E\left(\widetilde{r} - r_f\right) > 0$ *and* $\sigma^2 = Var\left(\widetilde{r} - r_f\right)$. *Furthermore*

$$\frac{\partial a}{\partial W_0} > 0 \text{ iff } b'(W_0) < -4(1 + r_f)\left(b(W_0)\right)^2 \tag{26}$$

This shows that under certain circumstances we can have an increase in the allocation to equity as we get wealthier when using quadratic utility. This corresponds to the earlier result on decreasing risk aversion. We get a similar result for general exponential utility.

Theorem 6. *For the general exponential utility function (18) the optimal fraction invested in equity,* a*, increases in initial wealth if and only if*

$$\frac{\partial\left(W_0 b(W_0)\right)}{\partial W_0} < 0 \tag{27}$$

Consequently the exponential utility function will no longer produce wealth-independent optimal asset allocations, which has been its main shortcoming in earlier theory. Since it is immediate that this generalisation is at no extra cost in terms of computational or algebraic complexity, we have essentially merged one of the most fundamental properties of investor behaviour with the exponential utility representation without incurring undue complications. Given the popularity of the exponential utility in Financial Economics precisely due to its mathematical tractability, we consider the generalised exponential utility function (18) one which merits the attention of theorists and practitioners alike.

4. The Principal-Agent Problem

As a further example of the extra explanatory power of our model, we consider how the principal-agent model can, by adapting such generalised preference representations, explain some stylized facts regarding the behaviour of pension funds and asset management companies, as of yet unexplained by the traditional model. This draws upon the work of Ross ([33] and [34]) and Mirrlees [25], whilst our explanation closely follows that of Rees [32].

For illustrative purposes we shall consider the principal (P) as a pension fund and the agent (A) as an asset management company. In the simplest version we wish to find a schedule of fees that P agrees to pay A to manage the assets of P. We shall assume that A's action is fixed at a_0 (for other possible structures, see Rees [32]). There is a random variable θ with p.d.f. $f(\theta)$, where $0 \leq \theta \leq 1$, corresponding to the state of the world. P has utility function $u(\cdot)$ whose argument depends on $(x - y)$, where $x = x(a_0, \theta)$ is the random return resulting from A's actions and $y = y(\theta)$ is the fee rate paid to A. A has a utility function $v = v(a_0, y(\theta))$. For ease of exposition, we assume that no other additional sources of income are available and that terminal wealth and income are equivalent. P chooses fees for A so as to maximise P's utility subject to the constraint that A achieves some fixed level of utility, $\overline{v}$. Formally, this corresponds to solving

$$\max_{y(\theta)} \int_0^1 u(x(a_0, \theta) - y(\theta)) pdf(\theta) d\theta$$

subject to

$$\int_0^1 v((a_0, y(\theta)) pdf(\theta) d\theta \geq \overline{v}$$

Standard results show that for the solution to this problem, y^*, satisfies

$$\frac{dy^*}{d\theta} = \frac{r_P}{r_P + r_A} * \frac{dx}{d\theta}$$

where r_p is the conventional Arrow-Pratt risk aversion coefficient for P and similarly, r_A is the conventional Arrow-Pratt risk aversion coefficient for A. If we assume exponential utility for both P and A so that r_P and r_A are constants, the pay schedule is thus linear, i.e.

$$y^*(\theta) = \beta + \frac{r_P}{r_P + r_A} x(a_0, \theta) \tag{28}$$

Interestingly, linear fee schedules are often what are found in practice, at least for $x \geq 0$. What is also interesting is the tendency for small pension funds to pay fixed fees and for the industry to be attracted to larger asset management companies, especially in recent periods when the market has been more volatile (see Pratten and Satchell [37] for details). As it stands, the above linear schedule (28) gives no insights into these stylised facts.

However, if we allow the parameters of the respective utility functions to additionally depend on the initial wealth levels (denoted W_0^P for the principal and W_0^A for the agent) in such a way that $\frac{dr_P}{dW_0^P} > 0$ and $\frac{dr_A}{dW_0^A} < 0$, conditions for which were deduced in Theorem 3, we get the following implications: firstly, as the wealth of the pension fund increases, the coefficient on $x(a_0, \theta)$ in the optimal schedule (28) increases. This implies that larger pension funds will be more prepared to agree to variable fees whilst smaller pension funds will be more inclined to agree to fixed fees. Secondly, the larger the asset management company, the closer to risk-neutrality it will be, so the tendency will be for the pension fund to accept a fixed return from larger asset managers while leaving the variable return as payment for the asset manager's services - so in more volatile markets larger asset management companies would be more attractive to pension funds as a source of reducing risk.

Finally, we note that this example has focused only on risk-sharing. Whilst simplistic in that issues such as moral hazard and incentive compatibility are ignored, it illustrates the extent to which explicit solutions to key economic problems can be extended by the use of initial-wealth dependent utility functions. We leave to future research the examination of what impact this may have on more complicated principal-agent frameworks..

5. Conclusion

The paper has considered how to extend utility functions to incorporate initial wealth into the risk parameters. Whilst the mathematics of the extensions are straightforward, the conclusions that follow seem to be helpful in eliminating some of the unrealistic features of the small number of tractable utility functions available to students of decision making under uncertainty. Our work is not exhaustive; the list of previous examples of the use of initial-wealth dependent utility functions in the literature - together with the new examples presented the paper - does not cover all useful applications. For instance, in a multi-period setting our utility functions are related to those used in works on habit formation (initially popularised by Pollak [31], see more recent references in Alessie and Lusardi [1]), where utility functions are defined over not only current consumption but also the fraction of wealth not consumed in the previous period. This is also a feature of the household technology preference structure of Hansen and Sargent [14], who make the bliss-points of their multi-period quadratic utility functions depend upon past consumption. We hope that since our framework still yields closed-form solutions to the corresponding optimisation problems in the traditional framework, extensions made immediate by using our new class of utility functions will enable researchers to resolve possible anomalies with regards to changes in wealth in these and other areas of Economics.

6. APPENDIX

PROOF OF THEOREM 1

By using a Taylor expansion of both sides of (11) around $\widetilde{X}=0$ (or equivalently $\widetilde{W}=W_0$), we get

$$\begin{aligned}
& \left(U_{0,1}\left(W_0,W_0\right)+U_{1,0}\left(W_0,W_0\right)\right)\pi\left(W_0,\widetilde{X}\right) \\
& +\frac{1}{2}\left(U_{2,0}\left(W_0,W_0\right)+2U_{1,1}(W_0,W_0)+U_{0,2}(W_0,W_0)\right)\pi^2\left(W_0,\widetilde{X}\right) \\
= \; & E\left[\left(U_{0,1}\left(W_0,W_0\right)+U_{1,0}\left(W_0,W_0\right)\right)\widetilde{X}+\frac{1}{2}\left(U_{0,2}\left(W_0,W_0\right)\widetilde{X}^2\right)\right] \qquad (29)
\end{aligned}$$

Note that the terms $U_{2,0}(W_0,W_0)$ and $U_{1,1}(W_0,W_0)$ do not appear in the expansion of the right hand side of (11) since $\widetilde{X}$ only enters the second argument of the function. Taking expectations and - following the derivation of the conventional Arrow-Pratt coefficient - truncating the series at π^2 (see, for instance, Eeckhoudt and Gollier [10], pages $32-33$), recalling that $E[\widetilde{X}]=0$ and $E[\widetilde{X}^2]=\sigma^2$, (29) becomes

$$\left(U_{0,1}\left(W_0,W_0\right)+U_{1,0}\left(W_0,W_0\right)\right)\pi\left(W_0,\widetilde{X}\right)=\frac{1}{2}\sigma^2U_{0,2}\left(W_0,W_0\right)$$

which rearranges for the result ■

PROOF OF LEMMA 1

Let

$$U^*\left(W_0,\widetilde{W}\right)=\alpha+\beta U\left(W_0,\widetilde{W}\right)$$

Then, clearly

$$U^*_{i,,j}\left(W_0,\widetilde{W}\right)=\beta U_{i,,j}\left(W_0,\widetilde{W}\right)$$

for all i and j, so

$$\frac{U^*_{0,2}\left(W_0,W_0\right)}{U^*_{1,0}\left(W_0,W_0\right)+U^*_{0,1}\left(W_0,W_0\right)} = \frac{\beta U_{0,2}\left(W_0,W_0\right)}{\beta U_{1,0}\left(W_0,W_0\right)+\beta U_{0,1}\left(W_0,W_0\right)}$$

$$= \frac{U_{0,2}\left(W_0, W_0\right)}{U_{1,0}\left(W_0, W_0\right) + U_{0,1}\left(W_0, W_0\right)}$$

as required ■

PROOF OF THEOREM 2

Differentiating (11) with respect to W_0, and writing π for $\pi(W_0, \widetilde{X})$, we get

$$\begin{aligned} & \left[U_{1,0}\left(W_0 - \pi, W_0 - \pi\right) + U_{0,1}\left(W_0 - \pi, W_0 - \pi\right)\right] * \left(1 - \frac{\partial \pi}{\partial W_0}\right) \\ = \;& E\left[U_{1,0}\left(W_0, W_0 + \widetilde{X}\right) + U_{0,1}\left(W_0, W_0 + \widetilde{X}\right)\right] \end{aligned}$$

which solves for

$$\frac{\partial \pi}{\partial W_0} = 1 - \frac{E\left[U_{1,0}\left(W_0, W_0 + \widetilde{X}\right) + U_{0,1}\left(W_0, W_0 + \widetilde{X}\right)\right]}{U_{1,0}\left(W_0 - \pi, W_0 - \pi\right) + U_{0,1}\left(W_0 - \pi, W_0 - \pi\right)}$$

Hence, $\frac{\partial \pi}{\partial W_0} < 0$ iff

$$\begin{aligned} & E\left[U_{1,0}\left(W_0, W_0 + \widetilde{X}\right) + U_{0,1}\left(W_0, W_0 + \widetilde{X}\right)\right] \\ > \;& U_{1,0}\left(W_0 - \pi, W_0 - \pi\right) + U_{0,1}\left(W_0 - \pi, W_0 - \pi\right) \end{aligned}$$

as required ■

PROOF OF THEOREM 3

For notational simplicity, we shall write $U_{i,j}$ for $U_{i,j}(W_0, W_0)$ and b for $b(W_0)$. For the general quadratic (17), we get $U_{1,0} = \widetilde{W} - b'\widetilde{W}^2$, $U_{0,1} = 1 - 2b\widetilde{W}$ and $U_{0,2} = -2b$. Hence, (13) becomes

$$R_A^G(W_0) = \frac{2b}{W_0 - b'W_0^2 + 1 - 2bW_0}$$

and so

$$\begin{aligned} \frac{\partial R_A^G(W_0)}{\partial W_0} &= \frac{2b'\left(W_0 - b'W_0^2 + 1 - 2bW_0\right) - 2b\left(1 - W_0^2 b'' - 2W_0 b' - 2b'W_0 - 2b\right)}{\left(W_0 - b'W_0^2 + 1 - 2bW_0\right)^2} < 0 \\ &\Leftrightarrow 2b'\left(W_0 - b'W_0^2 + 1 - 2bW_0\right) - 2b\left(1 - W_0^2 b'' - 2W_0 b' - 2b'W_0 - 2b\right) < 0 \\ &\Leftrightarrow 2bb'W_0 + W_0^2 bb'' + W_0 b' - W_0^2\left(b'\right)^2 + b' - b + 2b^2 < 0 \end{aligned}$$

Also, the conventional Arrow-Pratt measure (10) is

$$R_A(W_0) = \frac{2b}{1-2bW_0}$$

so

$$\begin{aligned} \frac{\partial R_A(W_0)}{\partial W_0} &= \frac{2b'\left(1-2bW_0\right)-2b\left(-2W_0b'-2b\right)}{\left(1-2bW_0\right)^2} < 0 \\ &\Leftrightarrow 2b'\left(1-2bW_0\right)-2b\left(-2W_0b'-2b\right) < 0 \\ &\Leftrightarrow b'+2\left(b'\right)^2 < 0 \end{aligned}$$

For the general exponential (18), $U_{1,0} = \widetilde{W}b'e^{-b\widetilde{W}}$, $U_{0,1} = be^{-b\widetilde{W}}$ and $U_{0,2} = -b^2e^{-b\widetilde{W}}$. Hence, (13) becomes

$$R_A^G(W_0) = \frac{b^2}{W_0+b'}$$

implying

$$\begin{aligned} \frac{\partial R_A^G(W_0)}{\partial W_0} &= \frac{2bb'\left(W_0+b'\right)-b^2(1+b'')}{\left(W_0+b'\right)^2} < 0 \\ &\Leftrightarrow 2bb'W_0+2b\left(b'\right)^2-b^2-b^2b'' < 0 \\ &\Leftrightarrow 2b'W_0+2\left(b'\right)^2-b-bb'' < 0 \end{aligned}$$

since $b > 0$ by assumption. Also, the conventional Arrow-Pratt measure (10) is

$$R_A(W_0) = b$$

so that

$$\frac{\partial R_A(W_0)}{\partial W_0} < 0 \Leftrightarrow b' < 0 \blacksquare$$

PROOF OF THEOREM 4

The first-order condition of (24) with respect to a is

$$E\left(U_{0,1}\left(W_0,\widetilde{W}\right)\left(\widetilde{r}-r_f\right)\right) = 0 \tag{30}$$

Recalling that $\widetilde{W} = W_0(1+r_f) + a(\widetilde{r} - r_f)$, differentiating (30) with respect to W_0 gives

$$E\left[U_{1,1}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)\right] + E\left[U_{0,2}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)\left((\widetilde{r} - r_f)\frac{\partial a}{\partial W_0} + (1+r_f)\right)\right] = 0$$

Noting that a is non-stochastic, we get

$$\frac{\partial a}{\partial W_0} = \frac{-\left(E\left[U_{1,1}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)\right] + E\left[U_{0,2}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)(1+r_f)\right]\right)}{E\left[U_{0,2}\left(W_0, \widetilde{W}\right)(\widetilde{r} - r_f)^2\right]}$$

We know that DCARA implies that $E\left[U_{0,2}(W_0, \widetilde{W})(\widetilde{r} - r_f)(1+r_f)\right] > 0$ (see Huang and Litzenberger [17], page 23, equations 1.21.2a-1.21.4. Also, $(\widetilde{r} - r_f)^2 > 0$ and since we have assumed that $U_{0,2}(W_0, \widetilde{W}) > 0$, we clearly get $\frac{\partial a}{\partial W_0} > 0$ if $E\left[U_{1,1}(W_0, \widetilde{W})(\widetilde{r} - r_f)\right] > 0$. Similarly, from the cited theorems in Huang and Litzenberger [17], we know that conventional ICARA implies that $E\left[U_{0,2}(W_0, \widetilde{W})(\widetilde{r} - r_f)(1+r_f)\right] < 0$. Hence, we get $\frac{\partial a}{\partial W_0} < 0$ if $E\left[U_{1,1}(W_0, \widetilde{W})(\widetilde{r} - r_f)\right] < 0$ under ICARA ■

PROOF OF THEOREM 5

The proof of (25) is standard and so omitted. Rearranging (25), one gets

$$a\left(\frac{\mu^2 + \sigma^2}{\mu}\right) = \frac{1}{2b(W_0)} - 2c$$

Hence,

$$\left(\frac{\mu^2 + \sigma^2}{\mu}\right)\frac{\partial a}{\partial W_0} = -\frac{b'(W_0)}{2\left(b(W_0)\right)^2} - 2(1+r_f)$$

so that $\frac{\partial a}{\partial W_0} > 0$ iff

$$b'(W_0) < -4(1+r_f)\left(b(W_0)\right)^2 \ ■$$

PROOF OF THEOREM 6

Clearly, expected utility can be written as

$$E\left[U\left(\widetilde{W}\right)\right] = -E\left[\exp\left(-b(W_0)\widetilde{W}\right)\right]$$

$$= \quad -\exp\left(-b(W_0)W_0(1+r_f)\right) E\left[\exp\left(-b(W_0)aW_0\left(\widetilde{r}-r_f\right)\right)\right]$$

Noting that $E\left[\exp\left(-b(W_0)aW_0\left(\widetilde{r}-r_f\right)\right)\right] = \Phi\left(-b(W_0)aW_0\right)$, where $\Phi(s)$ is the characteristic function of $\left(\widetilde{r}-r_f\right)$ and that $\exp\left(-b(W_0)W_0(1+r_f)\right)$ is a constant, it is clear that the first-order conditions for the maximisation problem $\max\limits_{a} E\left[U(\widetilde{W})\right]$ can be written

$$\frac{d\left[\Phi\left(-b(W_0)\widehat{a}W_0\right)\right]}{d\widehat{a}} = 0 \tag{31}$$

where $\widehat{a}$ is the utility-maximising value of a. Totally differentiating (31) with respect to W_0, one obtains

$$\frac{d^2\left[\Phi\left(-b(W_0)\widehat{a}W_0\right)\right]}{d\widehat{a}^2}\left(-\frac{\partial b(W_0)}{\partial W_0}\widehat{a}W_0 - b(W_0)\left(\widehat{a} + W_0\frac{\partial \widehat{a}}{\partial W_0}\right)\right) = 0$$

This can be rearranged for

$$\frac{\partial a}{\partial W_0} = -\frac{\left(\frac{\partial b(W_0)}{\partial W_0}W_0 + b(W_0)\right)\widehat{a}}{b(W_0)W_0}$$

Hence, if we have a positive amount allocated to equity, i.e. $\widehat{a} > 0$, then $\frac{\partial a}{\partial W_0} > 0$ iff $\frac{\partial b(W_0)}{\partial W_0}W_0 + b(W_0) < 0$, which is equivalent to

$$\frac{d\left(W_0 b(W_0)\right)}{dW_0} < 0 \,\blacksquare$$

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