Risk, utility and switching between gambles

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1. Introduction

The risk premium of a gamble, first derived independently by Arrow [1] and Pratt [28], depends upon three characteristics, the form of the utility function, the distribution of the gamble considered (and so the distribution of final wealth) and the level of initial wealth. Several studies have analysed what are desirable features of these for a 'sensible' financial investor. Arrow and Pratt themselves argue that the link between the utility function and wealth should be that absolute risk aversion should decrease in wealth while, in some cases, relative risk aversion should increase. Likewise, Rothschild and Stiglitz [29] provide a characterisation of how changes in distributions affect the risk premium for the class of utility functions which are increasing and concave. This includes the notion of stochastic dominance, which is one of the most frequently used risk characterisations in finance. A good examination of stochastic dominance is given in Bawa et.al.[3] and Huang and Litzenberger [18]. Either one of these approaches provide a framework for evaluating the usefulness of a utility function. However, the link between final wealth (essentially through the choice of distribution of the gamble) and initial wealth has not, to the authors knowledge, been analysed in the present finance literature.

In a series of exciting and original papers, Bell ([5] and [7]) has used the fundamental idea of how an expected utility function can capture a switch in preferences between two risky gambles as the initial wealth of the investor changes, to identify the corresponding utility function. Switching would appear quite plausible. Consider an investor with £1,000, who does not consider a fair gamble for £500 more desirable than a fair gamble for £50. If you were to give this investor an additional £19,000 prior to the gambling decision being taken, so initial wealth is now £20,000, it would be quite feasible that she would prefer the gamble for larger stakes. There may even be cases where it is desirable to switch preferences for these gambles again for an even higher wealth level.

Using a small set of axioms, including a simple one-switch rule, Bell narrows down the class of utility functions satisfying these requirements to the quadratic and certain types of exponential functions. By imposing additional requirements on the form of

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the utility function, such as decreasing absolute risk aversion (DARA), he is left with only one candidate utility function,

$$U(W) = aW - be^{-cW}$$

where W is wealth and a, b and c are positive constants. He also gives the general form of the utility functions for higher order ($n-$ switch) rules.

In the latter paper [7], he links the switching property to the separation of risk and return in preferences over gambles and derives the class of utility functions which adequately provide separation. Jia and Dyer [20] provide a similar result, although they restrict their attention to mean-zero gambles. These ideas are surveyed and brought together in Sarin and Weber [31]. The difficulty with this literature for financial economists is that the perspective is too broad. Here, we shall concentrate only on those issues we believe are important in financial economics. This was also the focus of more recent work by Bell [8], who derived risk measures from this approach, implicitly using multiplicative gambling in the process.

We will scrutinize the arguments employed in these papers by initially changing the analysis to multiplicative rather than additive gambles. This is the appropriate gamble in investment problems. We derive results analogous to those of both Bell and Jia and Dyer in this case, restricting our attention to switches between two gambles. Secondly, we look at which functions can adequately separate risk and return, using a link to one-switching. We also look at mixed gambles, which have an additive and a multiplicative component. These are motivated by their resemblence to financial investment strategies, where one seeks to secure some lower threshold level of wealth. It transpires that the zero-switch functions for this class of gambles are in the HARA class of utility functions (see Huang and Litzenberger [18], Eeckhoudt and Gollier [12] or Ingersoll [19]). Finally, we recall the theory of Constant Proportional Portfolio Insurance introduced in Black and Jones [9], and give sufficient conditions for their utility function to be zero-switch.

The paper is organised as follows : Section 2 introduces notation and preliminary definitions, and discusses the motivation for looking at multiplicative gambles and more complicated risk exposures. We also give some basic results on fundamental expected utility theory under these conditions. We then give the main theorems on preference switching and specific utility functions in section 3. Risk-return separation is addressed in full in section 4 and conclusions follow in section 5.

2. Definition, motivation and basic results

First we present the notion of risk-premium with additive risks. As is standard, we let $U(W_0)$ be the individual's utility function over initial wealth and denote the gamble by the random variable $\widetilde{X}$, which will remain independent of W_0. It is assumed throughout that $U(.)$ is defined only for positive arguments. The **risk premium** of the gamble is $\pi = \pi(W_0, \widetilde{X})$, defined as the solution to the equation

$$U(W_0 - \pi) = E(U(W_0 + \widetilde{X})) \tag{1}$$

When $U(.)$ is continuous and strictly increasing we can solve (1) for a unique value of π. Final wealth is implicitly assumed to be defined as

$$\widetilde{W} = W_0 + \widetilde{X} \tag{2}$$

which, in turn, requires the assumption $\widetilde{X} > -W_0$, since we do not define utility over negative wealth. This assumption is inappropriate for at least two reasons. Firstly, it implies that we can not analyse all gambles at all levels of wealth. For instance, if $W_0 = 1000$ and $\widetilde{X}$ can take the value -500 or 500 with equal probabilities, we can solve for the risk premium. However, at initial wealth $W_0 = 300$, this is not possible, as the evaluation of the right hand side of (1) involves the undefined term $U(-200)$.

One might think that this could be resolved by using truncated gambles $\widetilde{X}^T = \widetilde{X}$ such that $\widetilde{X} > -W_0$, which has a probability density function (pdf) defined by

$$pdf(\widetilde{X}^T) = \frac{pdf(\widetilde{X})}{\Pr(\widetilde{X} > -W_0)} \tag{3}$$

However, under this specification, $E(\widetilde{X}^T)$ depends explicitly upon initial wealth, while the derivation of the risk premium in (1) requires that the moments and pdf of $\widetilde{X}$ are independent of W_0.

Secondly, as we will be looking at this problem from a financial angle, we need to consider the bets that are taken in the financial markets. Most financial investments have payoff components which are multiples of the initial capital invested. Coupons on bonds, dividends on shares, and total returns on portfolios of any instruments, are examples of this. Whilst it is possible to purchase an over-the-counter option that provides any desired pattern of payout, the authors are unaware of financial gambles involving a purely additive outcome, with exception of lotteries. These are not regarded as a major asset class.

These two reasons also explain why Bell's utility functions have not found favour empirically in a financial context. To be tractable empirically as models of investor behaviour would require us to be able to include cases where $\widetilde{X}$ is normally distributed, which would violate the lower bound condition on $\widetilde{X}$.

The above discussion suggests the use of multiplicative gambles as an alternative to the additive effects commonly used in the literature. Multiplicative gambles were introduced in the original article by Arrow [1] and discussed by Eeckhoudt and Gollier [12], who rederive the Arrow-Pratt results including a detailed argument. In a recent paper [8], Bell also motivates the use of multiplicative gambles in a financial context. Mathematical psychologists (e.g. Luce [24] and Sarin [30]) have used multiplicative gambles to derive explicit risk measures. The link between the psychology literature and the management science literature was highlighted in both Sarin and Weber [31] and Pedersen and Satchell [27], and will resurface in later sections.

Suppose we define final wealth as

$$\widetilde{W} = W_0\widetilde{X} \tag{4}$$

and restrict $\widetilde{X} \geq 0$. Note that $\widetilde{X}$ now only involves restrictions that are independent of W_0. Additionally, if a financial investor holds a portfolio which pays rate of return $\widetilde{r}_p$, her final wealth, given initial investment W_0, will be given by (4) where $\widetilde{X} = 1+\widetilde{r}_p$. This model is also easy to implement empirically, as the assumption that $\widetilde{X}$ is log-normally distributed can be made without having to worry about what happens if returns are negative. First we introduce some basic results.

2.1. Basic Results. In this section, we recall the basic analysis of Arrow [1] and Pratt [28], offering an an intuitive way of looking at absolute and relative risk aversion coefficients. We derive a basic result relating the different coefficients of risk-aversion which arise from using additive and multiplicative gambles. The Arrow-Pratt coefficient of absolute risk aversion $r_A(W_0)$ is traditionally defined as

$$r_A(W_0) = -\frac{U''(W_0)}{U'(W_0)} \tag{5}$$

The relationship between (5) and (1) is fundamental. By expanding both sides of (1) and truncating appropriately (see Huang and Litzenberger [18] for details), one derives the following approximation

$$\pi(W_0, \widetilde{X}) \approx \frac{1}{2} Var(\widetilde{X}) r_A(W_0) \tag{6}$$

An equivalent approach applied to the multiplicative case (4) will yield a candidate for a correponding measure of risk aversion. This was done in Eeckhoudt and Gollier [12]. We present an alternative derivation. We need to solve the following equation for π :

$$U(W_0(1-\pi)) = E(U(W_0\widetilde{X}))$$

where now $\widetilde{X}$ is a small risk with mean 1. Expanding the left hand side around $\pi = 0$, truncating after the linear term, while expanding the right hand side around $\widetilde{X} = 1$, truncating at the quadratic term, yields

$$U(W_0) - \pi W_0 U'(W_0) = E\{U(W_0) + (\widetilde{X} - 1) W_0 U'(W_0) + \frac{1}{2}(\widetilde{X} - 1)^2 W_0^2 U''(W_0)\}$$

Taking expectations and cancelling, one can rearrange to get

$$\pi(W_0, \widetilde{X}) \approx -\frac{1}{2} Var(\widetilde{X}) \frac{W_0 U''(W_0)}{U'(W_0)}$$

Hence, if we define

$$r_M(W_0) = -\frac{W_0 U''(W_0)}{U'(W_0)}$$

we obtain

$$\pi(W_0, \widetilde{X}) \approx \frac{1}{2} Var(\widetilde{X}) r_M(W_0) \tag{7}$$

which, like (6), says that the risk premium rises in the variance of the gamble and the (now multiplicative) risk aversion. This similarity to the commonly accepted equation (6), leads us to treat multiplicative risk aversion as

$$r_M(W_0) = -\frac{W_0 U''(W_0)}{U'(W_0)} \tag{8}$$

It is worth noting that this is in fact the relative risk aversion coefficient (see [1]). It seems intuitive to view risk aversion by (8) in a multiplicative framework, since as the gamble is believed to be a multiple of wealth, so logically should risk aversion. We will say that an individual is **relatively risk averse** when $r_M(W_0) > 0$. Note that as wealth is positive, an individual is relatively risk-averse if and only if she is risk averse in the Arrow-Pratt sense. However, the risk characteristics may change differently as wealth changes for multiplicative and additive risk-averse individuals. Their precise relationship was discussed in detail in Liu [23] and Eeckhoudt and Gollier [12]. We return to this point in a later section.

3. Preference switching

Bell [5] defined the notion of an **n−switch utility function** as a function which has the property that, as initial wealth increases, we switch preferences between any two gambles at most n times. We also denote by **n-switch preferences** the preferences underlying an n-switch utility function, i.e. preferences which will not switch more that n times between any pair of gambles as wealth increases. We do not present his work in detail, but include the theorems which are most relevant to this paper. Pedersen [26] provides a more thourough examination of his theorems.

It is not at all clear whether zero, one or $n-$ switching is a natural property for an investor. One-switching is certainly not implausible. We prefer lottery tickets to traded funds when poor (at low W_0) but most likely reverse this preference as we become richer. Indeed, in support of two-switching, it may well be the case that the urge to gamble on the lottery reasserts itself at very high wealth levels. The proofs of the following two theorems can be found in [5].

Theorem 1. ***(Bell [5], Prop. 1)***
The only zero-switch utility functions are the exponential and the linear functions.

Theorem 2. ***(Bell [5], Prop. 2)***
The only one-switch utility functions are

$$\begin{array}{llcl} (a) & U(W) & = & aW^2 + bW + c \\ (b) & U(W) & = & ae^{bW} + ce^{dW} \\ (c) & U(W) & = & aW + be^{cW} \\ (d) & U(W) & = & (aW + b)e^{cW} \end{array} \tag{9}$$

The utility functions (9) are identical to the utility functions found by Farquhar and Nakamura [13] to satisfy their augmented constant exchange risk condition. As Bell notes, although both this and the one-switch conditions are conditions defined over all wealth levels, they are not obviously identical. It will be shown later that these functions can be increasing or decreasing, risk-averse or risk-loving and have decreasing or increasing risk aversion depending on restrictions on the parameters.

Bell's proofs of Theorems 1 and 2 both assume that $U(W)$ is infinite differentiable everywhere, which justifies the use of Taylor expansions. While economists rarely require popular functions to be more than twice differentiable for analytical purposes, almost all functions used are in fact infinitely differentiable, so this may not seem unreasonable. Indeed, in his most recent work on this topic [8], Bell argues, in support of this assumption, that a utility function should be smooth and not contain target levels of wealth.

In addition, Bell [5] contains a section to excluding trick solution (see the elegant Lemma in [5], Section 6, pages 1423-1424), which results in a claim that no other utility functions, however obscurely constructed, may be a one-switch function. However, a closer examination of the proof of Bell's main theorem reveals that the statement on switching are for individual levels of wealth, i.e. while it is true that the functions in his theorems are sufficient for switching, they are only necessary if we assume an individual has the same utility function at ALL levels of initial wealths. This has been implicitly assumed in his proof. A one-switch utility function must take one of the five forms given at every level of wealth, but not necessarily the SAME functional forms at all wealth levels, in other words it is a statement of local properties.

We will argue that firstly, there are functions which can be zero-switch and not infinitely differentiable and, secondly, that these need not be obscure at all. Consider the following theorem.

Theorem 3. *The utility function*

$$U(W) = \begin{cases} \lambda W & W > \eta \\ \lambda W - (\eta - W)^{\alpha} & W \leq \eta \end{cases} \tag{10}$$

for positive α, λ , $(1 < \alpha < 2)$ *and where* $\eta = W_0 + \tau$ *for some returns target* τ, *is an additive zero-switch utility function.*

Proof. To see this is an additive zero-switch, consider two gambles $\widetilde{X}$ and $\widetilde{Y}$ and initial wealth W_0. Clearly

$$\begin{aligned}
&EU(W_0 + \widetilde{X}) > EU(W_0 + \widetilde{Y}) \\
&\Leftrightarrow \Pr\{W_0 + \widetilde{X} > \eta\} E[\lambda(W_0 + \widetilde{X}) \mid W_0 + \widetilde{X} > \eta] \\
&+ \Pr\{W_0 + \widetilde{X} \leq \eta\}(E[\lambda(W_0 + \widetilde{X}) - (W_0 + \tau - W_0 - \widetilde{X})^{\alpha}] \mid W_0 + \widetilde{X} \leq \eta) \\
&> \Pr\{W_0 + \widetilde{Y} > \eta\} E[\lambda(W_0 + \widetilde{Y}) \mid W_0 + \widetilde{Y} > \eta] \\
&+ \Pr\{W_0 + \widetilde{Y} \leq \eta\}(E[\lambda(W_0 + \widetilde{Y}) - (W_0 + \tau - W_0 - \widetilde{Y})^{\alpha} \mid W_0 + \widetilde{Y} \leq \eta]) \\
&\Leftrightarrow \lambda E[\widetilde{X}] - \Pr\{\widetilde{X} \leq \tau\} E[(\tau - \widetilde{X})^{\alpha} \mid \widetilde{X} \leq \tau] > \\
&\lambda E[\widetilde{Y}] - \Pr\{\widetilde{Y} \leq \tau\} E[(\tau - \widetilde{Y})^{\alpha} \mid \widetilde{Y} \leq \tau]
\end{aligned}$$

This is independent of W_0 and so preferences can not switch as wealth increases. Hence, (10) is an additive zero-switch utility function which is not infinitiely differentiable everywhere ■

In this function, η is a wealth target. The utility function (10) is the mean-lower partial moment utility function, introduced in Fishburn [15]. This function is twice differentiable, locally risk averse and decreasingly locally risk averse, but the third derivative is not defined at $W = \eta$. The mean-lower partial moment decision rule congruent to (10) was related to second degree stochastic dominance in [4]. We should also mention the work of Holthausen [17] in this context, who gives another piecewise utility function which can be shown to be additive zero-switch, (shown in Pedersen [26]) and is congruent to a similar decision rule. We do not pursue this point further in this paper, but these observations further motivate our study of the switching property and Bell's analysis in particular.

The intuitive appeal of the switching property is made more clear with the application to discounted cash flows. Here the notion of switching is simply that of impatience where wealth becomes analogous to the timing of satisfaction. Formally, if we define a discount function $d(t)$, $t = 1, 2, 3, ...$, where $d(0) = 1, d(\infty) = 0$ and $d'(t) < 0$ and define one consumption stream $x(t)$ preferred to another, $y(t)$, by

$$\sum_{i=0}^{\infty} x(t)d(t) > \sum_{i=0}^{\infty} y(t)d(t)$$

then we can define increasing/decreasing/constant impatience by $\frac{d(T+t)}{d(T)} > d(t)$, $\frac{d(T+t)}{d(T)} < d(t)$ and $\frac{d(T+t)}{d(T)} = d(t)$ respectively, for positive T. In this context, a switch refers to changing your preference from $x(t)$ to $y(t)$ as T increases. Bell [5], page 1422, Proposition 8, shows that theorems about switching in wealth terms are analogous to switching in time.

Another motivation for switching is given in Bell [6]. This refers to issues of contextual uncertainty, i.e. the posiibility of removing one of two uncertainties that affect your initial wealth prior to decision making. It is shown that there is a close relationship between contextual uncertainties and one-switching in [6], Theorem 1.

With these arguments in mind and the previous discussion favouring multiplicative risks, we now introduce the multiplicative gambles, which complement the existing analysis, all of which has assumed additivity.

3.1. Multiplicative gambles. The analysis of multiplicative shocks as opposed to additive shocks with respect to measuring risk and return was done in detail in [8]. We will take a closer look at that in a later section. In this section we provide a full characterisation of what functions admit switching when risks are mutiplicative rather than additive.

We start by introducing some further notation and definitions. For multiplicative gambles, zero-switching occurs whenever $\widetilde{X} \succeq \widetilde{Y}$ (or $EU(W_0\widetilde{X}) \geq EU(W_0\widetilde{Y})$) for all W_0. To follow Bell, we restrict our attention to utility functions in the class $D_{\infty,}$ i.e.

the functions for which all derivatives exist. We also require the validity of the Taylor series

$$EU(W_0\widetilde{X}) = \sum_{i=0}^{\infty} \frac{W_0^i U^i(W_0) m_i}{i!} \tag{11}$$

where U^i is the ith derivative of $U(.)$ and $m_i = E[(\widetilde{X}-1)^i]$, the ith central moment of $\widetilde{X}$. The conditions for the existence of this expansion are given in the following Lemma.

Lemma 4. *If we assume that*

$$\sum_{n=0}^{\infty} \frac{W_0^n U^n(W_0)(\widetilde{X}-1)^n}{n!} = U(W_0\widetilde{X}) \tag{12}$$

for all W_0 *and* $\widetilde{X}$ *almost everywhere and*

$$\sum_{i=0}^{\infty} \frac{W_0^n \left|U^n(W_0)\right| E\left|(\widetilde{X}-1)^n\right|}{n!} < \infty \tag{13}$$

then (11) is valid.

Proof. Note first that (13) gives

$$\begin{aligned} \infty &> \sum_{i=0}^{\infty} \frac{W_0^n \left|U^n(W_0)\right| E\left|(\widetilde{X}-1)^n\right|}{n!} \\ &= \sum_{i=0}^{\infty} \frac{\left|W_0^n U^n(W_0)\right| \left|\int \left|(\widetilde{X}-1)^n pdf(\widetilde{X})\right|\right|}{n!} \end{aligned}$$

since $E\left|(\widetilde{X}-1)^n\right|$ and $pdf(\widetilde{X})$ are both positive. This in turn implies that

$$\begin{aligned} \infty &> \sum_{i=0}^{\infty} \left| \frac{W_0^n U^n(W_0) \int (\widetilde{X}-1)^n pdf(\widetilde{X})}{n!} \right| \\ &= \sum_{i=0}^{\infty} \left| \int \frac{W_0^n U^n(W_0)(\widetilde{X}-1)^n pdf(\widetilde{X})}{n!} \right| \end{aligned} \tag{14}$$

We apply the Beppo-Levi theorem[1] in Asplund and Bungart [2] (page 64-66) directly. Letting

$$f_n(\widetilde{X}) = \frac{U^n(W_0)(\widetilde{X}-1)^n}{n!} pdf(\widetilde{X})$$

[1] The Beppo-Levi theorem is not commonly used in economics. Formally, it says that if $\sum_{n=1}^{\infty} f_n$ is a series of integrable functions such that

and

$$f(\widetilde{X}) = U(W_0\widetilde{X})pdf(\widetilde{X})$$

then (12) and (14) imply that the conditions of the Beppo-Levi theorem are satisfied. As a result, we get that $U(W_0\widetilde{X})pdf(\widetilde{X})$ is integrable and

$$\int U(W_0\widetilde{X})pdf = \sum_{n=0}^{\infty} \int \frac{U^n(W_0)(\widetilde{X}-1)^n}{n!} pdf(\widetilde{X})$$

which in turn implies that

$$E[U(W_0\widetilde{X})] = \sum_{n=0}^{\infty} \frac{U^n(W_0)\int(\widetilde{X}-1)^n pdf(\widetilde{X})}{n!} = \sum_{i=0}^{\infty} \frac{W_0^i U^i(W_0) m_i}{i!}$$

where U^i is the ith derivative of $U(.)$ and $m_i = E[(\widetilde{X}-1)^i]$, the ith central moment of $\widetilde{X}$ about 1. ■

Although Bell's proof relies upon the existence of a similar series, he does not derive explicitly the necessary mathematical conditions. Note also that no explicit relationship between W_0 and $\widetilde{X}$ is required, in contrast to what was shown to be necessary for Bell's analysis in the first section. It might be thought that these mathematical requirements are of little or no consequence. However, the conditions required for validity of (11) are quite stringent. As a counter-example to these conditions, consider constant absolute risk aversion (i.e. exponential utility) and gambles defined as log-normal multiplicative shocks. In this case the left hand side of (11) becomes $E[e^{-W_0\widetilde{X}}]$ where $\widetilde{X}$ is log-normal, i.e. the moment-generating function of a log-normal random variable. It is a known result in probability theory that this cannot be approximated by an infinite Taylor series over its moments (see Durrett [11], page 89-93 for details). Since log-normality and exponential utility are basic building blocks in finance, this counter-example is far from pathological.

We now present theorems analogous to those in Bell [5] for multiplicative gambles. Our new theorems start with the multiplicative zero-switch utility functions.

$$\sum_{n=1}^{\infty} \int \mid f_n \mid dx < \infty$$

then $\sum_{n=1}^{\infty} f_n(x)$ converges for almost all x, and if additionally f is a function that equals $\sum_{n=1}^{\infty} f_n$ almost everywhere, then f is integrable and

$$\int f = \sum_{n=1}^{\infty} \int f_n dx$$

It is closely related to the convergence theorems of Lebesque (see, for instance, Grimmett and Stirgazer [16], page 160).

Theorem 5. *In the class of infinitely differentiable utility functions defined upon multiplicative gambles, with finite moments about* 1 *such that (11) holds, the only zero-switch utility functions are* $\ln W$ *and* bW^{α}*, when* $\widetilde{X}$ *is a multiplicative gamble such that* $E[\ln(\widetilde{X})]$ *and* $E[\widetilde{X}^{\alpha}]$ *exists and* b *and* α *are constants.*

Proof. **(Sufficiency)** Consider any gamble $\widetilde{X}$. Clearly,

$$E[\ln(W_0\widetilde{X})] = \ln W_0 + E[\ln(\widetilde{X})]$$

and

$$E[b(W_0\widetilde{X})^{\alpha}] = bW_0^{\alpha}E[\widetilde{X}^{\alpha}]$$

so that, for all W_0,

$$E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] = E[\ln(\widetilde{X}) - \ln(\widetilde{Y})]$$

and

$$E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] = bW_0^{\alpha}\{E[\ln(\widetilde{X})] - E[\ln(\widetilde{Y})]\}$$

respectively. These are always positive or always negative, and so no preference switching can occur.

(Necessity)

Suppose that $\widetilde{X}$ and $\widetilde{Y}$ are gambles[2] and, without loss of generality, $\widetilde{X} \succeq \widetilde{Y}$. Clearly, for any wealth level W_0

$$\begin{aligned} & E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] \\ = & \sum_{i=1}^{\infty} U^i(W_0)W_0^i q_i = a(W_0) \end{aligned} \tag{15}$$

where $q_i = \frac{E[(\widetilde{X}-1)^i]-E[(\widetilde{Y}-1)^i]}{i!}$. If $U(.)$ exhibits zero-switching, it must be that for any pair of wealth levels, W_0 and W_1, $a(W_0)a(W_1) \geq 0$. In particular, the restricted system

$$\sum_{i=1}^{2} U^i(W_0)W_0^i q_i = a(W_0) \tag{16}$$

cannot have a solution for two different wealth levels W_0 and W_1 unless $a(W_0)a(W_1) \geq 0$. This implies that the system can not have a general solution, which requires the existence of a real non-zero solution θ to the differential equation

$$U''(W)W^2 + \theta U'(W)W = 0 \tag{17}$$

[2]Notice that unlike Bell [5] we do not need the gamble to be 'suitably small'. Instead, we need only have that gambles cannot take a non-positive value.

for all W, since otherwise we could use (15) to find solutions to (16) for $a(W_0) < 0$ and $a(W_1) > 0$ which would imply a switch in preferences. Solving (17), we get

$$U(W) = \begin{cases} a\ln W & \theta = -1 \\ bW^{\theta-1} & \theta \neq -1 \end{cases} \tag{18}$$

for constants a, b and θ.

Note that these are the additive zero-switch functions where $\ln W$ replaces W. We will encounter this point again later, but now turn to the one-switch functions.

Theorem 6. *Under equivalent conditions to the previous theorem, and assuming all relevant expectations exist, the one-switch multiplicative risk utility functions are*

$$\begin{array}{llcl} (a) & U(W) & = & a(\ln W)^2 + b\ln W + c \\ (b) & U(W) & = & aW^b + cW^d \\ (c) & U(W) & = & a\ln W + bW^c \\ (d) & U(W) & = & (a\ln W + b)W^c \end{array} \tag{19}$$

Proof. **(Sufficiency)**

To show that the four functions all satisfy the one-switch rule, we again consider each one in turn. It suffices to show that $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{X})] = 0$ has at most one real solution for W_0.

(a) $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{X})]$

$$\begin{aligned} &= E\{a(\ln(W_0\widetilde{X}))^2 + b\ln(W_0\widetilde{X}) + c - a(\ln(W_0\widetilde{Y}))^2 + b\ln(W_0\widetilde{Y}) - c\} \\ &= (2a\ln W_0 + b)(E(\ln\widetilde{X}) - E(\ln\widetilde{Y})) + a(E(\ln\widetilde{X})^2 - E(\ln\widetilde{Y})^2) \\ &= 2a(E(\ln\widetilde{X}) - E(\ln\widetilde{Y}))\ln W_0 + b(E(\ln\widetilde{X}) - E(\ln\widetilde{Y})) + a(E(\ln\widetilde{X})^2 - E(\ln\widetilde{Y})^2) \\ &= 0 \Leftrightarrow W_0 = e^{\frac{-b(E(\ln\widetilde{X}) - E(\ln\widetilde{Y})) - a(E(\ln\widetilde{X})^2 - E(\ln\widetilde{Y})^2)}{2a(E(\ln\widetilde{X}) - E(\ln\widetilde{Y}))}} \end{aligned}$$

(b) $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})]$

$$\begin{aligned} &= aE(W_0\widetilde{X})^b + cE(W_0\widetilde{X})^d - aE(W_0\widetilde{Y})^b - cE(W_0\widetilde{Y}) \\ &= aW_0^b(E(\widetilde{X}^b) - E(\widetilde{Y}^b)) + cW_0^d(E(\widetilde{X}^d) - E(\widetilde{Y}^d)) \\ &= 0 \Leftrightarrow W_0 = \left(\frac{-c(E(\widetilde{X}^d) - E(\widetilde{Y}^d))}{a(E(\widetilde{X}^b) - E(\widetilde{Y}^b))}\right)^{\frac{1}{b-d}} \end{aligned}$$

(c) $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})]$

$$\begin{aligned} &= aE(\ln(W_0\widetilde{X})) + cE(W_0\widetilde{X})^d - aE(\ln(W_0\widetilde{Y})) - cE(W_0\widetilde{Y})^d \\ &= a(E(\ln\widetilde{X}) - E(\ln\widetilde{Y})) + cW_0^d(E(\widetilde{X}^d) - E(\widetilde{Y}^d)) \\ &= 0 \Leftrightarrow W_0 = \left(\frac{-a(E(\ln\widetilde{X}) - E(\ln\widetilde{Y}))}{c(E(\widetilde{X}^d) - E(\widetilde{Y}^d))}\right)^{\frac{1}{d}} \end{aligned}$$

and finally,

(d) $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})]$

$$
\begin{aligned}
&= E\{(a\ln(W_0\widetilde{X})) + b)(W_0\widetilde{X})^c\} - \{E(a\ln(W_0\widetilde{Y})) + b)(W_0\widetilde{Y})^c\} \\
&= aW_0^c(\ln W_0)E(\widetilde{X}^c) + bW_0^cE(\widetilde{X}^c) + aW_0^cE(\widetilde{X}^c\ln\widetilde{X}) \\
&\quad -aW_0^c(\ln W_0)E(\widetilde{Y}^c) - bW_0^cE(\widetilde{Y}^c) - aW_0^cE(\widetilde{Y}^c\ln\widetilde{Y}) \\
&= W_0^c\{(a\ln W_0 + b)(E(\widetilde{X}^c) - E(\widetilde{Y}^c)) + a(E(\widetilde{X}^c\ln\widetilde{X}) - E(\widetilde{Y}^c\ln\widetilde{Y}))\} \\
&= 0 \Leftrightarrow W_0 = e^{\frac{-a(E(\widetilde{X}^c\ln\widetilde{X}) - E(\widetilde{Y}^c\ln\widetilde{Y})) - b(E(\widetilde{X}^c) - E(\widetilde{Y}^c))}{a(E(\widetilde{X}^c) - E(\widetilde{Y}^c))}}
\end{aligned}
$$

(Necessity)

Suppose that $H(V)$ is a one-switch utility function. Since (11) holds, this implies that for three distinct wealth levels, $V_1 > V_2 > V_3$, the corresponding numbers $a(V_1), a(V_2)$ and $a(V_3)$ (as defined by (15)), cannot satisfy $a(V_1) > 0, a(V_2) < 0$ and $a(V_3) > 0$ or $a(V_1) < 0, a(V_2) > 0$ and $a(V_3) < 0$. A necessary condition for this is that the $H'(V), H''(V)$ and $H'''(V)$ are collinear (since if not, one could solve the sub-system of equations (15) ($i = 1, 2, 3$) for ANY values of $a(V_i)$ by simple matrix invertion. This means we can solve

$$\theta_1 H'(V)V + \theta_2 H''(V)V^2 + H'''(V)V^3 = 0 \tag{20}$$

for θ_1 and θ_2. Note the simple extension to (17) offered by (20). To solve this, we use a simple transformation motivated by the observation that the four sufficient functions we seek are in fact equal to Bell's one-switch functions when V is replaced by ln$V$. By letting $W = \ln V$ and $H(V) = U(\ln V)$, we get

$$H'(V) = U'(W) * \frac{dW}{dV} = U'(W) * \frac{1}{V} \tag{21}$$

Thus,

$$H''(V) = U''(W) * \frac{dW}{dV} * \frac{1}{V} - \frac{1}{V^2} * U'(W)$$

so that

$$H''(V) = \frac{1}{V^2}[U''(W) - U'(W)] \tag{22}$$

Finally, it can be shown by a similar argument that

$$H'''(V) = \frac{1}{V^3}[U'''(W) - 3U''(W) + U'(W)] \tag{23}$$

Substituting (21), (22) and (23) into (20) and simplifying then gives

$$U'(W)[2 - \theta_1 - \theta_2] + U''(W)[\theta_2 - 3] + U'''(W) = 0$$

Setting $\phi_1 = 2 - \theta_1 - \theta_2$ and $\phi_2 = \theta_1 - 3$, which is a one-to-one mapping, this can be simplified to

$$\phi_1 U'(W) + \phi_2 U''(W) + U'''(W) = 0$$

as required. The real solutions to this equation are given by (9) and setting $W = \ln W$ yields the result. ■

We thus have a class of one-switch utility functions with respect to multiplicative gambles. It can also be shown that in this case we find additional one-switch functions by relaxing the assumption of infinite differentiability. Recall (10), but now assume $\eta = W_0\tau$, (τ now being a multiplicative returns target). We get

$$\begin{aligned}
&EU(W_0\widetilde{X}) > EU(W_0\widetilde{Y})\\
&\Leftrightarrow \Pr\{W_0\widetilde{X} > \eta\}E[\lambda(W_0\widetilde{X}) \mid W_0\widetilde{X} > \eta]\\
&+\Pr\{W_0\widetilde{X} < \eta\}(E[\lambda(W_0\widetilde{X}) - (\eta - W)^\alpha \mid W_0\widetilde{X} < \eta])\\
&> \Pr\{W_0\widetilde{Y} > \eta\}E[\lambda(W_0\widetilde{Y}) \mid W_0\widetilde{Y} > \eta]\\
&+\Pr\{W_0\widetilde{Y} < \eta\}(E[\lambda(W_0\widetilde{Y}) - (\eta - W)^\alpha \mid W_0\widetilde{Y} < \eta])\\
&\Leftrightarrow \lambda E[\widetilde{X}] - W_0^\alpha \Pr\{\widetilde{X} < \tau\}E[(\tau - \widetilde{X})^\alpha \mid \widetilde{X} < \tau]\\
&> \lambda E[\widetilde{Y}] - W_0^\alpha \Pr\{\widetilde{Y} < \tau\}E[(\tau - \widetilde{Y})^\alpha \mid \widetilde{Y} < \tau]\\
&\Leftrightarrow W_0^\alpha > \tfrac{\lambda(E[\widetilde{Y}] - E[\widetilde{X}])}{\Pr\{\widetilde{Y}<\tau\}E[(\tau-\widetilde{Y})^\alpha|\widetilde{Y}<\tau] - \Pr\{\widetilde{X}<\tau\}E[(\tau-\widetilde{X})^\alpha|\widetilde{X}<\tau]}
\end{aligned}$$

Hence, there is a unique wealth level at which preferences could switch, and (10) with $\eta = W_0\tau$ is a multiplicative one-switch. There are naturally many other piece-wise functions which would be $n-$switch functions for either additive or multiplicative gambles, for various values of n. Details of such examples ar given in Pedersen [26].

Under infinite differentiability, the result can be generalised to $n-$switch utility functions by employing the same transformation as that utilised in the previous theorem. We state the following result without proof. The proof in its entirety can be found in Pedersen [26].

Theorem 7. *An infinitely differentiable utility function satisfying the conditions of the above theorem is an $n-$switch utility function if and only if it can be written in the form*

$$f_0(\ln W) + \sum_{i=0}^{k} W^{c_i} f_i(\ln W)$$

where $f_i(.)$ is a polynomial of order n_i such that

$$\sum_{i=0}^{k} n_i \leq n + 1 - k$$

We now turn to the isolation of desirable utility functions with the multiplicative gamble structure and the $n-$ switch rules. To use multiplicative switching functions as a selection criteria for utility, we initially recall the following result.

Theorem 8. ***(Bell [5], Prop. 3)***

If a decision maker

(a)	*prefers more to less*
(b)	*wishes to obey the axioms of expected utility*
(c)	*is risk averse at all wealth levels*
(d)	*is decreasingly risk averse at all wealth levels*
(e)	*wishes to obey the additive one-switch rule*
(f)	*will approach risk-neutrality for small gambles when extremely rich*

then the only feasible utility function which is infinitely differentiable everywhere, is

$$U(W) = W - be^{-cW} \tag{24}$$

for positive numbers b and c.

These properties are almost all accepted as stylized facts on investor behaviour. It would be inconcievable that as we should ever be saturated completely in monetary terms. This case is excluded in (a). The expected utility axioms of Von-Neumann Morgernstern, described in almost all microeconomic textbooks, have been generalised by Machina [25] in response to a large body of empirical criticism (surveyed well in Fishburn [14]), but the algebraic simplicity of the original framework has reinforced its position as the main theory of expected utility. Thus, axiom (b) is deemed reasonable. While (f) is a somewhat superficial condition, included mainly to enable Bell to separate (24) from (9b), (e) is the obvious application of the additive one-switch rule we now wish to substitute for multiplicative switching. Similar theorems could be proved for zero-, two- or $n-$ switch rules, by examining their propertes in detail. Some possibilities along this line are examined in detail in Pedersen [26]. (c) and (d) involve two of the fundamental concepts in expected utility theory. A good discussion of the absolute and relative risk aversions and their behaviour as wealth increases is found in Eeckhoudt and Gollier [12]. For a mathematical illustration of the relationship between them, one should consult Liu [23].

It is generally accepted that investors will always pay a positive monetary amount to avoid absolute or relative risks. Hence (c) is fundamental to accepted risk behaviour and the term "absolute" could be replaced with "relative" without affecting the result, since they differ only by a positive multiple, W_0. It is also deemed reasonable that absolute risk aversion should be decreasing, supporting property (d). As people get wealthier, they care less about the risks they face and consequently, the risk premium is smaller. Alternatively, one could say that the utility function gets less concave as wealth increases.

However, it is not so clear whether relative risk aversion should be increasing or decreasing in wealth. Arrow [1] argued that relative risk aversion should be increasing in wealth. Eeckhoudt and Gollier (pages 46-47) split the effects of increasing W_0 into the 'wealth effect' and a 'risk effect'. The wealth effect says that as wealth increases, the investor feels wealthier and if risk is kept constant, risk aversion would have to decrease. The risk effect, on the other hand, says that since risk is multiplicative, an increase in wealth implies an increase in risk which increases the aversion to the risk faced. Hence, there are in fact two opposing factors in determining whether or not relative risk aversion increses or decreases as wealth increases. They put forward a hypothesis which is consistent with Arrow's beliefs, i.e. that the latter effect outweighs the former and we can assume that relative risk aversion increases in wealth.

One could thus impose alternative criteria upon desirable utility functions, and even prove theorems of non-existence of utility functions with a given set of properties. We leave this for future research. Our main objective is the parallel theorem to the above with multiplicative gambles rather than additive. We elect to omit the superficial property (f), and change property (e) accordingly. We get the following result.

Theorem 9. ***(Parallel to Bell [5], Prop. 3)***
If a decision maker

- (a) *prefers more to less*
- (b) *wishes to obey the axioms of expected utility*
- (c) *is risk averse at all wealth levels*
- (d) *has decreasing absolute risk aversion at all wealth levels*
- (e) *wishes to obey the multiplicative one-switch rule*

then the only feasible utility functions which are infinitely differentiable everywhere, are

$$U(W) = a \ln W - bW^{-c} \tag{25}$$

and

$$U(W) = -aW^{-b} - cW^{-d}$$

for positive numbers a, b, c and d.

Proof. By Theorem 5, only utility functions in (19) satisfy the one-switch rule. All of these obey the axioms of expected utility. Hence (b) and (e) are satisfied. Now consider the properties $(a), (c)$ and (d) for each function in turn. For $U(W) = a(\ln W)^2 + b \ln W + c$, $U'(W) = \frac{2a \ln W + b}{W}, U''(W) = \frac{2a(1-\ln W)-b}{W^2}, r(W) = \frac{1}{W}\left[1 - \frac{2a}{(2a \ln W + b)}\right]$ and $r'(W) = \frac{1}{W^2}\left[\frac{4a^2}{(2a \ln W+b)^2} + \frac{2a}{(2a \ln W + b)} - 1\right]$. $U(W)$ will be increasing and risk-averse for $W > e^{1-\frac{b}{2a}}$. However, this would imply $r'(W) > 0$. Hence, this function can not be both risk-averse and have decreasing risk aversion. For $U(W) = aW^b + cW^d$, we have $U'(W) = abW^{b-1} + cdW^{d-1}$, $U''(W) = ab^2W^{b-2} + cd^2W^{d-2}, r(W) = \frac{-ab^2W^{b-2} - cd^2W^{d-2}}{abW^{b-1}+cdW^{d-1}}$ and $r'_M(W) = \frac{a^2b^3W^{2b-4}+c^2d^3W^{2d-4}+abcd(b+d)W^{b+d-4}}{(abW^{b-1}+cdW^{d-1})}$
Hence, we can get $U'(W) > 0, r_M(W) > 0$ and $r'_M(W) < 0$ for all $W \geq 0$ by selecting a, b, c, d all negative. When $U(W) = a \ln W + bW^c, U'(W) = \frac{a}{W} + bcW^{c-1}, U''(W) =$

$\frac{-a}{W^2}+bc(c-1)W^{c-2}, r(W)=\frac{1}{W}\left[\frac{a-bc(c-1)W^c}{a+bcW^c}\right]=\frac{1}{W}\left[1-\frac{bc^2W^c}{a+bcW^c}\right]$ and $r'(W)=\frac{1}{W}\left[-\frac{1}{W}+\frac{bc^2W^c}{(a+bcW^c)}-\frac{abc^3W^c}{(a+bcW^c)^2}\right]$. This means we again can ensure $U'(W)>0, r_M(W)>0$ and $r'_M(W)<0$ for all $W\geq 0$, this time by selecting $a>0, b<0$ and $c<0$. The fourth and final case is $U(W)=(a\ln W+b)W^c$, for which $U'(W)=W^{c-1}[a(1+\ln W)+b], U''(W)=W^{c-2}[a(2+\ln W)+b], r(W)=-\frac{[a(2+\ln W)+b]}{W[a(1+\ln W)+b]}=\frac{1}{W}\left[1-\frac{a}{[a(2+\ln W)+b]}\right]$ and $r'(W)=\frac{1}{W^2}\left[\left(\frac{a}{a(2+\ln W)+b}\right)^2+\left(1-\frac{a}{[a(2+\ln W)+b]}\right)\right]$. Hence $U(W)$ is increasing and risk averse only when $W>e^{-\frac{b}{a}}$. However, this is exactly the condition which makes the second bracketed term in $r'(W)$ positive. Hence, this function cannot be increasing, risk-averse and decreasingly risk averse over any wealth region. Thus, only the two utility functions mentioned in the theorem satisfies all the axioms ■

These two functions appear in Bell [8] as the only two increasing, risk-averse and decreasingly risk averse utility functions which separate risk and return for multiplicative risks. Indeed, as we will see later, the conditions for separation are the same as one-switching and so the above result can be looked at in parallel to Bell [8], Theorem 1. An interesting question is whether there is a link between the relative risk aversion of the multiplicative $n-$switch utility functions and the absolute risk aversion of the additive $n-$switch utility functions. The answer lies in the transformation between the two types of functions, as proved in the following Lemma.

Lemma 10. *Consider a utility function $U(W)$, which has Arrow-Pratt absolute risk aversion coefficient $r(W)=-\frac{U''(W)}{U'(W)}$. If $W=\ln V$ and $U(\ln V)=H(V)$, then $H(V)$ has relative risk aversion measured by $r_M(W)=-\frac{U''(W)}{U'(W)}$, where*

$$r_M(W)=1+r(W)$$

Proof. Consider a utility function $U(W)$. If $W=\ln V$ and $U(\ln V)=H(V)$, then

$$H'(V)=U'(W)*\frac{dW}{dV}=U'(W)*\frac{1}{V}$$

and

$$H''(V)=U''(W)*\frac{dW}{dV}*\frac{1}{V}-\frac{1}{V^2}*U'(W)$$

so that

$$H''(V)=\frac{1}{V^2}[U''(W)-U'(W)]$$

and so the relative risk coefficient for $H(V)$ can be written as $r_M(W)=-\frac{VH''(V)}{H'(V)}=-\frac{U''(W)-U'(W)}{U'(W)}=1+r(W)$ ■

In words, the relative risk aversion of a multiplicative risk one-switch function is equal to one plus the absolute risk aversion of the corresponding additive risk one-switch utility function. From this we can deduce two interesting results in relation to switching. Firstly, if an additive function displays absolute risk aversion, the multiplicative function will be risk averse - the converse holds when relative risk aversion of the multiplicative function is larger than -1. Secondly, the relative risk aversion of the multiplicative one-switch utility function is increasing (decreasing) if and only if absolute risk aversion of the corresponding additive function of Bell is increasing (decreasing).

We now turn our attention to a third type of gambles, the affine risk exposure, which use the multiplicative gambles to define risks more complex than simple additive or multiplicative ones.

3.2. Affine risk exposures. Our next application and extension of preference switching is to a type of gambles one would most expect to see in the financial world. Rather than arguing for a straight multiplicative gamble, as was done in the previous sections, we will assume that a financial investor with initial wealth W_0 to invest, elects (or is forced) to hold a pre-determined fixed quantity $\overline{W} < W_0$ in a riskless asset paying interest rate r but takes a multiplicative gamble $\widetilde{X}$ with the remaining $(W_0 - \overline{W})$. One may interpret this situation in several ways. A fund manager can either be viewed as passive if he is instructed to secure the wealth $\overline{W}$ and so cannot invest it, but alternatively, if the money is available and he chooses to guarantee $\overline{W}$ by investing it safely, perhaps for fear of losing his job, he is more active but careful. In both cases, the same rule is followed - in the former case by management, in the latter case by the fund manager.

We will set $r = 0$ for convenience without loss of generality. In this case, $\overline{W}$ is a benchmark wealth level which results if the gamble returns a zero contribution to total wealth. Recall that multiplicative gambles are restricted to be positive.

This makes our strategy look very similar to portfolio insurance, since at the end of an investment period, we have insured against wealth falling below $\overline{W}$. We call such an exposure to risk a **Lower Bound Affine Risk Exposure.** Algebraically, our final wealth is given by

$$W = (W_0 - \overline{W})\widetilde{X} + \overline{W} \tag{26}$$

Note that the restriction we need on $\widetilde{X}$ is again $\widetilde{X} \geq 0$. The following results rely upon a similar mathematical condition to that proved in Lemma 4, found by replacing W_0 by $W_0 - \overline{W}$. We get the following results.

Theorem 11. *For the class of Lower Bound Affine Risk Exposures given in (26), the zero-switch infinitely differentiable functions are*

$$\begin{array}{lrcl} (a) & U(W) & = & \ln(W - \overline{W}) \\ (b) & U(W) & = & (W - \overline{W})^b \end{array} \tag{27}$$

for $W > \overline{W}$ and real number b. In the region $W \leq \overline{W}$, the function can take any value.

Proof. We prove this theorem directly, although one can employ the multiplicative zero-switch functions for am alternative approach.

(Sufficiency)

For (a), $E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})]$

$$
\begin{aligned}
&= E[\ln[(W_0 - \overline{W})\widetilde{X}][-E[\ln[(W_0 - \overline{W})\widetilde{Y}]] \\
&= \ln(W_0 - \overline{W}) + \ln(\widetilde{X}) - \ln(W_0 - \overline{W}) - \ln(\widetilde{Y}) \\
&= \ln(\widetilde{X}) - \ln(\widetilde{Y})
\end{aligned}
$$

which is independent of W_0. Likewise for (b), we get $E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})]$

$$
\begin{aligned}
&= E[[(W_0 - \overline{W})\widetilde{X}]^b] - E[[(W_0 - \overline{W})\widetilde{Y}]^b] \\
&= (W_0 - \overline{W})^b(E[\widetilde{X}^b] - E[\widetilde{Y}^b])
\end{aligned}
$$

which will not change sign as W_0 increases, since $\overline{W} < W_0$.

(Necessity)

Following Theorem 5, expanding using a Taylor series around $\widetilde{X} = 1$ gives

$$
\begin{aligned}
& E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] - E[U((W_0 - \overline{W})\widetilde{Y} + \overline{W})] \\
= & \sum_{i=1}^{\infty} U^i(W_0 - \overline{W})(W_0 - \overline{W})^i m_i
\end{aligned} \tag{28}
$$

where $m_i = \frac{E[(\widetilde{X}-1)^i] - E[(\widetilde{Y}-1)^i]}{i!}$. A necessary condition for $U(.)$ to be zero-switch is that the system

$$
\begin{aligned}
\sum_{i=1}^{\infty} U^i(W_0 - \overline{W})(W_0 - \overline{W})^i m_i &= a(W_0) \\
\sum_{i=1}^{\infty} U^i(W_1 - \overline{W})(W_1 - \overline{W})^i m_i &= a(W_1)
\end{aligned}
$$

does not have a solution where $a(W_0)a(W_1) < 0$. Obviously this will then also have to hold for the smaller set

$$
\begin{aligned}
\sum_{i=1}^{2} U^i(W_0 - \overline{W})(W_0 - \overline{W})^i m_i &= a(W_0) \\
\sum_{i=1}^{2} U^i(W_1 - \overline{W})(W_1 - \overline{W})^i m_i &= a(W_1)
\end{aligned}
$$

which requires that we cannot find a general solution, i.e. that the first two derivatives of $U(.)$ are collinear. This means that

$$
U''(W - \overline{W}) + \theta U'(W - \overline{W}) = 0
$$

which has as solution the two functions in (27) ■

As may have been observed by the reader, the function $(W-\overline{W})^b$ is one of the popular HARA functions (to be defined). A later section will illustrate how the HARA functions can be linked to different switching rules, including these affine gambles.

Next we give the one-switch functions.

Theorem 12. *For the class of Lower Bound Affine Risk Exposures given by (26), the infinitely differentiable one-switch utility functions are*

$$\begin{array}{llll} (a) & U(W) & = & a\ln(W-\overline{W})^2 + b\ln(W-\overline{W}) + c \\ (b) & U(W) & = & a(W-\overline{W})^b + c(W-\overline{W})^d \\ (c) & U(W) & = & a\ln(W-\overline{W}) + b(W-\overline{W})^c \\ (d) & U(W) & = & \{a\ln(W-\overline{W}) + b\}\{W-\overline{W}\} \end{array} \quad (29)$$

for $W > \overline{W}$. In the region $W \leq \overline{W}$, the function can take any value.

Proof. **(Sufficiency)**

To show sufficiency, we substitute for wealth as defined in (26) and show there is at most one switch between two different gambles $\widetilde{X}$ and $\widetilde{Y}$. To simplify proceedings, we note the following relationship : If $U(W)$ is a multiplicative function, then $U(W) = U(W_0\widetilde{X})$. If $U(W)$ is a function defined on mixed gambles where final wealth is one of the above, then by definition $U[W-\overline{W}] = U[(W_0-\overline{W})\widetilde{X}]$. Hence, we can simplify our analysis by referring to the sufficiency conditions in Theorem 5 and replace W_0 by $(W_0-\overline{W})$. Since $W_0 > \overline{W}$ all logs and fractional powers are still defined. We now solve

$$\begin{array}{ll} (a) & E[U((W_0-\overline{W})\widetilde{X}+\overline{W})] - E[U((W_0-\overline{W})\widetilde{Y}+\overline{W})] \\ & = (2a\ln(W_0-\overline{W})+b)(E(\ln\widetilde{X}) - E(\ln\widetilde{Y})) + a(E(\ln\widetilde{X})^2 - E(\ln\widetilde{Y})^2) \\ (b) & E[U((W_0-\overline{W})\widetilde{X}+\overline{W})] - E[U((W_0-\overline{W})\widetilde{Y}+\overline{W})] \\ & = a(W_0-\overline{W})^b(E(\widetilde{X}^b) - E(\widetilde{Y}^b)) + c(W_0-\overline{W})^d(E(\widetilde{X}^d) - E(\widetilde{Y}^d)) \\ (c) & E[U((W_0-\overline{W})\widetilde{X}+\overline{W})] - E[U((W_0-\overline{W})\widetilde{Y}+\overline{W})] \\ & = a(E(\ln\widetilde{X}) - E(\ln\widetilde{Y})) + c(W_0-\overline{W})^d(E(\widetilde{X}^d) - E(\widetilde{Y}^d)) \\ (d) & E[U((W_0-\overline{W})\widetilde{X}+\overline{W})] - E[U((W_0-\overline{W})\widetilde{Y}+\overline{W})] \\ & = (W_0-\overline{W})^c\{(a\ln(W_0-\overline{W})+b)(E(\widetilde{X}^c) - E(\widetilde{Y}^c)) + a(E(\widetilde{X}^c\ln\widetilde{X}) - E(\widetilde{Y}^c\ln\widetilde{Y}))\} \end{array}$$

By the arguments in Theorem 6, all of these will switch sign at most once as W_0 increases.

(Necessity)

We again appeal to a connection to Theorem 6. When final wealth is defined as $W = (W_0-\overline{W})\widetilde{X}+\overline{W}$, consider

$$\begin{aligned} & E[U((W_0-\overline{W})\widetilde{X}+\overline{W})] - E[U((W_0-\overline{W})\widetilde{Y}+\overline{W})] \\ = & \sum_{i=1}^{\infty} U^i(W_0-\overline{W})(W_0-\overline{W})^i m_i \end{aligned} \quad (30)$$

by the Taylor series around $\widetilde{X} = 1$. A necessary condition for a utility function of this type to be one-switch is that the system

$$\begin{aligned}\sum_{i=1}^{\infty} U^i(W_0 - \overline{W})(W_0 - \overline{W})^i m_i &= a(W_0)\\ \sum_{i=1}^{\infty} U^i(W_1 - \overline{W})(W_1 - \overline{W})^i m_i &= a(W_1)\\ \sum_{i=1}^{\infty} U^i(W_2 - \overline{W})(W_2 - \overline{W})^i m_i &= a(W_2)\end{aligned}$$

has no solution for a restricted range of the $a(W_i)-$ functions and different wealth levels W_0, W_1 and W_2. Following the proof of Theorem 5, we need to ensure the subsystem consisting of the lowest derivative terms (replace $\sum_{i=1}^{\infty}$ by $\sum_{i=1}^{3}$) cannot have a general solution, which in turn requires us to have the following linearity condition

$$\phi_1 U'(W-\overline{W})(W-\overline{W})+\phi_2 U''(W-\overline{W})(W-\overline{W})^2+U'''(W-\overline{W})(W-\overline{W})^3 = 0 \quad (31)$$

Now let $V = W - \overline{W}$ and $H(V) = U(W - \overline{W})$. Note that $H'(V) = U'(W - \overline{W})\frac{d(W_I-\overline{W})}{dV} = U'(W - \overline{W})$, $H''(V) = U''(W - \overline{W})\frac{d(W-\overline{W})}{dV} = U''(W - \overline{W})$ and $H'''(V) = U'''(W - \overline{W})\frac{d(W-\overline{W})}{dV} = U'''(W - \overline{W})$, since $\frac{d(W-\overline{W})}{dV} = 1$. Hence, (31) becomes

$$\phi_1 H'(V)V + \phi_2 H''(V)V^2 + H'''(V)V^3 = 0$$

which is identical to (20). We know the real solutions to this equation are the functions in (19). Transforming these back in terms of $U(W-\overline{W}) = H(V)$, and using (26), we get the solution as $U(W - \overline{W})$ as required. The functions which give one-switching when final wealth is defined as in (26) are those in (29) ■

These functions are interesting in that they are not defined below the target wealth level. Consequently, there are an uncountably infinite number of one-switch utility functions for this type of risk exposures. We can extend the proof to the $n-$ switch case, thus getting the following result, proved in Pedersen [26].

Theorem 13. *For the class of affine investment rules given by (26), the infinitely differentiable one-switch utility functions are*

$$f_0[\ln(W - \overline{W})] + \sum_{i=0}^{k}(W - \overline{W})^{c_i} f_i[\ln(W - \overline{W})]$$

where $W > \overline{W}$ and $f_i(.)$ is a polynomial of order n_i such that

$$\sum_{i=0}^{k} n_i \leq n + 1 - k$$

They can take any from on the region $W \leq \overline{W}$.

A different but similar exposure is the **Upper Bound Affine Risk Exposure**

$$W = (\overline{W} - W_0)\widetilde{X} + (2W_0 - \overline{W}) \tag{32}$$

which ensures that wealth stays below $(2W_0 - \overline{W})$. This rule implies that we take on less risk as our initial wealth gets closer to the upper target. We will short the gamble and go long the risky asset. In other words, if we are a financial investor facing the choice of equity and bonds, we have a bliss point and as we approach it, we shift out of equity and into bonds. By symmetry to the lower bound affine investment rule, we get the following corollary. The proof of this is exactly the same as for the previous theorems, replacing $(W - \overline{W})$ by $(\overline{W} - W)$.

Corollary 14. *For the investment rule (32), the zero-switch utility functions are*

$$\begin{array}{lrcl} (a) & U(W) & = & \ln(\overline{W} - W) \\ (b) & U(W) & = & (\overline{W} - W)^b \end{array} \tag{33}$$

the one-switch

$$\begin{array}{lrcl} (a) & U(W) & = & a(\ln(\overline{W} - W))^2 + b\ln(\overline{W} - W) + c \\ (b) & U(W) & = & a(\overline{W} - W)^b + c(\overline{W} - W)^d \\ (c) & U(W) & = & a\ln(\overline{W} - W) + b(\overline{W} - W)^c \\ (d) & U(W) & = & (a\ln(\overline{W} - W) + b)(\overline{W} - W) \end{array} \tag{34}$$

and the $n-$ switch

$$f_0[\ln(\overline{W} - W)] + \sum_{i=0}^{k} (\overline{W} - W)^{c_i} f_i[\ln(\overline{W} - W)] \tag{35}$$

where $W < \overline{W}$ and $f_i(.)$ is a polynomial of order n_i such that

$$\sum_{i=0}^{k} n_i \leq n + 1 - k$$

The functions are all arbitrary at or above $\overline{W}$.

These are all the same as before only defined on the lower part of the distribution. These functions, while interesting in their own right, also cast light on a close relationship between switching and a popular general class of utility functions, which we now introduce.

3.3. Switching and the HARA functions. In order to get a better overview of the large number of functions we encountered so far, it will be convenient to introduce the **Hyperbolic Absolute Risk Aversion (HARA)** functions, defined by

$$U_{HARA}(W) = \frac{1-\gamma}{\gamma}\left[\frac{aW}{1-\gamma} + b\right]^{\gamma} \tag{36}$$

where $W > \frac{b(\gamma-1)}{a}$. This class is presented and analysed in depth in Ingersoll [19] and Eeckhoudt and Gollier [12]. A large body of commonly used functions are special cases of the HARA class. In particular, (36) contains the constant absolute risk aversion functions

$$U_{CARA}(W) = -\left(\frac{1}{\alpha}\right)e^{-\alpha W} \tag{37}$$

derived from (36) by setting $\frac{a}{b} = \alpha$ and letting $\gamma \to \infty$, and the constant relative risk aversion functions

$$U_{CRRA}(W) = \begin{cases} \frac{W^{1-\alpha}}{1-\alpha} & \alpha > 0, \alpha \neq 1 \\ \ln W & \alpha = 1 \end{cases} \tag{38}$$

which we get from (36) by setting $a = 1-\alpha, b = 0$ and $\gamma = \alpha$ (letting $\gamma \to 1$ gives the logarithm).

The zero-switch additive functions (see Theorem 1) are the constant absolute risk aversion HARA functions (37), while the zero-switch multiplicative utility functions (Theorem 4) are the constant relative risk aversion HARA functions (38). Apart from the limit functions mentioned above, the HARA class contains several polynomial functions, including the quadratic, which was shown to be additive one-switch (see 9). In general, there is an $n-$ degree polynomial function in the HARA class which will be an additive $(n-1)-$ switch function. The polynomial functions have the property that, in a financial setting, an investor with an $n-$degree polynomial utility function will care only about the first n moments of the return distribution. Kraus and Litzenberger [22] derive a $3-$moment Capital Asset Pricing Model which uses a cubic utility function, for example. Hence, the number of maximum switches allowed between any pair of additive gambles is equal to the order of the highest moment of the return distribution affecting the investment decisions less 1.

Additionally, the other one-switch additive functions are all defined as sums of, or sums of products of, separate HARA functions. The last major type of functions which are special cases of the HARA's can be derived by rearranging (36). It should be clear that simple manipulation of (36) leads to $U(W) = \frac{(1-\gamma)^{1-\gamma}}{\gamma}\left[aW + b(1-\gamma)\right]^{\gamma}$ and by setting $a = 1$ and $b(1-\gamma) = -\overline{W}$ (or $a = -1$ and $b(1-\gamma) = \overline{W}$) we recover the zero-switch functions corresponding to the risk exposures (26) (or (32)) from the previous section. Note that the logarithmic functions are again obtained in the limit. It has thus been shown that all the HARA functions can be identified as either zero-switch multiplicative functions, $n-$ switch additive functions for some non-negative integer n or zero-switch affine risk exposure functions. The following table sums up this discussion - ARA stands for Absolute Risk Aversion and DARA for Decreasing Absolute Risk Aversion. Parameters refer to (36). The double (x, y) under Rule denotes gamble type x (either additive (A), multiplicative (M) or affine risk exposure (ARE)), and whether it is a zero- or one-switch, i.e. $y = 0$ or 1.

Table 1. The HARA functions and their switching rules

	Parameters	ARA	DARA	Rule
Log	$a = 1, b = 0, \gamma = 0$	$\frac{1}{W}$	Always	(M,0)
Power	$a = 1 - \beta, b = 0, \gamma = \beta$	$\frac{(1-\beta)}{W}$	$\beta < 1$	(M,0)
Quadratic	$a = 2\beta, b = 1, \gamma = 2$	$\frac{2\beta}{(1-2\beta W)}$	Never	(A,1)
Exponential	$\frac{a}{b} = \beta, \gamma = -\infty$	-β	Never	(A,0)
$(W - \overline{W})^\beta$	$a = 1, b(1 - \gamma) = \overline{W}, \gamma = \beta$	$\frac{(1-\beta)}{(W-\overline{W})}$	$\beta < 1$	(ARE,0)
$(\overline{W} - W)^\beta$	$a = -1, -b(1 - \gamma) = \overline{W}, \gamma = \beta$	$\frac{(1-\beta)}{(\overline{W}-W)}$	$\beta < 1$	(ARE0)

(39)

In addition, the $n-$ switch additive gamble functions which are not in (36) can be modelled by the appropriate sum or product of HARA functions. As the HARA's are vastly popular in economics, this would add support to the use of preference switching as a property of desirable utility functions.

Next we introduce a known financial utility function which lies outside this popular domain, and analyse its relationship to preference switching.

3.4. Constant Proportional Portfolio Insurance. Black and Perold [10] discuss the theory of **Constant Proportional Portfolio Insurance (CPPI).** This is a multi-period investment rule in which you invest a certain fraction of your excess wealth (with respect to some target wealth level) in a risky asset and the rest in a riskless asset. Initially suggested as a plausible investment scenario by Black and Jones [9], CPPI works as follows : For a fixed wealth floor, let the cushion be the present level of wealth minus the floor. If we can invest in two assets, a riskless and/or a risky one, the exposure is defined as the investment in the risky asset. CPPI then requires that the exposure be a constant multiple of the cushion. Hence, when we are below the floor and the cushion is negative, we short the risky asset, when we are above the floor, we short the bond. Their main result, (see [10], Proposition 11) proves that the rule can be derived as an optimal steady-state policy for a multi-period optimisation of the following utility function, which is linear in the below-target returns but a power function in above-target wealth, i.e.

$$U_{bp(\alpha)}(W) = \begin{cases} \frac{1}{1-\alpha} W^{1-\alpha} & W \geq \eta \\ A + BW & W < \eta \end{cases} \tag{40}$$

where selecting $A = U_{bp(\alpha)}(\eta) - \eta U'_{bp(\alpha)}(\eta)$ and $B = WU'_{bp(\alpha)}(\eta)$ will guarantee continuity and differentiability.

Note that this is almost the reverse of the lower partial moment utility function (10). However, the fact that the power term on the above target section of the (40) is not in terms of deviations from the target means that we can not apply similar arguments to prove it is an $n-$ switch function for any of the earlier types of gambles. We need some other type of gamble to capture this function.

We will examine the conditions under which this utility function is zero-switch. Since we initially would like to employ as few distributional restrictions on the actual process for the gambles, it is worth observing that the section of (40) below the target is an additive zero-switch, while the above-target piece is a multiplicative zero-switch. Hence, as we would require the function to be zero-switch over both pieces, in the light of the sufficient conditions for zero-switching given in the previous sections, we could work with a piece-wise gamble rule to avoid restrictive distributional assumptions. In fact, consider the following definition of final wealth

$$W = \begin{cases} W_0\widetilde{X} & W_0 \geq \eta \\ W_0 + \widetilde{X} & W_0 < \eta \end{cases} \tag{41}$$

At low levels, according to this rule, gambles are additive, while at high levels gambles are multiplicative. This definition seems at least as plausible as the purely additive definition most commonly used, especially since most gambles available at very low wealth levels are lotteries, which are additive, while almost all other gambles are by definition multiplicative. Given this structure, we can prove the following theorem.

Theorem 15. *Given (41), (40) is a zero-switch function if we restrict the gambles to*

$$\left\{ \begin{array}{ll} W \geq \eta, & \text{if } W_0 \geq \eta \\ W < \eta, & \text{if } W_0 < \eta \end{array} \right\} \tag{42}$$

Proof. Consider two distinct gambles $\widetilde{X}$ and $\widetilde{Y}$. Clearly, if $W_0 < \eta$, $E[U(W_0 + \widetilde{X})] - E[U(W_0 + \widetilde{Y})] =$

$$\begin{aligned} \Pr\{W_0 + \widetilde{X} &< \eta\}E[A(W_0 + \widetilde{X}) + b] + \Pr\{W_0 + \widetilde{X} \geq \eta\}E\left[\frac{1}{1-\alpha}(W_0\widetilde{X})^{1-\alpha}\right] \\ - \Pr\{W_0 + \widetilde{Y} &< \eta\}E[A(W_0 + \widetilde{Y}) + b] - \Pr\{W_0 + \widetilde{Y} \geq \eta\}E\left[\frac{1}{1-\alpha}(W_0\widetilde{Y})^{1-\alpha}\right] \end{aligned}$$

However, the restrictions in (42) imply that if $W_0 < \eta$ then $\Pr\{W_0 + \widetilde{X} \geq \eta\} = 0$, and the above simplifies to

$$E[A(W_0 + \widetilde{X}) + b] - E[A(W_0 + \widetilde{Y}) + b] = E[\widetilde{X}] - E[\widetilde{Y}]$$

which is clearly sufficient for zero-switching. Likewise, when $W_0 < \eta$, $E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] =$

$$\begin{aligned} \Pr\{W_0\widetilde{X} &< \eta\}E[A(W_0 + \widetilde{X}) + b] + \Pr\{W_0\widetilde{X} \geq \eta\}E\left[\frac{1}{1-\alpha}(W_0\widetilde{X})^{1-\alpha}\right] \\ - \Pr\{W_0\widetilde{Y} &< \eta\}E[A(W_0 + \widetilde{Y}) + b] - \Pr\{W_0\widetilde{Y} \geq \eta\}E\left[\frac{1}{1-\alpha}(W_0\widetilde{Y})^{1-\alpha}\right] \end{aligned}$$

However, the restrictions in (42) imply that if $W_0 \geq \eta$ then $\Pr\{W_0 + \widetilde{X} < \eta\} = 0$, and the above simplifies to

$$E\left[\frac{1}{1-\alpha}(W_0\widetilde{X})^{1-\alpha}\right] - E\left[\frac{1}{1-\alpha}(W_0\widetilde{X})^{1-\alpha}\right] = \frac{W_0^{1-\alpha}}{1-\alpha}\left[E\left(\widetilde{X}^{1-\alpha}\right) - E\left(\widetilde{Y}^{1-\alpha}\right)\right]$$

which is also sufficient for zero-switching ■

We make no claim that these are necessary conditions, nor indeed that necessary conditions with such simple gamble restrictions exist. The weaknesses with the above arguments is just the weaknesses of additive gambles mentioned in section 2. Next we investigate the possibility of restricting the domain of the gambles over which switching takes place.

3.5. Restricting the domain of gambles. Up to now, we have concentrated mainly upon a single paper, Bell [5]. Before moving on to his extensions to risk-return separation, we temporarily turn to the related work of Jia and Dyer [20] and [21]. For this, we need some further notation. For the additive or multiplicative gambles considered so far, we introduce two important subclasses. Let P be the set of probability distributions (or lotteries) on a non-empty set of outcomes. Denote by $\widetilde{X}$ a general member of P. Let

$$P^A = \{\widetilde{X}^A \mid \widetilde{X}^A = \widetilde{X} - E[\widetilde{X}], \widetilde{X} \in P\} \tag{43}$$

be defined as the 'normalised distributions with respect to additive risk' and

$$P^M = \{\widetilde{X}^M \mid \widetilde{X}^M = \frac{\widetilde{X}}{E[\widetilde{X}]}, \widetilde{X} \geq 0, \widetilde{X} \in P\} \tag{44}$$

the 'normalised distributions with respect to multiplicative risk'. Note that all members of P^A have mean zero, while members of P^M have the property that the mean of their logarithms is zero. Note further that since we must have positive final wealth, in the multiplicative case the only gamble with $E[\widetilde{X}] = 0$ is degenerate. Consequently, we have no trouble of division by zero in (44). Define preferences $\succ_P^A$ over lotteries in P^A (additive risk), which is the case studied by Jia and Dyer [20], and preferences $\succ_P^M$over P^M (multiplicative risk), presented in [21]. In either case, preferences are the inverse of risk preferences, $\succ_R^A$and $\succ_R^M$respectively.

Definition 16. *The risk preferences and preferences over lotteries satisfy consistency conditions such that* $\widetilde{X}^A \succeq_P^A \widetilde{Y}^A \Leftrightarrow \widetilde{X}^A \preceq_R^A \widetilde{Y}^A$ *and* $\widetilde{X}^M \succeq_P^M \widetilde{Y}^M \Leftrightarrow \widetilde{X}^M \preceq_R^M \widetilde{Y}^M$ *for additive and multiplicative gambles respectively.*

Jia and Dyer also introduce the notion of **Risk Independence.** In the next section, we investigate its precise connection with Bell's preference switching.

Equivalence of Risk independence and zero-switching. Bearing in mind that Jia and Dyer work with gambles of equal wealth only, we recall their definition of risk independence.

Definition 17. *A gamble $\widetilde{X}^A$ is **Risk Independent** if for all $\widetilde{Y}^A$,*

$$\widetilde{X}^A \succeq_P^A \widetilde{Y}^A \Rightarrow \widetilde{X}^A + \overline{W} \succeq_P^A \widetilde{Y}^A + \overline{W} \tag{45}$$

for all $\overline{W} \in \mathrm{Re}$.

This could be defined analogously for multilpicative gambles. However, we stay with the additive framework. Risk independence with multiplicative gambles is investigated in Pedersen [26]. We now show that this is in fact equal to zero-switching.

Theorem 18. *All gambles $\widetilde{X}^A$ are **Risk Independent** if and only if $\succeq_P^A$ is an additive zero-switch preference relation over gambles with identical means.*

Proof.

($\Rightarrow$) Suppose that $\succeq_P^A$ admits a switch in preferences between two gambles $\widetilde{X}^A$ and $\widetilde{Y}^A$, where $\widetilde{X}^A \succ_P^A \widetilde{Y}^A$ and $\widetilde{Y}^A$ and $\widetilde{X}^A$ have identical means. This implies there exists two distinct wealth levels W_0 and $\overline{W}$ such that (without loss of generality) $W_0 + \widetilde{X}^A \succ_P^A W_0 + \widetilde{Y}^A$ and $\overline{W} + \widetilde{X}^A \prec_P^A \overline{W} + \widetilde{Y}^A$. Consider the gambles $\widetilde{Y}_1^A = W_0 + \widetilde{Y}^A$ and $\widetilde{X}_1^A = W_0 + \widetilde{X}^A$. Since $E[\widetilde{X}^A] = E[\widetilde{Y}^A]$, we have $E[\widetilde{Y}_1^A] = E[\widetilde{X}_1^A]$, so that $\widetilde{X}_1^A$ and $\widetilde{Y}_1^A$ are in the class of identical mean gambles. Then we have that $\widetilde{X}_1^A \succ_P^A \widetilde{Y}_1^A$ and $(\overline{W} - W_0) + \widetilde{X}_1^A \prec_P^A (\overline{W} - W_0) + \widetilde{Y}_1^A$, where $(\overline{W} - W_0) \in \mathrm{Re}$, which violates the risk independence condition.

($\Leftarrow$) Suppose now that $\succeq_P^A$ admits no switches in preferences between any two gambles. Suppose that $\widetilde{X}^A \succ_P^A \widetilde{Y}^A$ without loss of generality. Then for all $\overline{W} \in \mathrm{Re}$, the zero-switching implies that $\overline{W} + \widetilde{X}^A \succ_P^A \overline{W} + \widetilde{Y}^A$ ■

This link enables us to translate the main results of Jia and Dyer with reference to Bell. Indeed, the difference in the two approaches is that the family of gambles considered by Jia and Dyer is restricted to identical mean gambles. To confirm this, we first show that the Jia and Dyer analysis can be replicated by adapting the original proofs in Bell, and then use this result to motivate an extension of the Bell analysis to allow for not only multiplicative risk, but also to the possibility of restricting the domain to gambles whose first k moments are identical.

Jia and Dyer show that the only utility functions which satisfy risk independence are

$$\begin{array}{llll} (a) & U(W) & = & aW - bW^2 \\ (b) & U(W) & = & aW - be^{-cW} \end{array} \tag{46}$$

Their proof utilises a result on utility independence in multiattribute utility theory from Keeney and Raiffa (pages 224-229). Note that this contains the linear and exponential utility functions which Bell showed to be the only zero-switch utility functions with no restrictions on the dsitribution moments. Indeed, as the following result will confirm, this result can be derived from the Bell proofs by utilising the equivalence between risk independence and zero-switching presented above.

Theorem 19. *The set of additive zero-switch utility functions is enlarged from the linear and exponential to (46) if we allow only gambles of identical mean.*

Proof. (Sufficiency)

Consider each function in turn. Note that it is sufficient to show that the expression $E[U(W_0+\widetilde{X})]-E[U(W_0+\widetilde{Y})]=0$ has no non-trivial solution in W_0.

(a) $E[U(W_0+\widetilde{X})]-E[U(W_0+\widetilde{Y})]$

$$\begin{aligned} &= E\{a(W_0+\widetilde{X})-b(W_0+\widetilde{X})^2-a(W_0+\widetilde{Y})+b(W_0+\widetilde{Y})^2\} \\ &= aW_0(E(\widetilde{X})-E(\widetilde{Y}))-b(E(W_0+\widetilde{X})^2-E(W_0+\widetilde{Y})^2) \\ &= -b(E(\widetilde{X})^2-E(\widetilde{Y})^2) \end{aligned}$$

since $E(\widetilde{X})=E(\widetilde{Y})$. This is independent of W_0.

(b) $E[U(W_0+\widetilde{X})]-E[U(W_0+\widetilde{Y})]$

$$\begin{aligned} &= -E\{be^{-c(W_0+\widetilde{X})}-be^{-c(W_0+\widetilde{Y})}\} \\ &= -be^{-cW_0}(E(e^{-c\widetilde{X}})-E(e^{-c\widetilde{Y}})) \end{aligned}$$

which will not be zero for non-trivial values of W_0 and general values of b and c (the linear terms cancel since $E(\widetilde{X})=E(\widetilde{Y})$).

(Necessity)

Suppose that $\widetilde{X}$ and $\widetilde{Y}$ are gambles. Following the proof of Theorem 5, for any wealth level W_0, suppose $E[U(W_0+\widetilde{X})]-E[U(W_0+\widetilde{Y})]=$

$$\sum_{i=1}^{\infty} U^i(W_0)m_i = a(W_0) \tag{47}$$

where $m_i=\frac{E[\widetilde{X^i}]-E[\widetilde{Y^i}]}{i!}$. If $U(.)$ exhibits zero-switching, it must be that for any pair of wealth levels, W_0 and W_1, $a(W_0)a(W_1)\geq 0$. This implies that $U(W)$ satisfies, for real non-zero θ, the differential equation

$$U''(W)+\theta U'(W)=0 \tag{48}$$

We know that the possible solution to this equation are the linear and the exponential utility functions. Suppose we instead impose a restriction that $E(\widetilde{X})=E(\widetilde{Y})$. In this case, $m_i=0$ and (47) becomes

$$\sum_{i=2}^{\infty} U^i(W_0)m_i = a(W_0) \tag{49}$$

and the minimal necessary condition (48) is

$$U'''(W) + \theta U''(W) = 0 \tag{50}$$

which simply solves for the functions in (46) ■

A general preference switching theorem. Immediate extensions become apparent. It may be in our interest to consider the specific question of how preferences change between two gambles which can not be distinguished by their first two moments. When dealing with non-normal distributions, the skewness and kurtosis measures become crucial to decision making. Kraus and Litzenberger [22] provide arguments as to why this may be favourable in finance to motivate their three-moment Capital Asset Pricing Model. When distributions are such that the first two moments are virtually indistinguishable, we will rely exclusively upon the higher moments to provide input for investment decisions. This leads us to the following extension, which allows us to admit more utility functions as n-switch under more restrictive assumptions on the gambles. In particular, by following the analysis of the previous proof, one can conclude the following.

Theorem 20. *A continuously differentiable $n-$ switch utility function for all gambles whose first k moments are identical necessarily satisfies the differential equation*

$$\sum_{i=k+1}^{n+k+2} \lambda_i U^i(W) = 0 \tag{51}$$

if gambles are additive, and

$$\sum_{i=k+1}^{n+k+2} \lambda_i W^i U^i(W) = 0 \tag{52}$$

if gambles are multiplicative. In both cases we require that λ_i are real for all i and $\lambda_i \neq 0$ for at least one i.

Proof. We do the additive case. The multiplicative case follows similarly. Recall the expansion

$$\begin{aligned} & E[U(W_k + \widetilde{X})] - E[U(W_k + \widetilde{Y})] \\ = & \sum_{i=1}^{\infty} U^i(W_k) m_i = a_a(W_k) \end{aligned} \tag{53}$$

where $m_i = \frac{E[\widetilde{X}^i] - E[\widetilde{Y}^i]}{i!}$, which holds for additive gambles.

We have already seen the necessary conditions for zero-switching and one-switching. Clearly, you need at least two different wealth levels to give a counter-example to zero-switching (i.e. exhibit at least one single switch) and similarly three wealth levels to find a counter-example to one-switching. In each case, this equates to finding solutions to (53) for restricted values of $a_a(W_k)$. Hence, to found a counter-example to $n-$ switching, we should need to solve the above for $(n+1)$ different wealth levels and some restricted set of values $a_a(W_k)$. A necessary condition for $n-$ switching is therefore that it should NOT be possible to find a general solution to the system

$$\begin{aligned} \sum_{i=1}^{\infty} U^i(W_1)m_i &= a_a(W_1) \\ \sum_{i=1}^{\infty} U^i(W_2)m_i &= a_a(W_2) \\ &\dots\dots\dots\dots \\ \sum_{i=1}^{\infty} U^i(W_{n+1})m_i &= a_a(W_{n+1}) \end{aligned} \tag{54}$$

for distinct wealth levels $W_1 < W_2 < .... < Wn < W_{n+1}$, since if a general solution can be found, so can any specific solution. But we are restricting the first k moments of $\widetilde{X}$ and $\widetilde{Y}$ to be identical. This corresponds to having $m_i = 0$ for $i = 1$ to k. Hence, (54) becomes

$$\begin{aligned} \sum_{i=k+1}^{\infty} U^i(W_1)m_i &= a_a(W_1) \\ \sum_{i=k+1}^{\infty} U^i(W_2)m_i &= a_a(W_2) \\ &\dots\dots\dots\dots \\ \sum_{i=k+1}^{\infty} U^i(W_{n+1})m_i &= a_a(W_{n+1}) \end{aligned} \tag{55}$$

for which we need to ensure no general solution can be found. This in turn requires at the minimum that the sub-system

$$\begin{aligned} \sum_{i=k+1}^{n+k+2} U^i(W_1)m_i &= a_a(W_1) \\ \sum_{i=k+1}^{n+k+2} U^i(W_2)m_i &= a_a(W_2) \\ &\dots\dots\dots\dots \\ \sum_{i=k+1}^{n+k+2} U^i(W_{n+1})m_i &= a_a(W_{n+1}) \end{aligned} \tag{56}$$

has no general solution. Note that this has $(n+1)$ equations, $(n+1)$ unknowns and involves terms in only the $(k+1), (k+2)....(n+k+2)$ derivatives of $U(W)$. The subsystem will not have a general solution if these derivatives are colinear, i.e. if

$$\sum_{i=k+1}^{n+k+2} \lambda_i U^i(W) = 0 \tag{57}$$

where λ_i are real for all i and $\lambda_i \neq 0$ for at least one i. This proves the additive case.

The multiplicative case is identical except the initial expansion is

$$\begin{aligned} & E[U(W_0\widetilde{X})] - E[U(W_0\widetilde{Y})] \\ = & \sum_{i=1}^{\infty} U^i(W_0)W_0^i m_i = a(W_0) \end{aligned} \tag{58}$$

Following exactly the steps as those in the additive case then gives the result ■

This theorem encompasses all Bell's theorems as well as extensions to Jia and Dyer's work, and is studied in more detail in Pedersen [26], where the solution to (57) is investigated. However, for the purposes of this paper, we include the result to show how the switching approach captures alternative notions of risk and return, which was a major part of Jia and Dyer's work. We now turn our attention to the question of risk-return separation in greater detail.

4. One-switching and risk-return separation

This section deals with the link between one-switching and risk-return separation. Bell addresses this issue in detail for in [7]. For our purposes, we use just the following Lemma.

Lemma 21. *Suppose preferences $\preceq$ are strictly continuous and monotone. Then the following are equivalent.*

(i) If $\widetilde{X} \sim \widetilde{Y}$ at distinct wealth levels W_0 and W_1, then $\widetilde{X} \sim \widetilde{Y}$ at all wealth levels.

(ii) $\preceq$ is a one-switch preference relation.

Proof.

$(i) \Rightarrow (ii)$ Suppose that $\preceq$ is not a one-switch preference relation. Then there exists two distinct gambles $\widetilde{X}$ and $\widetilde{Y}$, such that for three distinct wealth levels, $W_1 > W_2 > W_3$, we have $\widetilde{X} \succ \widetilde{Y}$ at W_1, $\widetilde{X} \prec \widetilde{Y}$ at W_2 and $\widetilde{X} \succ \widetilde{Y}$ at W_3. By continuity, there will be two further wealth levels, W_{12} and W_{23}, such that $W_1 > W_{12} > W_2 > W_{23} > W_3$, at which $\widetilde{X} \sim \widetilde{Y}$. (i) then implies that $\widetilde{X} \sim \widetilde{Y}$ at all wealth levels. But $\widetilde{X} \succ \widetilde{Y}$ at W_1, a contradiction.

$(i) \Leftarrow (ii)$ Suppose that (i) is not true, i.e. $\widetilde{X} \sim \widetilde{Y}$ for wealth levels W_1 and W_2, where $W_1 \neq W_2$, but $\widetilde{X} \succ \widetilde{Y}$ at some W_3. However, if preferences are one-switch, there is a unique wealth level W_u above which (without loss of generality) $\widetilde{X} \succ \widetilde{Y}$ and below which $\widetilde{X} \prec \widetilde{Y}$. In other words at exactly one wealth level do we have $\widetilde{X} \sim \widetilde{Y}$. But $\widetilde{X} \sim \widetilde{Y}$ both W_1 and W_2 ■

Note that this Lemma applies regardless of whether risk is defined additively or multiplicatively and so generalises part of Bell's result. Also its proof goes explicitly via the one-switch property, utilising the earlier Lemma, as opposed to Bell's, which, due to his more general statement, employs difference equations which yield the same solutions as the one-switch differential equations.

From this theorem it follows that if we can define a model which satisfies condition (i) of the theorem, we immediately know the form of possible utility functions congruent to that model. One such model is the important risk-value models of Bell [7] and [8], which we now present.

4.1. Risk-value models. In this section, we look at which utility functions adequately separate risk and return. In [7], Bell examines specific risk-value models and derives several results from treating risk as a seperate entity. His motivation for this was the apparent absence of explicit formulations of such seperation in the original Von Neumann-Morgernstern analysis. A later paper [8] rederives the results for multiplicative risk without using the relationship to one-switching explicitly. However, both these derivations assume the utility functions are infinitely differentiable, which excludes cases we consider important.

Formally, he assumes people evaluate an uncertain prospect $\widetilde{X}$ by the function

$$f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right)$$

where $r(\widetilde{X})$ is a return function, $R(\widetilde{X})$ is a risk function (both to be defined) and W_0 the wealth level. An expected utility maximiser is consistent with such a two-parameter decision model if

$$E[U(W_0 + \widetilde{X})] = f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right) \tag{59}$$

As an example, consider (24). Clearly, $E[U(W_0 + \widetilde{X})]$

$$\begin{aligned} &= E\left[a(W_0 + \widetilde{X}) - e^{-c(W_0+\widetilde{X})}\right] \\ &= aW_0 + aE[\widetilde{X}] - e^{-cW_0}E\left[e^{-c\widetilde{X}}\right] \end{aligned}$$

Hence, if $r(\widetilde{X}) = E[\widetilde{X}]$ and $R(\widetilde{X}) = E\left[e^{-c\widetilde{X}}\right]$, then (59) is satisfied for $f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right) = aW_0 + ar(\widetilde{X}) - e^{-cW_0}R(\widetilde{X})$. Note that $r(\widetilde{X})$ and $R(\widetilde{X})$ are not unique. The induced risk measure, $E\left[e^{-c\widetilde{X}}\right]$, is one of the measures derived in the psychology literature (see Pedersen and Satchell [27] or Sarin and Weber [31] for details). The model (59) assumes additive risk. The equivalent model for multiplicative risk would be

$$E[U(W_0\widetilde{X})] = f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right) \tag{60}$$

For an example of the advantage of this approach over (59) in some cases, consider the case of the following linear/logarithmic utility function, $U(W) = aW + \ln W$. Clearly, $E[U(W_0 + \widetilde{X})]$

$$\begin{aligned} &= E\left[a(W_0 + \widetilde{X}) + \ln(W_0 + \widetilde{X})\right] \\ &= aW_0 + E[\widetilde{X}] + E[\ln(W_0 + \widetilde{X})] \end{aligned}$$

which will not reduce to $f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right)$ for a sensible risk measure $R(\widetilde{X})$. A good selection for return would be $r(\widetilde{X}) = E(\widetilde{X})$, but this still leaves the term $E[\ln(W_0 + \widetilde{X})]$, which is not separable in terms of W_0 and $\widetilde{X}$ for general gambles. However, the model (60) would give $E[U(W_0\widetilde{X})] = E[a(W_0\widetilde{X}) + \ln(W_0\widetilde{X})] = W_0E(\widetilde{X}) + \ln W_0 + E[\ln(\widetilde{X})]$, which is of the required form and we can select as a measure of risk the function $R(\widetilde{X}) = E[\ln(\widetilde{X})]$, which is a risk measure derived in Luce [24] from a mathematical axiomatisation approach.

By introducing different axioms on the functions $R(.), r(.), U(.)$ and $f(.,.,.)$ Bell derives conditions under which he can apply the above Lemma to identify the corresponding utility functions for the additive case. We will focus upon the application of multiplicative risk, and so fix a basic set of axioms with which to work. The application of multiplicative gambles to risk-return models was touched upon in Sarin and Weber [31]. The following theorem[3] is similar to that in Bell [8], but the proof is our own, which adapts a link to switching.

Axiom 1. *(Monotonicity)*
We assume that $U(.)$ is strictly increasing in W_0 and f strictly increases with $r(\widetilde{X})$.

Axiom 2. *(Relative Risk Aversion)*
We assume that $U(.)$ displays relative risk aversion and that f decreases in $R(\widetilde{X})$.

Axiom 3. *(Decreasing Relative Risk Aversion)*
We assume that $U(.)$ displays decreasing relative risk aversion. For f we assume that if $\widetilde{X} \sim \widetilde{Y}$ for some wealth level W_0, then $\widetilde{X} \succ \widetilde{Y}$ at higher values of wealth if and only if $R(\widetilde{X}) > R(\widetilde{Y})$.

Following the approach of Bell, we present the following theorem :

Theorem 22. *Suppose that $U(W)$ is continuous, gambles are defined multiplicatively and the three axioms above are satisfied. Then if*

$$E[U(W_0\widetilde{X})] = f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right)$$

either $U(W) = a\ln W - bW^{-c}$ or $U(W) = -aW^{-b} - cW^{-d}$, where a, b, c, d, are all positive.

[3]We thank David Bell for discussing his latest results with us.

Proof. By Axiom 3, it is clear that two gambles willl be indifferent at two different wealth levels only if $R(\widetilde{X}) = R(\widetilde{Y})$. Since f is strictly increasing in $r(.)$, this will imply that $r(\widetilde{X}) = r(\widetilde{Y})$. But then $\widetilde{X}$ must be indifferent to $\widetilde{Y}$ at all wealth levels, since preferences are only a function of $R(.)$ and $r(.)$. By Lemma 14, this implies that $U(.)$ must be one-switch, i.e. be one of (19). The only functions in (19) which are increasing, relative risk averse and has decreasing relative risk aversion are $U(W) = a \ln W - bW^{-c}$ and $U(W) = -aW^{-b} - cW^{-d}$, where a, b, c, d, are all positive. This we saw in the proof of Theorem 6 ■

These two utility functions give us two other utility based risk measures than the additive ones already given by Bell and an alternative return measure than the expected value. Several risk-value model with return measures restricted to the expected value were discussed briefly in Sarin and Weber [31]. For $U(W) = a \ln W - bW^{-c}$, we have

$$\begin{aligned} E[U(W_0\widetilde{X})] &= E[a\ln(W_0\widetilde{X}) - b(W_0\widetilde{X})^{-c}] \\ &= a\ln W_0 + aE[\ln\widetilde{X}] - bW_0^{-c}E[\widetilde{X}^{-c}] \end{aligned}$$

and we can let $r(\widetilde{X}) = E[\widetilde{X}^{-c}], R(\widetilde{X}) = E[\ln \widetilde{X}]$. For $U(W) = -aW^{-b} - cW^{-d}$, we get

$$\begin{aligned} E[U(W_0\widetilde{X})] &= E[-a(W_0\widetilde{X})^{-b} - c(W_0\widetilde{X})^{-d}] \\ &= -aW_0^{-b}E[\widetilde{X}^{-b}] - cW_0^{-d}E[\widetilde{X}^{-d}] \end{aligned}$$

and we can elect $r(\widetilde{X}) = E[\widetilde{X}^{-b}], R(\widetilde{X}) = E[\widetilde{X}^{-d}]$. Both risk measures arrived at, i.e. $R(\widetilde{X}) = E[\ln \widetilde{X}]$ and $r(\widetilde{X}) = E[\widetilde{X}^{-d}]$ were derived from an axiomatisation by Luce [24], and complement the measures derived by Bell [5]. The return measure $R(\widetilde{X}) = E[\ln \widetilde{X}]$ is important. For multiplicative risks, Bell [8] argues convincingly that this is the correct measure of return, rather than the commonly used $E[\widetilde{X}]$.

Note also that the risk and return measures which derive from the augmented risk-return model

$$E[U((W_0 - \overline{W})\widetilde{X} + \overline{W})] = f\left(r(\widetilde{X}), R(\widetilde{X}), W_0, \overline{W}\right) \tag{61}$$

corresponding to the affine risk exposures will be the same as those from the multiplicative risk-value model, when applied to the one-switch functions (29). This follows from the definition of final wealth and the relation to the multiplicative structure. The additional parameter $\overline{W}$ in the model is the minimum wealth requirement. Its effect is simply a shift in the wealth expression, and it will not affect the measures of volatility of the gambles or the volatility of final wealth which emerge from the theorem.

As a final note, it is worth pointing out that we can use this approach to verify the variance measure of risk, the most widely used in financial models and the basis for much criticism directed at the traditional CAPM (see e.g. [27]), is consistent with the quadratic utility function in the additive setting. As we have already seen, the quadratic utility function has increasing risk aversion, which is not empirically acceptable. Hence, the risk-return model is further evidence against the use of the variance as a measure of risk. We should rather favour those emerging from the theorem if we believe the theory.

The above analysis still implicitly assumes infinite differentiability everywhere. Recall the example (10) discussed earlier, derived from the mean-lower partial moment decision rule, which was shown to be either a zero-switch additive function or a multiplicative one-switch, depending upon the way the target is defined. In either case, it is not infinitely differentiable everywhere. We would expect a sensible risk-value model to associate this utility function with the lower partial moment risk measure, $R(\widetilde{X}) = E[(\tau - \widetilde{X})^\alpha \mid \widetilde{X} < \tau]$, which has been extensively studied by finance researchers (see Pedersen and Satchell [27] for references). To verify this, consider the multiplicative representation of (10). Recall that this gives $EU(W_0\widetilde{X})$

$$\begin{aligned}&= \Pr\{W_0\widetilde{X} > \eta\}E[\lambda(W_0\widetilde{X}) \mid W_0\widetilde{X} > \eta]\\ &+ \Pr\{W_0\widetilde{X} < \eta\}(E[\lambda(W_0\widetilde{X}) - (\eta - W)^\alpha] \mid W_0\widetilde{X} < \eta)\\ &= \lambda E[\widetilde{X}] - W_0^\alpha \Pr\{\widetilde{X} < \tau\}E[(\tau - \widetilde{X})^\alpha \mid \widetilde{X} < \tau]\end{aligned}$$

Consequently, if we let $R(\widetilde{X}) = E[(\tau - \widetilde{X})^\alpha \mid \widetilde{X} < \tau]$ and $r(\widetilde{X}) = E[\widetilde{X}]$, we can write $EU(W_0\widetilde{X}) = f\left(r(\widetilde{X}), R(\widetilde{X}), W_0\right)$ and if we keep $\lambda > 0$, expected utility is increasing in the mean but decreasing in the lower partial moment, thus recovering the underlying mean-lower partial moment decision rule.

5. Conclusion

Previous literature has used risk aversion measures and distributional asumptions to characterise favourable utility functions. We link several papers in the management science and operational research literature to modern finance theory, most notably those of Bell ([5] and [7]), and propose an alternative way of characterising utility functions by how many switches they allow between any pair of gambles as initial wealth increases. Given the potential complexity of all interactions between W_0, the distribution of gambles and the nature of the utility function, low order switching seems a desirable simplification. We get very specific results rather than the broader theorems of earlier works.

We have derived the forms of utility functions which allow at most n switches between any pair of gambles for both additive (as done in Bell [5]) and multiplicative gambles, as well as for a more complicated class of risk exposures. It is shown that any utility function in the HARA class (36), which covers vitually all applications of utility functions to financial problems, is either an additive or multiplicative gamble zero-switch utility function, an $n-$ switch additive utility function or a risk exposure zero-switch utility function. The HARA class contains exponential, polynomial, power and logarithmic functions, as well as functions defined in deviations from a target.

Additionally, we have illustrated that by relaxing the assumption of infinite differentiability everywhere, the piece-wise mean-lower partial moment utility function (10) of Fishburn [15] can be either a zero-switch additive or a one-switch multiplicative function, depending upon the definition of the target. We do not pursue this point further, but note that Pedersen [26] extends this to cover other piece-wise functions. The Constant Proportional Portfolio Insurance utility function (40) is analysed in detail. It is shown that we can make this a zero-switch gamble for a piece-wise gamble rule subject to severe restrictions on the gambles themselves.

It is proved that the risk independence condition of Jia and Dyer [20] is equal to the zero-switch property, so that their analysis, which concentrates on gambles with identical mean, is closely related to Bell's work. A general theorem is proved which gives the necessary conditions for functions which allow at most n switches between any pair of gambles whose first k moments are identical.

Finally, we use the fact that the one-switch property is equivalent to a necessary condition for a utility function to separate risk and return to analyse which risk measures can be recovered from utility functions. It is shown that in the multiplicative case, we recover several risk measures presented and axiomatised in the psychology literature (see, e.g Luce [24]) and those derived in Bell [8], while we can recover the lower partial moment risk measure in this structure by relaxing the assumption of infinite differentiability.

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