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Relative Risk-Value Models

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Abstract

In this paper we propose a relative risk-value model and derive a relative measure of risk for the lotteries with positive-outcomes. Under a condition called relative risk independence, a decision could be made by explicitly trading off between the relative measure of risk and a measure of value, which can either be consistent with some expected utility models or represent nonexpected utility preferences. Specifically, this type of risk-value model is associated with power (or linear plus power) and logarithmic (or linear plus logarithmic) functions. We address some prescriptive and descriptive implications of our relative risk-value framework, and show that our generalized relative risk-value model is very flexible for modeling individuals' preferences and can explain many decision paradoxes.

Keywords: *Utility theory, risk measure, risk-value models, decision paradoxes.*

1. Introduction

Risk-value frameworks have proven to be useful in providing additional insights into decision making under risk (Dyer, 1987; Bell, 1988, 1995; Sarin and Weber, 1993; Jia and Dyer, 1994, 1995). In our previous study (Jia and Dyer, 1994), we proposed a standard measure of risk that is based on normalized lotteries with zero-expected values, obtained by converting the outcomes of a lottery into differences from the mean of the lottery. Under a condition called risk independence, an expected utility model (von Neumann and Morgenstern, 1947) can be decomposed into a value measure, a tradeoff factor, and a standard measure of risk. This standard risk-value model allows a decision to be made by explicitly trading off between risk and value. Specifically, the standard risk-value model is associated with the quadratic, exponential and linear plus exponential utility models. In Jia and Dyer (1995), we further generalized our risk-value framework for nonexpected utility preferences and showed that our generalized risk-value models can explain many decision paradoxes.

When decision problems involve lotteries with nonnegative outcomes only, we may have an alternative decomposition for these lotteries. Suppose that someone invests in the stock market, and the total return of the investor's portfolio is a random variable $\tilde{X} \geq 0$ (i.e., the worst case is to lose all money invested). We can represent the return variable in the form, $\tilde{X} = \bar{X} * (\tilde{X}/\bar{X})$, which decomposes the return variable into the average return $\bar{X}$ and a risk related factor $(\tilde{X}/\bar{X})$. If risk is not involved, then $(\tilde{X}/\bar{X}) = (\bar{X}/\bar{X}) = 1$ for any $\tilde{X} = \bar{X} > 0$; otherwise, $(\tilde{X}/\bar{X})$ is a random variable. The quantity $(\tilde{X}/\bar{X})$ provides a measure of the risk associated with the possible return relative to the anticipated return. For this case, the risk discussed should have a different meaning from our standard measure of risk in decision making. Thus, a new type of risk-value framework should be established.

The rest of this paper is organized as follows. In the next section, we propose a relative risk-value model that is consistent with the expected utility model under a relative risk independence condition, and derive a relative measure of risk from this risk-value model. We also discuss some financial implications of our framework. In Section 3, we generalize our relative risk-value framework for non expected utility preferences and propose some useful generalized relative risk-value models for descriptive decision studies. Section 4 shows that our generalized relative risk-value models are very flexible in modeling risky preferences and can explain many decision paradoxes. Finally, in Section 5, we summarize our studies on this topic. All proofs of theorems are provided in the appendix.

2. Relative risk-value models and a measure of risk

In this section, we discuss under what condition an expected utility model can be decomposed into an explicit form of relative risk-value tradeoffs, from which we also derive a relative measure of risk. We show some implications of our risk-value model in prescriptive studies in decision making under risk.

2.1 Basic assumption and results

Let P_+ be a convex set of all simple probability distributions or lotteries $\{X, Y, Z, ...\}$ on a nonempty set $\mathbf{X}^+$ of nonnegative outcomes. (Degenerate lotteries are required to be positive.) For convenience, we will use X, Y and Z to refer to probability distributions and random variables interchangeably. As motivated in our introduction, we define a relative risk set P_+^* as follows:

$$P_+^* = \{X^* \mid X^* = X / \bar{X},\ X \in P_+\}$$

Thus, P_+^* is a set of relatively normalized probability distributions or random variables. We will call $X^* \in P_+^*$ the relative risk variable of a lottery X. It is important to note that all relative risk variables in P_+^* are dimensionless, and have the same expected value $E(X^*) = 1$. A risk variable can be understood as a return rate relative to the expected return. It reflects the relative riskiness of a lottery with respect to the mean return.

The relative risk variable can be incorporated into expected utility models explicitly or implicitly. Using the simple transformation discussed in the introduction, an expected utility model can be written as $\mathrm{E}[u(X)] = \mathrm{E}[u(\bar{X} * \frac{X}{\bar{X}})] = \mathrm{E}[u(\bar{X} * X^*)]$, where the symbol E represents expectation over the probability distribution of a lottery. The question we will address is under what condition can we decompose the expected utility model into a separable form in terms of relative risk and expected return.

Let $>_p$ be defined as a binary preference relation on P_+. Similar to our risk independence condition for the standard risk-value model (Jia and Dyer, 1994), we define a relative risk independence condition as follows.

Definition 1. $X^* \in P_+^*$ is relatively risk independent of $\bar{w} \in Re_+$ if for any $Y^* \in P_+^*$ and some $\bar{w}_o \in Re_+$ such that $\bar{w}_o * X^* >_p \bar{w}_o * Y^*$, then $\bar{w} * X^* >_p \bar{w} * Y^*$ for all $\bar{w} \in Re_+$, where Re_+ is the set of positive real numbers.

Here $\bar{w}$ can be interpreted as an expected return or an expected wealth level. The representation of $\bar{w} * X^*$ just means that all outcomes of the normalized lottery X^* are multiplied by the constant $\bar{w}$. Relative risk independence implies that if two investment options have the same expected return, then an investor's preference over the two options will not change by increasing or decreasing the expected return or the expected wealth level by an equal amount. This also implies that the investor's preference between two options can be determined simply by the ranking of their relative riskiness defined on P_+^* if the two options have the same expected return.

THEOREM 1. Assume the existence of an expected utility model on $\mathbf{X}^+$. Then

$$\mathrm{E}[u(X)] = u(\bar{X}) - j(\bar{X})[R(X^*) - R(1)] \qquad (1)$$

if and only if the relative risk independence condition holds, where $j(\bar{X}) > 0$ and $R(X^*) = -\mathrm{E}[u(X / \bar{X})]$.

Model (1) has intuitive appeal in risky decision making. If a decision maker is risk averse, then his utility function u is concave; thus $R(X^*) > R(1)$ (by Jensen's inequality, we should have $\mathrm{E}[u(X^*)] < u[\mathrm{E}(X^*)] = u(1)$, or $R(X^*) = -\mathrm{E}[u(X^*)] > -u(1) = R(1)$). In the utility evaluation of a lottery, the decision maker may first anchor his initial utility at $u(\bar{X})$, and then make an adjustment by reducing it an amount proportional to $[R(X^*) - R(1)]$. $\varphi(\bar{X})$ is a tradeoff factor that may depend on the expected value. We will refer to model (1) as the "relative risk-value model" and $R(X^*)$ as the "relative measure of risk." Because a utility function is unique up to a positive linear transformation, the relative measure of risk $R(X^*) = -\mathrm{E}[u(X/\bar{X})]$ is also unique up to a positive linear transformation. For instance, $[R(X^*) - R(1)]$ is also a relative measure of risk.

Note that in this paper we do not distinguish between a lottery and a lottery plus a wealth level, so that a "lottery" X may be composed of one's initial wealth level w and a "true" lottery Y, i.e., $X = w + Y$. For this case, the risk variable is defined as $X^* = (w+Y)/(w+\bar{Y})$, where $\bar{X} = (w+\bar{Y})$ is the expected wealth level. Substituting them into (1), we can have

$$\mathrm{E}[u(w+Y)] = u(w+\bar{Y}) - \varphi(w+\bar{Y})\{R[(w+Y)^*] - R(1)\} \tag{2a}$$

where $R[(w+Y)^*] = -\mathrm{E}[u[(w+Y)/(w+\bar{Y})]]$. For a concave utility function and a given lottery, the relative measure of risk $R[(w+Y)^*]$ decreases as wealth level w increases (because the variation of $(w+Y)/(w+\bar{Y})$ decreases as w increases). This also implies that this risk measure decreases as a constant amount is added to all outcomes of a lottery, which is consistent with empirical studies about risk (e.g., see Keller, Sarin and Weber, 1986).

In financial modeling, people often speak of investments as a proportion of assets. For this case, we have a different structure for a lottery, say $X = w * Z$, where Z is a random return rate variable and w is the level of assets or initial wealth. Then the corresponding form of risk-value model can be obtained as follows:

$$\mathrm{E}[u(w_* Z)] = u(w_* \bar{Z}) - \varphi(w_* \bar{Z})\{R(Z^*) - R(1)\} \qquad (2b)$$

Note that in this risk-value structure, the relative measure of risk $R(Z^*)$ is not explicitly related to the level of assets or initial wealth.

2.2. A functional family of relative risk-value models

If, in addition, the utility function is continuous and differentiable everywhere, we will further have the following result.

THEOREM 2. If a utility function is increasing, risk averse and continuously differentiable, then it can be represented in the form of the relative risk-value model (1) if and only if it is one of the following functions:

i) $u(x) = \mathrm{b}\log(x)$ $\theta = 1$;

ii) $u(x) = \mathrm{a}x + \mathrm{b}\log(x)$ for $\theta = 1$;

iii) $u(x) = \mathrm{a}x - \mathrm{b}x\log(x)$ for $\theta = 0$ and $x < \exp(-1 + \mathrm{a}/\mathrm{b})$;

iv) $u(x) = \mathrm{b}x^{1-\theta}$ for $0 < \theta < 1$;

v) $u(x) = -\mathrm{b}x^{1-\theta}$ for $\theta > 1$;

vi) $u(x) = \mathrm{a}x + \mathrm{b}x^{1-\theta}$ for $0 < \theta < 1$;

vii) $u(x) = \mathrm{a}x - \mathrm{b}x^{1-\theta}$ for $\theta > 1$; or
for $\theta < 0$ and $x < (\mathrm{b}(1-\theta)/\mathrm{a})^{1/\theta}$;

where $x > 0$, $\theta = -u'''(1)/u''(1) - 1$, $u'''(1)$ and $u''(1)$ are the third and second order derivatives at $x = 1$ respectively, and a and b are positive constants.

The relative risk-value model is associated with logarithmic (or linear plus logarithmic) and power (or linear plus power) utility functions. In the proof (see the appendix) we have also identified the tradeoff factor for the relative risk-value models in Theorem 2 as $\varphi(x) = x^2u''(x)/u''(1)$.

If we assume that the underlying preference of a decision maker is captured by the logarithmic utility model (i) or linear plus logarithmic model (ii), then the corresponding relative measure of risk is

$$R(X^*) = a - b\, E[\log(X/\bar{X})] \tag{3}$$

where $a \geq 0$ and $b > 0$ are constants. If the utility function is power (iv) or linear plus power (vi), then the relative measure of risk is

$$R(X^*) = a - b\, E[(X/\bar{X})^{1-\theta}] \tag{4}$$

where $0 < \theta < 1$, $a \geq 0$ and $b > 0$ are constants. For the negative power utility model (v) or linear plus negative power model (vii), the relative measure of risk becomes

$$R(X^*) = a + b\, E[(X/\bar{X})^{1-\theta}] \tag{5}$$

where $\theta > 1$ (or $\theta < 0$), $a \geq 0$ and $b > 0$ are constants. Sarin and Weber (1993) also suggested $E[(X/\bar{X})^{1-\theta}]$ as a measure of risk, which is a positive linear transformation of (5).

As a special case when $\theta = -1$, the model (vii) becomes a quadratic utility model, and by a positive linear transformation the relative measure of risk (5) can be written as $R(X^*) = E[(X/\bar{X})^2] - 1 = E[(X-\bar{X})^2] / \bar{X}^2$, which is the square of the coefficient of variation, $\sigma/\bar{X}$, where σ is a standard deviation. The coefficient of variation is often used to compare the relative riskiness of risky prospects in statistics and financial management. The quadratic utility model illustrates a relationship between the standard measure of risk (variance) and the relative measure of risk (the coefficient of variation).

Luce (1980) proposed four possible measures of risk, two of which he judged to be superior: $E[\log(X)]$ and $E(X^\alpha)$, where $\alpha > 0$. For different degenerate lotteries, however, these measures will have different values. This is contrary to our intuition that outcomes received with certainty should be judged to be riskless (or at least judged to equivalent in terms of riskiness). Our models (3) - (5) may be viewed as revisions of Luce's risk measures.

To see that the utility models in Theorem 2 can be represented in the form of the relative risk-value models (1), we provide the following examples.

a) Linear plus logarithm model

The relative measure of risk for the utility model (ii) in Theorem 2 is given by (3). The relative risk-value form can be obtained in the following way:

$$
\begin{aligned}
E[u(X)] &= a\bar{X} + b\,E[\log(X)] \\
&= a\bar{X} + b\,E[\log(X)] + b\log(\bar{X}) - b\log(\bar{X}) \\
&= u(\bar{X}) + b\,E[\log(X/\bar{X})] \\
&= u(\bar{X}) - [R(X^*) - R(1)]. \qquad (6)
\end{aligned}
$$

For this relative risk-value model, the tradeoff factor $j(\bar{X}) = 1$ is a constant.

b) Linear plus power model

The relative measure of risk for the utility model (vii) is given by (5), and its relative risk-value model can be obtained as follows:

$$
\begin{aligned}
E[u(X)] &= a\bar{X} - b\,E[(X)^{1-q}] \\
&= a\bar{X} - b\,E[(X)^{1-\theta}] + b(\bar{X})^{1-q} - b(\bar{X})^{1-q} \\
&= u(\bar{X}) - (\bar{X})^{1-q}\, b\,E[(X/\bar{X})^{1-q} - 1] \\
&= u(\bar{X}) - j(\bar{X})[R(X^*) - R(1)] \qquad (7)
\end{aligned}
$$

where the tradeoff factor $j(\bar{X}) = (\bar{X})^{1-q}$, which is a decreasing function of the mean for $\theta > 1$.

c) A multiplicative form of the relative risk-value model

When $j(\bar{X}) = b'\,u(\bar{X})$, we obtain a special case of the relative risk-value model (1),

$$
E[u(X)] = u(\bar{X}) - b'\,u(\bar{X})[R(X^*) - R(1)] = u(\bar{X})\,R'(X^*) \qquad (8)
$$

where $R'(X^*) = [1 - b' R(X^*) + b' R(1)]$, which serves as a utility discount factor. For degenerate lotteries, the discount factor $R'(X^*)$ must be equal to 1. We can verify that the only solutions satisfying (8) among the utility models in Theorem 2 are the power model (iv) or the negative power model (v). For example, for the power utility model (iv), the relative measure of risk is (4) and $R'(X^*) = [1 - b' R(X^*) + b' R(1)] = \mathrm{E}[(X/\bar{X})^{1-q}]$, where b' = 1/b. Thus, its relative risk-value model can be written as

$$\mathrm{E}[u(X)] = u(\bar{X})\, R'(X^*) = b(\bar{X})^{1-q}\, \mathrm{E}[(X/\bar{X})^{1-q}]. \tag{9}$$

In fact, for the power (or negative power) model, a simple transformation is all that is required to obtain the multiplicative form. Here we should realize that $R'(X^*) = \mathrm{E}[(X/\bar{X})^{1-q}]$ has a different meaning from the relative measure of risk $R(X^*) = -\,\mathrm{E}[(X/\bar{X})^{1-q}]$. For a risk averse decision maker and a lottery with a higher relative riskiness $R(X^*)$, his or her utility value of the lottery would be discounted more; thus, the corresponding utility discount factor $R'(X^*)$ has a smaller value. That is why $R(X^*)$ and $R'(X^*)$ have an opposite sign. In the case of the negative power model (v), however, we can verify that the utility discount factor has the same sign as the relative measure of risk. Both the power and negative power risk-value models have desirable inverse properties, which have important implications in financial modeling, as we shall see in the next subsection.

2.3. Some certainty equivalent forms of relative risk-value models

Consider the following utility function family which is a subset of the utility models in our Theorem 2 (models (i), (iv) and (vii)):

$$u_q(x) = \begin{cases} \dfrac{1}{1-q}\, x^{1-q} & \text{for } q > 0, \text{ and } q \neq 1 \\ \log(x) & \text{for } q = 1 \end{cases} \tag{10}$$

This class of functions is also referred to as the iso-elastic utility functions (Ingersoll, 1987). When $\theta > 1$, we will have a negative power function which allows the modeling of more extreme degrees of risk aversion.

One important feature of the utility function family in (10) is constant proportional risk aversion (Pratt, 1964, Arrow, 1965). For an investor who has a utility function of this type, his or her optimal investment plan does not depend on his or her wealth level or the amount of money to be invested. Thus the investor can determine his preferences by just focusing on the return rate. Because of these appealing properties, this utility function family is widely used in financial modeling. Another operational advantage of the models (10) is that their inverse properties allow for explicit forms of the certainty equivalent.

The corresponding relative risk-value models of (10) can be obtained as follows:

$$E[u_\theta(X)] = \begin{cases} \frac{1}{1-\theta}\bar{X}^{1-\theta} * E[(X/\bar{X})^{1-\theta}] & \text{for } \theta > 0, \text{ and } \theta \neq 1 \\ \log(\bar{X}) + E[\log(X/\bar{X})] & \text{for } \theta = 1 \end{cases} \quad (11)$$

and their certainty equivalent forms are

$$CE_\theta = \bar{X} * D_\theta(X^*) \quad (12)$$

where

$$D_\theta(X^*) = \begin{cases} [R_\theta''(X^*)]^{\frac{1}{1-\theta}} & \text{for } \theta > 0, \text{ and } \theta \neq 1 \\ \exp[-R_1^*(X^*)] & \text{for } \theta = 1 \end{cases}$$

where $R_\theta''(X^*) = -(1-\theta)R_\theta^*(X^*) = E[(X/\bar{X})^{1-\theta}]$ is a utility discounting factor corresponding to the power model, $R_\theta^*(X^*) = -\frac{1}{(1-\theta)}E[(X/\bar{X})^{1-\theta}]$ and $R_1^*(X^*) = -E[\log(X/\bar{X})]$ are the relative measures of risk for the power and logarithmic models in (10) respectively. The certainty equivalent forms (12) of the risk-value models have an intuitive appeal in which $D_\theta(X^*)$ serves as a risk-discounting factor with respect to the mean return. The model of $D_\theta(X^*)$ provides a way to calculate the risk-adjusted discount rate (RADR), which is frequently used in financial management for risky asset pricing. It can be seen that the discounting factor is associated with the relative measures of risk. If there is no risk, $D_\theta(X^*)$ = 1 and $CE_\theta = \bar{X}$. This provides insight into the relationship between the relative measure of risk and the risk-discounting factor. The degree to which a risky prospect will be discounted depends on the individual's risk aversion (reflected by the parameter θ.

For some financial applications, we may consider an approximation for the risk-discounting factor $D_\theta(X^*)$ by moments. The utility discount factor for the power function can be written as

$$R_\theta''(X^*) = \mathrm{E}[(X/\bar{X})^{1-\theta}] = \mathrm{E}[(1+\frac{X-\bar{X}}{\bar{X}})^{1-\theta}]$$

where $(X-\bar{X})/\bar{X}$ is the rate of change with respect to the mean return. If $|(X-\bar{X})/\bar{X}|$ is much smaller than 100%, $R_\theta''(X^*)$ can be approximated by using the Taylor expansion:

$$\begin{aligned} R_\theta''(X^*) &= \mathrm{E}[(1+\frac{X-\bar{X}}{\bar{X}})^{1-\theta}] \\ &\approx \mathrm{E}[1+(1-\theta)(\frac{X-\bar{X}}{\bar{X}}) - \frac{(1-\theta)(\theta)}{2}(\frac{X-\bar{X}}{\bar{X}})^2 + \frac{(1-\theta)\theta(1+\theta)}{6}(\frac{X-\bar{X}}{\bar{X}})^3 \ldots] \\ &\approx 1-\frac{\theta(1-\theta)}{2}\frac{\sigma^2}{\bar{X}^2} + \frac{(1-\theta)\theta(1+\theta)}{6}\frac{S^3}{\bar{X}^3} \qquad (13) \end{aligned}$$

where σ^2 is the variance of the lottery X and S^3 is the third moment or the measure of skewness. If the distribution of a lottery is approximately symmetric, then we can neglect the third moment (i.e., S^3 = 0).

Similarly, the utility discount factor for the logarithmic model can be approximated by

$$\begin{aligned} R_1^*(X^*) &= -\mathrm{E}[\log\frac{X}{\bar{X}}] = -\mathrm{E}[\log(1+\frac{X-\bar{X}}{\bar{X}})] \\ &\approx \frac{1}{2}\frac{\sigma^2}{\bar{X}^2} - \frac{1}{3}\frac{S^3}{\bar{X}^3}. \qquad (14) \end{aligned}$$

Note that both $R_1^*(X^*)$ and $R_q^{''}(X^*)$ are associated with the coefficient of variation, $s/\bar{X}$. Models (13) and (14) have a computational advantage. They can be easily estimated by historic data even though distributions of risky assets are unknown.

3. Generalized relative risk-value models

In the previous section, we have shown that the expected utility model can be represented as a relative risk-value model if the relative risk independence condition holds. However, for descriptive purposes, the empirical validity of the expected utility theory has been called into question (e.g., see Kahneman and Tversky, 1979). In the past, a number of contributions have been made in developing non-expected utility models (for a survey, see Fishburn, 1988). In this section, we relax the restriction that the decision implied by a risk-value model is consistent with the decision implied by an expected utility model, and extend our relative risk-value framework for non-expected utility preferences.

3.1. A generalized relative risk-value framework

In Section 2, we decomposed a lottery X into its mean $\bar{X}$ and its relative risk variable, $X^* = X / \bar{X}$, and treated it as a special case of a two-attribute structure; i.e., $X = (\bar{X} * X^*)$. Thus, it is natural to extend this idea and consider a general two-attribute structure $(\bar{X}, X^*)$ for the evaluation of a lottery X in risky decision making. In this way we can separate the evaluation of a lottery into two attributes, its relative risk and mean, so that the risk-value tradeoffs are not necessarily consistent with the traditional expected utility decisions.

Let $\mathbf{P}_+$ be the set of all simple probability distributions (including positive degenerate distributions) on a product set, $\mathbf{X}_1^+ \times \mathbf{X}_2^+$, of nonnegative-outcomes. For our special case, the outcome of a lottery X on $\mathbf{X}_1^+$ is fixed, which is its mean $\bar{X}$; thus, we can think of the marginal distribution on $\mathbf{X}_1^+$ as a degenerate one with a singleton outcome $\bar{X} \in \mathbf{X}_1^+$. For the second attribute, the marginal distribution on $\mathbf{X}_2^+$ is $X^* \in \mathrm{P}_+^*$, where P_+^*, as previously defined, is the set of relatively normalized probability distributions. Therefore, $(\bar{X}, X^*)$ denotes the distribution in

$\mathbf{P}_+$ that yields $\bar{X} \in \mathbf{X}_1^+$ with probability 1 coupled with $x^* \in \mathbf{X}_2^+$ with probability $X^*(x^*)$, where x^* is a realization on the risk attribute. Note that because $\bar{X}$ is a constant, the two marginal distributions, $(\bar{X},\ X^*)$, are sufficient to determine a unique distribution in $\mathbf{P}_+$. Let $>_\mathbf{P}$ be a binary strict preference relation and $\sim_\mathbf{P}$ an indifference relation on $\mathbf{P}_+$. We assume the existence of a two-attribute expected utility model $U\colon \mathbf{X}_1^+ \times \mathbf{X}_2^+ \to$ R, such that for all $(\bar{X},\ X^*)$, $(\bar{Y},\ Y^*) \in \mathbf{P}_+$,

$$(\bar{X},\ X^*) >_\mathbf{P} (\bar{Y},\ Y^*) \quad \Leftrightarrow \quad \mathrm{E}[U(\bar{X},\ X^*)] > \mathrm{E}[U(\bar{Y},\ Y^*)]$$

with U unique up to a positive linear transformation.

It can be seen that the generalized relative risk-value model, $\mathrm{E}[U(\bar{X},\ X^*)]$, includes the traditional expected utility model, $\mathrm{E}[u(X)]$, as a special case when $(\bar{X},\ X^*) = (\bar{X} * X^*) = (X)$. In this case, $\mathrm{E}[U(\bar{X} * X^*)] = \mathrm{E}[U(X)]$, and by uniqueness, $\mathrm{E}[U(X)] = \mathrm{a}\,\mathrm{E}[u(X)] + \mathrm{b}$, where $\mathrm{a} > 0$ and b are constant. Generally speaking, however, the relative risk-value model allows the decision maker to deviate from the expected utility preference.

The generalized relative risk-value model makes restricted use of the axioms of the traditional expected utility theory only for the set of relatively normalized probability distributions with the same expected values (on the relative risk attribute). We are unaware of any empirical study that has demonstrated a systematic violation of the expected utility axioms on a normalized set of probability distributions with the same expected values.

To obtain a separable form of the risk-value model, we need a general relative risk independence condition stated as follows.

Definition 2. X^* is generally relatively risk independent of $\bar{X}$ if there exists a $\bar{w}_o \in \mathbf{X}_1^+$ for which $(\bar{w}_o,\ X^*) >_\mathbf{P} (\bar{w}_o,\ Y^*)$, where $X^*,\ Y^* \in \mathrm{P}_+^*$, then $(\bar{w},\ X^*) >_\mathbf{P} (\bar{w},\ Y^*)$ for all $\bar{w} \in \mathbf{X}_1^+$.

The general relative risk independence condition includes the relative risk independence (see Definition 1) as a special case, and it is also analogous to the utility independence condition for a multiattribute utility model (see Keeney and Raiffa, 1976, pp 224-229).

Based on the above assumption, we have the following result for a separable relative risk-value model.

THEOREM 3. The generalized relative risk-value model can be represented as

$$\mathrm{E}[U(\bar{X},\ X^*)] = V(\bar{X}) - \mathrm{y}\,(\bar{X})[R^*(X^*) - R^*(1)] \tag{15}$$

if and only if the relative risk independence condition holds, where $\mathrm{y}\,(\bar{X}) > 0$ and $R^*(X^*) = -\mathrm{E}[u_r(X^*)]$, and $u_r(\cdot)$ defined by $U(1,\ \cdot)$. Furthermore, three other functions, a value measure $F(\bar{X})$, a tradeoff factor $\mathrm{w}(\bar{X})$, and a relative measure of risk $G^*(X^*)$, also satisfy (15), if and only if there exist some constants a, c > 0 and b, d such that $F(\bar{X}) = \mathrm{a}V(\bar{X}) + \mathrm{b}$, $G^*(X^*) = \mathrm{c}R^*(X^*) + \mathrm{d}$, and $\mathrm{w}(\bar{X}) = (\mathrm{a}/\mathrm{c})\mathrm{y}\,(\bar{X})$.

If we state an individual's wealth level explicitly, we will have risk-value models similar to the structures of (2a) and (2b).

In the generalized relative risk-value model (15), we also define $R^*(X^*)$ as a relative measure of risk. From the proof of Theorem 3 in the appendix, we see $R^*(X^*) = -\mathrm{E}[u_r(X^*)] = -\mathrm{E}[U(\bar{w}_o,\ X^*)]$, where $\bar{w}_o$ = \$1. For any normalized lottery with the expected value \$1, we may require that preferences over the two-attribute structure $(\$1,\ X^*)$ and the single-attribute structure $(\$X^*)$ be consistent. In this case, we can choose the same utility function for $R^*(X^*)$ as for $R(X^*)$. For a risk averse individual, the function u_r for the risk measure must be concave, which implies $R^*(X^*) > R^*(1)$.

Rothschild and Stiglitz (1970) proposed a theory for increasing risk by mean preserving spreads. They established an equivalence theorem involving the expected utility model, stochastic dominance, and mean preserving spreads when lotteries have the same expected values. In the expected utility model, Rothschild and Stiglitz's definition of increasing risk is perhaps the most widely accepted one (Sarin and Weber, 1993). In nonexpected utility theory, Machina (1982) also used the mean preserving spread as a basic condition for his local utility function. According to Rothschild and Stiglitz's (1970) theorem, we can have the following result for our generalized relative risk-value models if the relative measure of risk R^* is defined by one of the utility functions in Theorem 2 (note that these utility models are increasing and concave).

THEOREM 4. For X, $Y \in$ P$_+$ with $\bar{X} = \bar{Y}$, let F and G be the cumulative distributions of X and Y respectively. Then the following statements are equivalent:

i) E[$u(X)$] $\geq$ E[$u(Y)$], where u is one of the utility functions in Theorem 2.

ii) For the relative risk-value model (15), E[$U(\bar{X},\ X^*)$] $\geq$ E[$U(\bar{Y},\ Y^*)$] or $R^*(X^*) \leq R^*(Y^*)$, where $R^*(X^*) = -$E[$u(X^*)$] and u is the same as in (i).

iii) There exists a random variable Z^o such that Y has the same distribution as $X + Z^o$, where E[$Z^o | X$] = 0 for all X.

iv) $T(x) = \int_0^x [G(t) - F(t)]dt \geq 0$ for all $x \in$ Re$_+$.

Statements (i) and (ii) mean that a relative risk-value model and an expected utility model in Theorem 2 will be consistent for lotteries with the same expected value if the relative measure of risk uses the same utility function. Statement (iii) says that if we add some uncorrelated noise to a lottery, the new lottery should be riskier then the original, and thus less preferable. Statement (iv) implies that if the distribution of lottery Y is obtained from the distribution of lottery X by taking some of the probability weight from the center of the distribution X and adding it to each tail of the distribution in such a way as to leave the mean unchanged, then the lottery Y is less preferable than lottery X. This statement is equivalent to the second degree stochastic dominance condition. The equivalence of these statements leads us to use the utility functions in Theorem 2 for the relative measure of risk in order to keep the intuitive property of mean preserving spreads for our relative risk-value model (15).

In the model (15) of Theorem 3, when there is no risk involved, then $U(\bar{X}, 1) = V(\bar{X})$ and the risk-value decision problem reduces to a single-attribute problem concerning only the certain consequences. Thus, $V(\bar{X})$ should be a value measure in the risk-value model. V is usually increasing and concave, which implies a decreasing marginal value. If we require the relative risk-value model to be consistent with the traditional expected utility model, as we have seen in model

(1), then $V = u$. However, many authors argue that a value measure V should be different from a utility measure u because the latter involves risk (see von Winterfeldt and Edwards, 1986). Our relative risk-value model (15) actually provides an explicit way to separate an individual's risk attitude from the strength of his or her preference.

Using the value function for riskless outcomes leads us to define the certainty equivalent for a lottery as $(CE,\ 1) \sim_{p} (\bar{X},\ X^{*})$; that is, $U(CE,\ 1) = V(CE) = \mathrm{E}[U(\bar{X},\ X^{*})]$. Thus,

$$CE = V^{-1}\{V(\bar{X}) - y(\bar{X})[R^{*}(X^{*}) - R^{*}(1)]\}. \qquad (16)$$

This certainty equivalent model can be used for the assessment of a generalized relative risk-value model. When $V(\bar{X}) = \bar{X}$, (16) becomes $CE = \bar{X} - y(\bar{X})[R^{*}(X^{*}) - R^{*}(1)]$. For this case, $y(\bar{X})[R^{*}(X^{*}) - R^{*}(1)]$ is the risk premium.

When $y(\bar{X}) = k$, where k is a positive constant, the model (15) becomes an additive risk-value model,

$$\mathrm{E}[U(\bar{X},\ X^{*})] = V(\bar{X}) - k[R^{*}(X^{*}) - R^{*}(1)]. \qquad (17)$$

When $y(\bar{X}) = cV(\bar{X}) > 0$, where c is a constant, then we have a multiplicative risk value model,

$$\mathrm{E}[U(\bar{X},\ X^{*})] = V(\bar{X})R^{''}(X^{*}) \qquad (18)$$

where $R^{''}(X^{*}) = 1 + cR^{*}(X^{*}) - cR^{*}(1)$. In this model, $R^{''}(X^{*})$ serves as a value discount factor due to risk. If c is positive, i.e., $V(\bar{X}) > 0$, then $R^{''}(X^{*})$ and $R^{*}(X^{*})$ have the same sign. Otherwise, they have different signs.

3.2. Examples of generalized relative risk-value models

In the generalized relative risk-value model (15) (or its special cases (17) and (18)), the relative measure of risk, the value function, and the tradeoff factor can be determined independently. Thus, we can properly choose functional forms for each of them according to different theoretical and empirical considerations. This makes the generalized relative risk-value model extremely flexible. In order to maintain the property of mean preserving spreads discussed earlier, we constrain our choice of the functions for relative measures of risk within the set of utility functions in Theorem 2. Some examples of the generalized relative risk-value model are provided as follows.

a) A multiplicative power model

For the multiplicative form of the relative risk-value model (9) implied by a power utility function, the value measure and the relative measure of risk have the same exponential power because a consistency condition is required between the relative risk-value model and the expected utility model. According to the multiplicative risk-value model (18), if we choose $V(\bar{X}) = (1/b)\psi(\bar{X}) = a(\bar{X})^{1-\lambda}$ and $R(X^*) = -bE[(X/\bar{X})^{1-\theta}]$, where $1 > \lambda > 0$, $1 > \theta > 0$, a, b > 0 (or $\lambda, \theta > 1$ and a, b < 0), then we can have the following model:

$$E[U(\bar{X}, X^*)] = a(\bar{X})^{1-\lambda}\, E[(X/\bar{X})^{1-\theta}]\ . \qquad (19)$$

This generalized relative risk-value model includes the model (9) as a special case when $\lambda = \theta$. Thus, it will be more flexible for modeling individuals' preferences. According to (16), the certainty equivalent form can be found as $CE = \bar{X} * \{E[(X/\bar{X})^{1-\theta}]\}^{\frac{1}{1-\lambda}}$, which can be used in modeling risk-discounting behavior for nonexpected utility preferences.

b) A general power model

If we choose different power functions for the relative measure of risk, the value measure, and the tradeoff factor, then the generalized risk-value model can be written as:

$$E[U(\bar{X}, X^*)] = a(\bar{X})^{1-\alpha} + b(\bar{X})^{1-\lambda}\, E[(X/\bar{X})^{1-\theta} - 1] \qquad (20)$$

where $0 < \theta < 1$, $0 \leq \alpha < 1$, a, b > 0 (or $\alpha, \theta > 1$ and a, b < 0), and $\lambda > 0$ are constant. When $\alpha = \lambda$ and a = b, this model is equivalent to model (19). With model (20), we can obtain richer descriptions of risky choice behavior, as we shall see in the following section.

c) A linear plus power model

Another relative risk-value model can be constructed as follows:

$$\mathrm{E}[U(\bar{X},\ X^{*})] = \bar{X}\ \text{- b}(\bar{X})^{1-l}\ \mathrm{E}[(X/\bar{X})^{1-q}\ \text{- d}] \quad (21)$$

where b , l , q > 0 and d are constant. When d = 0 and l =q , this risk-value model reduces to the linear plus power utility model (7). When d = 1, the value function is $V(\bar{X}) = \bar{X}$, and then model (21) gives the certainty equivalent of a lottery directly; and when d _ 1, $V(\bar{X}) = \bar{X}$ - b (1-d)$(\bar{X})^{1-l}$.

d) A logarithm model

If we use logarithmic functions for both the value measure and the risk measure, and a positive constant for the tradeoff factor, then we have the following additive logarithmic risk-value model:

$$\mathrm{E}[U(\bar{X},\ X^{*})] = \log(\bar{X}) + k\,\mathrm{E}[\log(X/\bar{X})] \quad (22)$$

where k > 0. When k = 1, this model reduces to a traditional expected logarithm utility model. Model (22) also has an explicit form of certainty equivalent, $CE = \bar{X} * \exp\{k\,\mathrm{E}[\log(X/\bar{X})]\}$.

4. Risk-value interpretation of some paradoxes and decision models

The expected utility model has been shown to be a poor descriptor of empirically observed decision making behavior. In this section, we show that our generalized relative risk-value models can explain many paradoxes and provide a new justification for them.

For convenience, we will use the following parametric form of the general power risk-value model (20) for this study:

$$\mathrm{E}[U(\bar{X},\ X^{*})] = (\bar{X})^{a} + \mathrm{b}(\bar{X})^{l}\ \mathrm{E}[(X/\bar{X})^{q}\ \text{- 1}] \quad (23)$$

where 0 < q < 1 and b > 0 (or q > 1 and b < 0), 0 < a ² 1 and l are constant.

4.1. Utility dependence on probability

A traditional assessment procedure for an individual's utility function $u(x)$ is to use the certainty equivalent method. For an elementary lottery (x, p) (i.e., p chance of winning \$$x$ and 1 - p chance of \$0), when the certainty equivalent of the lottery has been assessed, say $CE(x,p)$, then we should have $u(CE(x,p)) = pu(x)$, or $u(x) = u(CE(x,p))/p$. According to the classic expected utility theory, the elicitation of an individual's utility function $u(x)$ should not depend on the choice of probability levels used in the assessment. However, Karmarkar (1978), McCord and de Neufville (1984) and others have found that higher levels of probabilities lead to the "recovery" of higher valued utility functions.

Suppose that an individual's "true" preference can be represented by our relative risk-value model (23). Thus, according to model (16) the certainty equivalent of an elementary lottery (x, p) should be determined by

$$CE(x,p) = \{(px)^{\alpha} + b(px)^{\lambda}[p(\frac{x}{px})^{\theta} - 1]\}^{1/\alpha}$$
$$= [(px)^{a} - b(px)^{l}(1 - p^{1-q})]^{1/a}.$$

However, if we use the classic power expected utility model to assess the individual's preference, then we will have $(x)^{b} = (CE(x,p))^{b}/p$, where b is to be assessed. Thus, when $p \neq 1$, $b = log(p)/log[CE(x,p)/x]$, where b may depend on p and x, say $b(x,p)$. Substituting $CE(x,p)$ into this, we obtain

$$b(x,p) = \frac{log(p)}{log\{[(px)^{a} - b(px)^{l}(1 - p^{1-q})]^{1/a}/x\}}$$

$$= \frac{log(p)}{log\{[p^{a} - bp^{l}x^{l-a}(1 - p^{1-q})]^{1/a}\}}$$

$$= a\frac{log(p)}{log[p^{a} - bp^{l}x^{l-a}(1 - p^{1-q})]} \qquad (24)$$

where $p \neq 1$. Only under the special case of $a = l$ and b = 1, will $b(x,p)$ not be related to p and x (i.e., $b = a/(1 + a - q)$). In general, $b(x,p)$ depends on p and x. Thus, the assessed power utility model, in fact, is $u(x, p) = (x)^{\beta(x,p)}$. Normalizing the utility model such that $u(x^*, p)$ = 1 for the maximum value x^*, then we have the following recovered utility model:

$$u(x, p) = \left(\frac{x}{x^*}\right)^{\beta(x,p)}. \qquad (25)$$

When $\beta(x, p)$ is a decreasing function of p, model (25) increases in risk aversion with increasing probability level; i.e., the higher the probability level used in the assessment, the higher the recovered utility value. In fact, $\beta(x, p)$ in (24) is flexible enough so that an appropriate choice of parameters can make $\beta(x, p)$ decreasing. For example, if α = 0.7, λ = 1, θ = 0.5, b = 1 and x = 4000 (This is one of the attribute values McCord and Neufville (1984) used in their experiment), then for p = 0.1, β = 0.752; for p = 0.5, β = 0.590; and for p = 0.9, β = 0.544.

Model (25) also illustrates a discrepancy between assessed utility functions and the value function. For a simple example, let us consider the special case when $\alpha = \lambda$ and b = 1; then β = $\alpha/(1+\alpha-\theta)$ and the corresponding risk-value model is E[$U(\bar{X}, X^*)$] = $(\bar{X}/x^*)^{\alpha}$ E[$(X/\bar{X})^{\theta}$] (normalized). The assessed utility function $u(x) = (x/x^*)^{\beta}$ and the value function $V(x) = (x/x^*)^{\alpha}$ are obviously different.

According to Dyer and Sarin's (1982) relative risk framework, we can find a relationship between a utility measure $u(x)$ and a value measure $V(x)$; i.e., $u(x) = (V(x))^{\beta/\alpha}$. Thus, when $\alpha > \theta$, then $\beta < \alpha$ and this relationship indicates relative risk aversion; and when $\alpha < \theta$, then $\beta > \alpha$, indicating relative risk proneness. In an experimental study, Krzysztofowicz (1983) found that people do not have any tendency toward one type of relative risk attitude (averse or seeking). To allow for this observation, we do not restrict α (i.e., either $\alpha > \theta$ or $\alpha < \theta$) in our later analysis.

4.2. Decision weights

Kahneman and Tversky's (1979) prospect theory suggests that people replace probabilities by decision weights and replace utility functions by value functions when evaluating risky outcomes. For a simple lottery (x, p), the model has the form, $\pi(p)v(x)$, where $v(x)$ is the value function and $\pi(p)$ is the decision weight associated with probability p. Even though Kahneman and Tversky proposed that a typical value function is concave for gains and convex for losses (e.g., a two-piece power function (Tversky and Kahneman, 1990)), we just consider positive outcomes here. In a decision weight study, Hogarth and Einhorn (1990) also used a power function as the value measure. We assume our value function and the prospect theory value function are identical and that both can be modeled by a power function, i.e., $V(x) = v(x) = x^{\alpha}$. Now consider our power risk-value model (23). If the model (23) has an equivalent decision weight model, say $\pi(p,x)x^{\alpha}$, (where the equivalent decision weight $\pi(p,x)$ may depend on the outcome as we shall see in the following), then

$$\begin{aligned}\pi(p,x)x^{\alpha} &= (px)^{\alpha} + b(px)^{\lambda}(p^{1-\theta} - 1)\\ &= [p^{\alpha} - bp^{\lambda}x^{\lambda-\alpha}(1 - p^{1-\theta})]x^{\alpha}.\end{aligned}$$

Thus, we must have

$$\pi(p,x) = p^{\alpha} - bp^{\lambda}x^{\lambda-\alpha}(1 - p^{1-\theta}). \qquad (26)$$

This decision weight is well behaved at the endpoints, i.e., $\pi(0,x) = 0$ and $\pi(1,x) = 1$. (In a new version of prospect theory (Tversky and Kahneman, 1990), the decision weighting functions have this property. Also see the decision weights proposed by Hogarth and Einhorn (1990).) Model (26) shows that the outcome x generally has an effect on the decision weight. Only when $\alpha = \lambda$, does the decision weight not depend on the outcome. In this case, $\pi(p,x) = p^{\alpha}[1 - b(1 - p^{1-\theta})]$. Further, if b = 1, then $\pi(p,x) = p^{1+\alpha-\theta}$; when $\alpha > \theta$, the probability is discounted in the decision weight, which corresponds to relative risk aversion (Dyer and Sarin, 1982).

Tversky and Kahneman (1990) proposed different weighting functions for positive outcomes and negative outcomes. More general, Hogarth and Einhorn (1990) considered the effect of outcome sizes on decision weights. According to their theory, the decision weight for gains generally decreases as the size of the outcome x increases. This is consistent with our model (26) if we let $l > a$. Figure 1 illustrates some different types of decision weights determined by using different parameters. It can be seen that the decision weight implied in our risk-value model is very flexible, and can have all the shapes that have been considered.

Even though our risk-value model (23) has a similarity to some nonlinear decision weighting models, our framework is more general and has a more intuitive appeal; i.e., it is based on risk-value tradeoffs and can be used for lotteries with many outcomes.

4.3. Some decision paradoxes

As we have shown above, our risk-value model (23) implies a nonlinear decision weight so that it can capture some paradoxes that violate the independence axioms of expected utility theory (i.e., linearity in the probabilities). However, our interpretation is different from many previous explanations; we attribute the violation to "inconsistent" tradeoffs between risk and value.

a) Allais Paradox (the common consequence effect)

The well-known Allais Paradox (Allais, 1953) is based on the following pairs of lotteries:

X_1: win \$1 million for sure	versus	X_2: 0.10 chance of \$5 million
		0.89 chance 0f \$1 million
		0.01 chance 0f \$0
X_3: 0.11 chance of \$1 million	versus	X_4: 0.10 chance of \$5 million
0.89 chance of \$0		0.90 chance of \$0

Many experiments have shown that a majority of subjects have the preference pattern of X_1 over X_2 but X_4 over X_3, which is inconsistent with any form of expected utility models. Now for our risk-value model (23), let a = 0.01 (for the traditional utility model, $E[(X)^a]$, to make $X_1 >_p X_2$, we need a to be approximately smaller than 0.05), l = 0, q = 0.01 and b = 1; then $E[U(\bar{X}_1, X_1^*)] = 1 > E[U(\bar{X}_2, X_2^*)] = 0.992$ and $E[U(\bar{X}_3, X_3^*)] = 0.0906 < E[U(\bar{X}_4, X_4^*)] = 0.0954$, which predicts the preference pattern. In fact, the model will maintain this preference pattern if 0

< q < 0.41, and the other parameters are held constant at the same values. The Allais Paradox is now known to be a special case of a general empirical phenomenon called the common consequence effect (e.g., see Machina, 1987).

b) The common ratio effect

Another type of paradox is the common ratio effect that involves pairs of lotteries of the form illustrated by the following numerical example:

X_1: win $3000 for sure	versus	X_2: 0.80 chance of $4000
		0.20 chance 0f $0
X_3: 0.25 chance of $3000	versus	X_4: 0.20 chance of $4000
0.75 chance of $0		0.80 chance of $0

It has been observed that a majority of the subjects prefer X_1 to X_2 and X_4 to X_3 (e.g., see Kahneman and Tversky, 1979). This is also inconsistent with expected utility theory. Consider our risk-value model (19). If we let a = 0.1, l = -0.2, q = 0.1 and b = 1, then a simple calculation shows that $\mathrm{E}[U(\bar{X}_1, X_1^*)]$ = 2.227 > $\mathrm{E}[U(\bar{X}_2, X_2^*)]$ = 2.205 and $\mathrm{E}[U(\bar{X}_3, X_3^*)]$ = 1.749 < $\mathrm{E}[U(\bar{X}_4, X_4^*)]$ = 1.750, which predicts the subjects' preference pattern.

5. Conclusions

In this paper, we have explored risk-value models based on the decomposition of a return variable X into the average return $\bar{X}$ and a risk related factor ($X/\bar{X}$). When the relative risk independence condition holds, a relative risk-value model is obtained that is consistent with classical expected utility theory. This model also provides a measure of relative risk, and is related to power (or linear plus power) and logarithmic (or linear plus logarithmic) utility functions. In an earlier paper (Jia and Dyer, 1994), we obtained related results based on the decomposition of a return variable into the average return $\bar{X}$ and the risk related factor ($X - \bar{X}$).

When a condition called risk independence is satisfied, the corresponding risk-value model is related to the quadratic, exponential, and linear plus exponential utility functions. In general, the relative risk and the risk independence conditions cannot be satisfied simultaneously. They are suitable to different types of decision makers, those thinking of risk in terms of a relative measure, or those thinking of risk in terms of an absolute measure.

The definition of relative risk independence, stated formally in Definition 1, simply says that choices between normalized lotteries with outcomes of proportional returns relative to the mean of a lottery do not depend on a common mean. For example, suppose the decision maker prefers an even chance lottery with outcomes of $8.00 and $2.00 to a lottery with a 0.6 chance of an outcome of $6.00 and a 0.4 chance of $3.50. Both of these lotteries have a mean of $5.00, so the normalized lotteries corresponding to the measures of relative risk ($X/\bar{X}$) are (0.5, 1.6; 0.5, .4) and (0.6, 1.2; 0.4, 0.7), respectively. If relative risk independence is satisfied, the decision maker will prefer any lottery determined by multiplying the outcomes of the first normalized lottery by a constant to one determined by multiplying the second normalized lottery by the same constant. As an illustration, corresponding lotteries with means of $20.00 are (0.5, $32.00; 0.5, $8.00) and (0.6, $24.00; 0.4, $14.00), and the latter should be preferred to the former.

If the decision-maker's preferences over risky rates of return are independent of the amount of money invested, as is often assumed in the financial literature, then the decision-maker's utility function must be represented by either the power or the logarithmic model. However, our relative risk independence condition is a weaker assumption that results in a wider range of utility functions. Our assumption requires that the decision-maker's preferences over the *normalized* rates of return with a mean rate equal to 1.0 or 100% be independent of the levels of investment. From a prescriptive point of view, this seems to be a reasonable condition for a decision maker to accept.

From a descriptive point of view, we have also generalized our risk-value framework for nonexpected utility preferences. This extension of the model can explain many decision paradoxes and should provide much more flexibility in capturing preferences that violate the independence axiom of classical utility theory. Our generalization differs from previous nonexpected utility models that have been proposed because it is based on the intuitively appealing idea of risk-value tradeoffs. However, the generalized relative risk-value model does retain some desirable properties of expected utility theory, such as consistency with the theory of risk based on mean-preserving spreads.

6. Appendix

<u>Proof of Theorem 1</u>

Let x^* be a realization of $X^* \in P_+^*$. For $(\overline{w}, x^*) \equiv (\overline{w} * x^*) \in Re_+ \times Re_+$, the "two-attribute" utility function $u(\overline{w} * x^*)$ can be decomposed as follows:

$$u(\overline{w} * x^*) = g(\overline{w}) + \varphi(\overline{w})\, u(\overline{w}_o * x^*) \tag{a1}$$

if and only if the relative risk independence condition holds, where $\overline{w}_o$ is any constant and $\varphi(\overline{w}) > 0$. (Note that the relative risk independence condition can be thought as a special case of utility independence in multiattribute utility theory; see Keeney and Raiffa, 1976, pp 224-229, for the utility independence condition.) By choosing $\overline{w} = \overline{X}$ and $\overline{w}_o = 1$, and taking the expectation for (a1), we have

$$\begin{aligned} \mathrm{E}[u(\overline{X} * X^*)] = \mathrm{E}[u(X)] &= g(\overline{X}) + \varphi(\overline{X})\mathrm{E}[u(X^*)] \\ &= g(\overline{X}) - \varphi(\overline{X})\, R(X^*), \end{aligned} \tag{a2}$$

where $R(X^*) = -\mathrm{E}[u(X/\overline{X})]$. Letting $X = \overline{X}$, we must have

$$u(\overline{X}) = g(\overline{X}) - \varphi(\overline{X})\, R(1).$$

Thus,

$$g(\overline{X}) = u(\overline{X}) + \varphi(\overline{X})\, R(1). \tag{a3}$$

Substituting (a3) into (a2), we obtain the desired result.

<u>Proof of Theorem 2</u>

Because the utility function is continuously differentiable, for a lottery X with a small risk, we can expand the expected utility model around the mean $\bar{X}$ by the Taylor expansion as follows:

$$\mathrm{E}[u(X)] = u(\bar{X}) + \frac{u''(\bar{X})}{2}\mathrm{E}[(X-\bar{X})^2] + \frac{u'''(\bar{X})}{6}\mathrm{E}[(X-\bar{X})^3] + \ldots \qquad \text{(b1)}$$

Similarly, we expand the relative measure of risk in the risk-value model (1) at $\bar{X}$

$$u(\bar{X}) - \varphi(\bar{X})[R(X^*) - R(1)] = u(\bar{X}) + \varphi(\bar{X})\mathrm{E}[u(X/\bar{X}) - u(1)]$$

$$= u(\bar{X}) + \varphi(\bar{X})\left[\frac{u''(1)}{2}\frac{\mathrm{E}[(X-\bar{X})^2]}{\bar{X}^2} + \frac{u'''(1)}{6}\frac{\mathrm{E}[(X-\bar{X})^3]}{\bar{X}^3} + \ldots\right]. \qquad \text{(b2)}$$

Comparing (b1) with (b2), if the relative risk-value model is consistent with the expected utility model, then we must have

$$u^{(k)}(\bar{X}) = \varphi(\bar{X})\frac{u^{(k)}(1)}{\bar{X}^k} \qquad \text{for all } k \geq 2, \qquad \text{(b3)}$$

where (k) is the order of derivative.

Because $\bar{X}$ can be any positive real value, substituting $\bar{X}$ by $x \in \mathrm{Re}_+$, then (b3) can be written as

$$u^{(k)}(x) = \varphi(x)\frac{u^{(k)}(1)}{x^k} \qquad \text{for all } k \geq 2. \qquad \text{(b4)}$$

If $u^{(k)}(1) = 0$ for all $k \geq 2$, then (b4) becomes $u^{(k)}(x) = 0$, for all $k \geq 2$. Thus the only solution satisfying this is a linear utility function which is a noninteresting case (not a risk averse case).

If at least $u''(1) \neq 0$, then by (b4) we have

$$\varphi(x) = \frac{u''(x)}{u''(1)}x^2 \; . \qquad \text{(b5)}$$

Substituting (b5) into (b4), we obtain

$$u^{(k)}(x)\ \ x^{k-2} = \frac{u^{(k)}(1)}{u''(1)} u''(x) \text{ for all k ³ 3.} \qquad \text{(b6)}$$

A solution satisfying (b6) must also be a solution for the following special case (when k = 3):

$$u'''(x)x = \frac{u'''(1)}{u''(1)} u''(x)$$

or

$$u'''(x)\ \ x - \text{c } u''(x) = 0, \qquad \text{where c} = \frac{u'''(1)}{u''(1)}. \qquad \text{(b7)}$$

When c = -1, this differential equation has the following solution:

$$u(x) = \text{a}x + \text{b } x \log(x) + \text{d}, \qquad \text{(b8)}$$

when c = -2, its solution is

$$u(x) = \text{a}x + \text{b} \log(x) + \text{d}, \qquad \text{(b9)}$$

and when c ≠ -1 and c ≠ -2, its general solution is

$$u(x) = \text{a}x + \text{b } x^{\text{c}+2} + \text{d}, \qquad \text{(b10)}$$

where a, b and d are some integral constants.

It is easy to verify that (b8)-(b10) always satisfy the condition (b6). Thus they are also the only solutions for (b6). Let q = - c -1 and by the condition that *u(x)* is increasing and risk averse (Pratt, 1964; Arrow, 1965), then we can determine the sign of all constants and obtain the models listed in Theorem 2. The models (i), (iv), and (v) are special cases of models (b9) and (b10). They are constant proportional risk averse in the sense of the Arrow-Pratt definition.

Conversely, these models (i)-(vii) can be written in the form of relative risk-value model (1). Here we will only show one case, the model (iii), which is more difficult than the others. The other models will be shown in Sections 2.2 and 2.3 by examples. For the model (vii), its relative measure of risk is

$$R(X^*) = -\,\text{E}[u(X/\bar{X})] = -\text{ a } - \text{ d } + \text{ b E}[\frac{X}{\bar{X}} \log(\frac{X}{\bar{X}})]$$

and

$$R(X^*) - R(1) = \text{b E}[\frac{X}{\overline{X}} \log(\frac{X}{\overline{X}})].$$

The relative risk-value model can be obtained in the following way:

$$\begin{aligned}
\text{E}[u(X)] &= \text{a}\overline{X} - \text{b E}[X\log(X)] + \text{d} \\
&= \text{a}\overline{X} - \text{b E}[X\log(X)] + \text{d} + \text{b E}[\overline{X}\log(\overline{X})] - \text{b E}[\overline{X}\log(\overline{X})] \\
&= u(\overline{X}) - \text{b E}[X\log(X) - \overline{X}\log(\overline{X})] \\
&= u(\overline{X}) - \text{b E}[X\log(X) - X\log(\overline{X})] \\
&= u(\overline{X}) - \text{b E}[X\log(X/\overline{X})] \\
&= u(\overline{X}) - \overline{X}\,[R(X^*) - R(1)],
\end{aligned}$$

where $\text{j}\,(\overline{X}) = \overline{X}$.

<u>Proof of Theorem 3</u>

The proof of Theorem 3 is similar to that of Theorem 1. Let x^* be a realization of $X^* \in \text{P}^*_+$. For $(\overline{w}, X^*) \in \mathbf{X}_1^+ \times \mathbf{X}_2^+$, the two-attribute utility function $U(\overline{w}, x^*)$ can be decomposed as follows:

$$U(\overline{w}, x^*) = g(\overline{w}) + \psi(\overline{w})U(\overline{w}_o, x^*) \qquad \text{(c1)}$$

if and only if the general relative risk independence condition holds, where $\overline{w}_o$ is any constant and $\psi(\overline{w}) > 0$. By choosing $\overline{w} = \overline{X}$ and $\overline{w}_o = 1$, and taking the expectation for (c1), we have

$$\begin{aligned}
\text{E}[U(\overline{X}, X^*)] &= g(\overline{X}) + \text{y}\,(\overline{X})\text{E}[U(1, X^*)] \qquad \text{(c2)} \\
&= g(\overline{X}) - \psi(\overline{X})R^*(X^*)
\end{aligned}$$

where $R^*(X^*) = -\text{E}[U(1, X^*)] = -\text{E}[u_r(X^*)]$. Letting $X = \overline{X}$, we must have

$$U(\overline{X}, 1) = g(\overline{X}) - \text{y}\,(\overline{X})\,R^*(1)$$

Let $U(\overline{X}, 1) = V(\overline{X})$, then

$$g(\bar{X}) = V(\bar{X}) + \mathrm{y}\,(\bar{X})\, R^*(1). \qquad \text{(c3)}$$

Substituting (c3) into (c2), we obtain the desired result:

$$\mathrm{E}[\,U(\bar{X}, X^*)\,] = V(\bar{X}) - \psi(\bar{X})[R^*(X^*) - R^*(1)]. \qquad \text{(c4)}$$

If three other functions F, w and G^* also satisfy (c4), then by uniqueness, we must have

$$F(\bar{X}) - \mathrm{w}(\bar{X})[G^*(X') - G^*(1)] = \mathrm{a}\{V(\bar{X}) - \mathrm{y}\,(\bar{X})[R^*(X^*) - R^*(1)]\} + \mathrm{b} \qquad \text{(c5)}$$

where a > 0. When $X = \bar{X}$, we directly have $F(\bar{X}) = \mathrm{a}V(\bar{X}) + \mathrm{b}$. Thus, (c5) can be reduced to the following:

$$\mathrm{w}(\bar{X})[G^*(X') - G^*(1)] = \mathrm{a}\mathrm{y}\,(\bar{X})[R^*(X^*) - R^*(1)]. \qquad \text{(c6)}$$

Since both R^* and G^* are defined by the special case of the two-attribute utility function $-\mathrm{E}[U(1,\ X^*)]$, one must be a positive linear transformation of the other, i.e., $G^*(X') = \mathrm{c}R^*(X^*)$ + d, where c > 0. Substituting this into (c6), we can obtain $\mathrm{w}(\bar{X}) = (\mathrm{a}\,/\,\mathrm{c})\mathrm{y}\,(\bar{X})$.

Conversely, if $F(\bar{X}) = \mathrm{a}V(\bar{X}) + \mathrm{b}$, $G^*(X^*) = \mathrm{c}R^*(X^*) + \mathrm{d}$, and $\mathrm{w}(\bar{X}) = (\mathrm{a}\,/\,\mathrm{c})\mathrm{y}\,(\bar{X})$, where a, c > 0, then we can obtain (c5). Thus, $F(\bar{X}) - \mathrm{w}(\bar{X})[G^*(X') - G^*(1)]$ is a positive linear transformation of $V(\bar{X}) - \mathrm{y}\,(\bar{X})[R^*(X^*) - R^*(1)]$, and F, w and G^* satisfy (c4) when V, y and R^* satisfy (c4).

<u>Proof of Theorem 4</u>

The equivalence of statements (i), (iii) and (iv) was established by Rothschild and Stiglitz (1970) (note that our cases are special ones when lotteries X, $Y \in \mathrm{P}_+$). We only need to prove the equivalence between (i) and (ii). For X, $Y \in \mathrm{P}_+$ with $\bar{X} = \bar{Y}$,

$$V(\bar{X}) - \psi(\bar{X})[R^*(X^*) - R^*(1)] \;\geq\; V(\bar{Y}) - \psi(\bar{Y})[R^*(Y^*) - R^*(1)]$$

if and only if

$$R^*(X^*) \leq R^*(Y^*) \;\text{ or }\; \mathrm{E}[\,u(X\,/\,\bar{X})\,] \geq \mathrm{E}[\,u(Y\,/\,\bar{Y})\,].$$

It is easy to verify that for the set of the utility functions in Theorem 2, E[$u(X/\bar{X})$] $\geq$ E[$u(Y/\bar{Y})$] if and only if E[$u(X)$] $\geq$ E[$u(Y)$] when $\bar{X} = \bar{Y}$.

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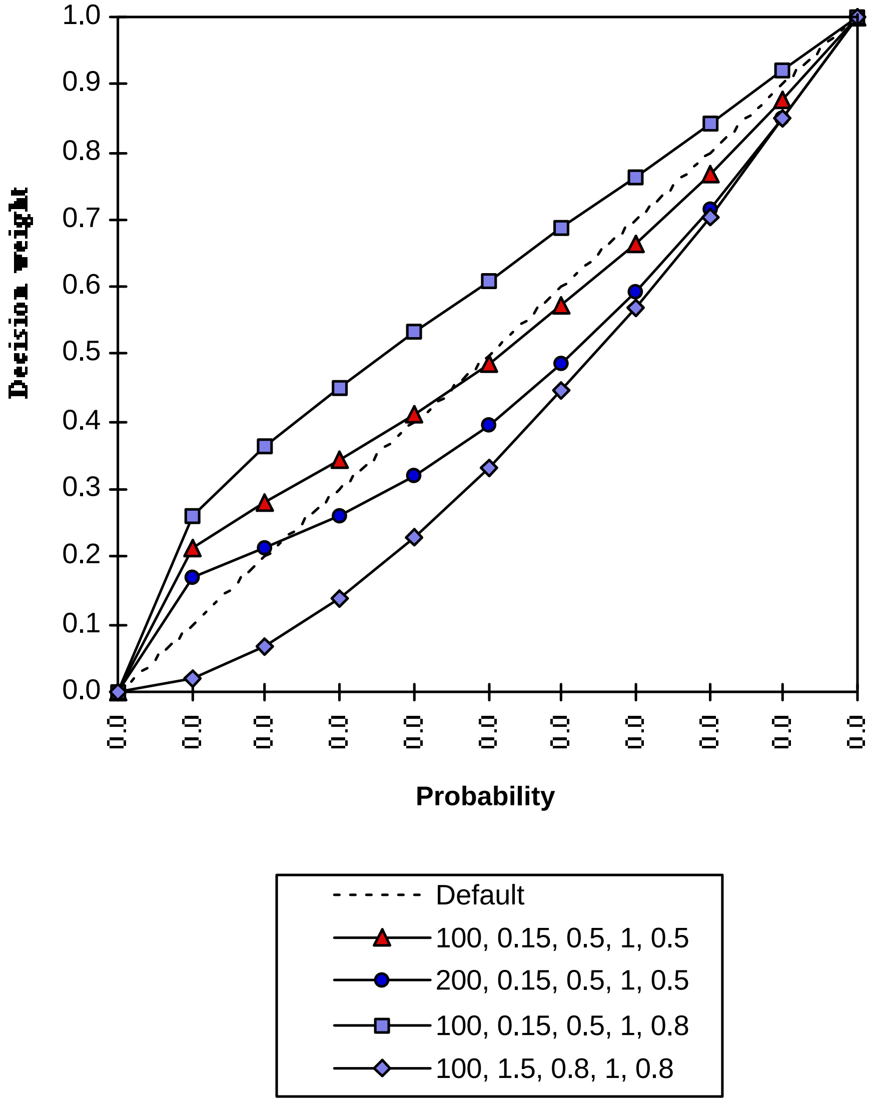
Figure 1. Decision weights based on model (26).



(The numbers in the series represent *x*, *b*, a, l and q in model (26) respectively.)