Utility Transversality: A Value-Based Approach

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Abstract

We examine multiattribute decision problems where a value function is specified over the attributes of a decision problem, as is typically done in the deterministic phase of a decision analysis. When uncertainty is present, a utility function is then assigned over the value function to represent the risk attitude of the decision maker towards value. With this structure, we derive a "chain rule" for deriving the risk attitude towards any attribute from the risk attitude of the utility function towards value combined with a "risk attitude" induced by the value function. This shows that the value function plays a major role in determining risk attitude towards an attribute.

We then define utility transversality equations, and show how the risk attitude of one attribute may be derived from the risk attitude of another through knowledge of the value function. In this sense there is only one dimension of risk attitude once the value function is determined. Specification of the risk attitude on one attribute thus determines the risk attitude of all other attributes, through transversality. Value functions that do not induce risk attitude are defined as risk neutral. This important class of value functions includes the multilinear family. We present several examples to illustrate how this approach unifies the derivation of several classical results.

Key words: multi attribute utility, value function, risk attitude, aspiration equivalent

1 – Introduction

In everyday decision situations, we are faced with multiple and sometimes conflicting attributes. When the decision situation is deterministic, the problem of choosing the best alternative is reduced to the problem of assigning a value function $V(x_1,...,x_n)$ over these attributes. The optimal alternative is then the one that has the largest value as determined by the value function. Perhaps the simplest approach to value modeling is to develop tradeoffs among attributes in a linear additive model. Figure 1 shows a model used in late 1960s to treat a decision regarding expansion of the electrical system of Mexico (Matheson 1970). The value model was linear in system profit, outage rates, balance of payments, dependence of foreign supply, pollution, public works, employment, and profit to Mexican industry. Using reasonable ranges of tradeoff of each attribute to monetary value, sensitivity analysis quickly showed that the only important attributes to the decision at hand were outage rates, balance of payments and pollution. Additional linear tradeoffs appeared in discounting the resulting time stream of monetary value to a single value measure, the net present value of Mexican "national profit." This measure included all of the important social tradeoffs this governmental body considered important in the context of this decision.

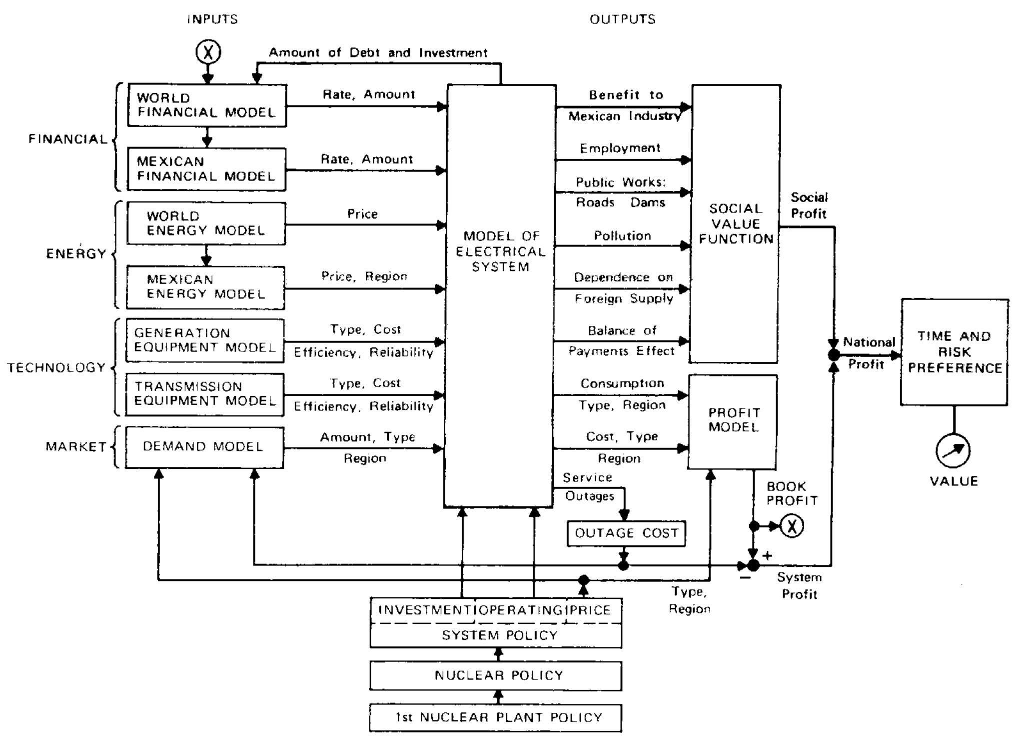

Figure 1 – A decision analysis model of the Mexican electrical subsystem.

For many situations a linear value model is adequate: in others it is often a good first approximation to use for sensitivity analysis and further refinement. However for some situations, for example when the attributes represent an individual's consumption, a multiplicative value model or a Cobb-Douglas function may be more appropriate. Our main point is that construction of value models are often the best first step in capturing and understanding the tradeoffs in multiattribute decision situations. Often the tradeoff values represent willingness-to-pay measures that are transparent and straightforward to assess, and when they are difficult to assess, sensitivity analysis can help pin down those dimensions most worthy of further study.

If the situation has significant uncertainty, the next step is to assess the uncertainty and use the model of the situation, including the value model, to generate probability distributions on the value measure resulting from alternative decisions. To judge the desirability of these decisions, we often assess a utility function for calculating expected utilities and certain equivalents. While linear functions seemed natural for value tradeoffs, linear utility is risk neutral, so we need a more complex function. When cardinal utility was first introduced, many people suggested the quadratic utility function as the simplest. However, it was soon realized that the exponential utility function was the natural first approximation, as it has constant risk aversion as a function of wealth (Pratt, 1964). Quadric utility was recognized as a poor choice as its risk aversion increases with wealth.

If a decision model is constructed in stages, such as by following the decision analysis cycle, by first considering value tradeoffs, including time preference), the natural multiattribute utility formulation is to place a single-dimensional utility function over a multiattribute value function (Matheson and Howard 1968). A real example treating time-risk preference in this manner is given in our table 1 (Spetzler & Zamora 1974-Table 5), which shows how the certain equivalent of net present value varies with the value tradeoff, expressed as three levels of discounting, and risk attitude, expressed as risk neutral and three levels of risk tolerance equal to 100, 33.3, and 12.5 million dollars respectively. The decision was whether to go ahead with the investment or to abandon the effort.

Combined Effect of Discount Rate and Risk Aversion
on the Certain Equivalent

		Certain Equivalent		
		Risk Aversion Level ($M)		
Discount Rate	Risk Neutral ($M)	1	2	3
7%	25.6	16.9	1.7	-46.6
10%	11.7	8.5	2	-15.5
13%	4.8	3.3	0.1	-8.3

Table 1- Combined effect of discount rate and risk aversion.

Additive multiattribute utility models, or even multilinear ones, rarely arise naturally in this approach. With value-based assessments, it is natural to calculate certain equivalents, value of information, and value of control, which is difficult using multiattribute formulations that are not derived from value functions.

In settings having monotonic (increasing) utility in each attribute and continuous tradeoffs between them, there is really only one dimension of risk attitude and only one utility function need be assessed once the value function is specified. On top of this value function, we assess the single dimension of risk attitude, resulting in the compound multiattribute utility function $U(V(x_1,...,x_n))$. This leads to multiattribute preference models that have rich interaction between dimensions. Realizing that there is only one dimension of risk attitude means that, in the presence of a multiattribute value function, specification of risk attitude on one attribute indirectly specifies it on all other attributes. For example, we will show that if the value function is linear, constant risk tolerance on one attribute implies constant risk attitude on all other dimensions, where these risk tolerances are related by the linear tradeoff coefficients.

In this paper, we derive a closed form relationship between the risk aversion coefficient and tradeoff coefficients derived from the value function. We present the notion of an induced risk attitude and induced utility function from one attribute to the other by knowledge of the trade-off coefficient between them. In section 2 we review the notion of a value function, indifference curves, and multiattribute utility functions. In section 3, we derive the chain rule for risk aversion, and in section 4, we present the notion of utility transversality. In sections 5 and 6 we define conjugate and risk neutral value functions and illustrate their use in facilitating the transversality conditions. In section 7 we present several examples to discuss the implications of the approach, in section 8 we discuss further implications of using the value-based approached and in section 9 we discuss the case of scaled attributes.

2 – Value Functions and Multiattribute Utility Functions

For simplicity of exposition, we limit this work to cases where preferences for each attribute are continuously increasing (more is preferred to less) and where at any point there always are tradeoffs among all attributes (no lexicographic ordering). Typical families of value functions that satisfy this condition are the Cobb-Douglas family, multiplicative value functions, and additive value functions. These results can easily be extended to cases where preferences are continuously increasing or decreasing over some region of attribute space.

One of the fundamental axioms of decision analysis requires ordering the prospects of the decision situation from the best to the worst. The extension of this axiom to the continuous multiattribute case is the assignment of isopreference contours or value functions over the attributes. Prospects on the same preference curve are equally preferred. An example of a two-dimensional Cobb-Douglas value function for two attributes, *x* and y, is

$$V(x,y) = yx^{\eta}, \tag{1}$$

where η is a measure of the trade-off between the two attributes and is equal to $-\frac{dy}{dx}\frac{x}{y}$.

In decision situations when uncertainty is present, we represent the decision maker's risk attitude towards value by assigning a utility function over the value function, which creates a compound multiattribute utility function

$$U(x_1,...,x_n) = U(V(x_1,...,x_n)) \tag{2}$$

where, $x_1,...,x_n$ are the *n* attributes of the decision situation. In our developments, we assume that both U and V are twice continuously differentiable with positive first derivatives. Also, for simplicity we will drop the arguments of functions when this is unambiguous, and we will often treat the two-dimensional case when the generalization to more dimensions is straightforward.

3 – Chain Rule for Risk Aversion

The risk aversion coefficient, $\gamma_{x_i}^U$, for any attribute, x_i, $i = 1,..,n$ was defined by Pratt (1964) as

$$\gamma_{x_i}^U \triangleq -\frac{U_{x_i}^{''}}{U_{x_i}^{'}} \tag{3}$$

where $U_{x_i}^{''} = \dfrac{\partial^2 U(x_1,...,x_n)}{\partial x_i^2}$, and $U_{x_i}^{'} = \dfrac{\partial U(x_1,...,x_n)}{\partial x_i}$.

In this section we will derive an expression for the risk aversion coefficient towards any attribute in terms of the risk aversion coefficient towards value as well as other parameters of the value function that we define. We will present the analysis for the case of a value function with two attributes, *x* and y, but the analysis extends directly to more than two attributes.

$$U(x,y) = U(V(x,y)) \tag{4}$$

Using the chain rule for partial derivatives, we take the first derivative of equation (4) with respect to x,

$$\frac{\partial U(x,y)}{\partial x} = \frac{\partial U(V(x,y))}{\partial x} = \frac{\partial U(V(x,y))}{\partial V}\frac{\partial V(x,y)}{\partial x} = U_V^{'}\frac{\partial V(x,y)}{\partial x}, \tag{5}$$

where $U_V^{'} = \dfrac{\partial U}{\partial V}$.

Now we use the chain rule again and evaluate the derivative of equation (5) with respect to *x*

$$\begin{aligned}\frac{\partial^2 U(x,y)}{\partial x^2} &= \frac{\partial^2 U(V(x,y))}{\partial V^2}\left(\frac{\partial V(x,y)}{\partial x}\right)^2 + \frac{\partial U(V(x,y))}{\partial V}\frac{\partial^2 V(x,y)}{\partial x^2} \\ &= U_V^{''}\left(\frac{\partial V(x,y)}{\partial x}\right)^2 + U_V^{'}\frac{\partial^2 V(x,y)}{\partial x^2},\end{aligned} \tag{6}$$

where $U_V^{''} = \dfrac{\partial^2 U}{\partial V^2}$.

From equations (3), (5), and (6), the risk aversion coefficient for attribute, *x*, is

$$\gamma_x^U = -\frac{\dfrac{\partial^2 U(x,y)}{\partial x^2}}{\dfrac{\partial U(x,y)}{\partial x}} = -\frac{U_V^{"}\left(\dfrac{\partial V(x,y)}{\partial x}\right)^2 + U_V^{'}\dfrac{\partial^2 V(x,y)}{\partial x^2}}{U_V^{'}\dfrac{\partial V(x,y)}{\partial x}} = -\frac{U_V^{"}}{U_V^{'}}\frac{\partial V}{\partial x} - \frac{\dfrac{\partial^2 V(x,y)}{\partial x^2}}{\dfrac{\partial V(x,y)}{\partial x}} \quad (7)$$

For simplicity of expressions, let us define the risk aversion coefficient of the utility function over the value function as, γ_V^U, and the risk aversion coefficient of value towards attribute x as γ_x^V, then

$$\gamma_V^U \triangleq -\frac{U_V^{"}}{U_V^{'}}, \qquad \gamma_x^V \triangleq -\frac{\dfrac{\partial^2 V(x,y)}{\partial x^2}}{\dfrac{\partial V(x,y)}{\partial x}} \quad (8)$$

Substituting equation (8) into equation (7) gives

$$\boxed{\gamma_x^U = \gamma_V^U \frac{\partial V}{\partial x} + \gamma_x^V} \quad (9)$$

Equation (9) relates the risk aversion coefficient of an attribute x as a sum of two terms. The first term, $\gamma_V^U \frac{\partial V}{\partial x}$, is a product of the risk aversion coefficient of utility towards value and the partial derivative of the value function with respect to the attribute of interest, x. The partial derivative, $\frac{\partial V}{\partial x}$, is a unit conversion factor. The second term, γ_x^V, is the risk aversion coefficient of the value function towards the attribute of interest. Note that this term is completely determined by the value function and not by any utility function over the value function or risk attitude. The sign of this value function risk aversion coefficient depends only on the concavity or convexity of the value function with respect to the attribute of interest.

Equation (9), which we call *the chain rule for risk aversion*, shows that the risk aversion coefficient for attribute, x, is completely specified by knowledge of the risk aversion coefficient of utility towards value and other parameters related only to the value function. Since the value function represents deterministic preferences, and can in principle be assessed with deterministic questions, the one-dimension of risk attitude is expressed by the utility function on value, which should be assessed using probabilistic questions. This assessment is easier if the value function, or at least one of its attributes, is expressed in units that are easy to relate to, such as money. Of course, any increasing monotonic function of a value function represents identical deterministic preferences, so there is a lot of freedom in selecting a desirable value measure. Monetary units, such as net present value, also have the advantage of permitting monetary value of information or control calculations.

4 – Utility Transversality

If we are given a value function and a risk aversion coefficient for one dimension, say *x*, we can re-arrange equation (9) for the risk attitude on value to obtain

$$\gamma_V^U = \frac{\gamma_x^U - \gamma_x^V}{\frac{\partial V}{\partial x}}. \tag{10}$$

Equating this expression with the similar one for attribute *y*, gives us the equation

$$\frac{\gamma_x^U - \gamma_x^V}{\frac{\partial V}{\partial x}} = \frac{\gamma_y^U - \gamma_y^V}{\frac{\partial V}{\partial y}}. \tag{11}$$

Equation (11) relates the risk aversion coefficient of attribute *x* to the risk aversion coefficient of attribute *y*. We call this relationship *the utility transversality condition* as it translates risk aversion coefficients across value attributes.

If we are given the value function and the risk aversion coefficient for *y* we can use the transversality condition to solve for the risk aversion coefficient on *x* as

$$\gamma_x^U = \left[\gamma_y^U - \gamma_y^V\right]\frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial y}} + \gamma_x^V \tag{12}$$

To further simplify this equation, we investigate the slope of the isopreference curve. We want to find the derivative along any isopreference contour by observing that along such a curve

$$V(x, y) = \text{Constant} \tag{13}$$

So setting the derivative of a value function across an isopreference contour to zero yields

$$dV(x,y) \triangleq \frac{\partial V(x,y)}{\partial x}dx + \frac{\partial V(x,y)}{\partial y}dy = 0 \tag{14}$$

Re-arranging gives the value tradeoff between the two attributes as

$$\frac{\dfrac{\partial V(x,y)}{\partial x}}{\dfrac{\partial V(x,y)}{\partial y}} = -\frac{dy}{dx}\Big|_{\text{isopreference contour}} \triangleq t(x,y) \tag{15}$$

where $t(x,y)$ defined above is the trade-off between attributes *y* and *x* along an indifference curve.

Thus the solution for the risk aversion coefficient on *y* given the coefficient on *x* and the value function is

$$\gamma_x^U(x,y) = \left[\gamma_y^U(x,y) - \gamma_y^V(x,y)\right]t(x,y) + \gamma_x^V(x,y)\,. \tag{16}$$

Simplifying the notation, we will drop the (x,y) terms form equation (16) keeping in mind that the parameters of the equation will generally depend on the values of both *x* and *y*. *The utility transversality condition* becomes

$$\boxed{\gamma_x^U = (\gamma_y^U - \gamma_y^V)t + \gamma_x^V} \tag{17}$$

Because additive utility functions are popular, people are led to believe that they can (or should) select arbitrary utility functions for each attribute and then combine them to get a multiattribute utility function. However, once a (deterministic) value function is established, there is only one dimension, or degree of freedom, of risk attitude in the sense that specifying a utility function for a single dimension determines the utility functions for all other dimensions through the value function. The transversality conditions make this translation.

Specifying an arbitrary utility function for each attribute as well as a multiattribute value function may result in inconsistency with the axioms of utility theory. For example, in a multi-period case, setting an identical constant risk tolerance in each period is inconsistent with a discounted net present value (NPV) time preference, unless the discount factor is 1.0. We will discuss this example in detail later.

Once a risk aversion coefficient, $\gamma_{x_i}^U$, has been translated by the transversality condition from one attribute to another, the utility function for this attribute can be determined by solving the differential equation of

$$\gamma_{x_i}^U(x_1,..,x_i,.,x_n) = -\frac{\frac{\partial^2 U(x_1,..,x_i,.,x_n)}{\partial x_i^2}}{\frac{\partial U(x_1,..,x_i,.,x_n)}{\partial x_i}} \tag{18}$$

Setting all attributes except x_i at a constant value, the solution to this equation has the form

$$U(x_i) = \int e^{-\int \gamma_{x_i}^U(x_1,..,x_i,.,x_n)dx_i} dx_i \tag{19}$$

Equations (17) and (19) show that a utility function over one attribute is specified completely by the utility function over another attribute and a given value function

5 – Conjugate Value Functions

Now we are ready to introduce the notion of a conjugate value function. A value function, V, is said to be a conjugate to a utility function on value, $U(V)$, if the marginal utility function induced on each attribute, $U(x_i)$, is of the same family as $U(V)$. A conjugate value function is related to the to the induced utility function on each attribute through the chain rule for risk aversion of equation (9). We give one example below and another section 7 regarding time preference.

Example: Product Value Function

Let us assume the utility function $U(V) = \log(V)$ is logarithmic, then from equation (8) $\gamma_V^U = \frac{1}{V}$.

Now if the value function is a product value function, then $V(x, y) = xy$, $\frac{\partial V}{\partial x} = y$ and $\gamma_x^V(x, y) = 0$. Substituting into the chain rule for risk aversion, we get

$$\gamma_x^U = \frac{1}{xy} y + 0 = \frac{1}{x} \tag{20}$$

Substituting equation (20) into (19) produces an induced logarithmic utility function on attribute *x,* where $U(x) = \log(x)$. Similar analysis shows the induced utility function on attribute *y* is also a logarithmic value function. Based on this result we say that the product value function is conjugate to the logarithmic utility function.

6 – Risk Neutral Value Functions

We define a *risk neutral value function* as one having risk aversion coefficient of the value function towards any attribute equal to zero

$$\gamma_{x_i}^V = -\frac{\frac{\partial^2}{\partial x_i^2} V(x_1, \ldots, x_i, \ldots, x_n)}{\frac{\partial}{\partial x_i} V(x_1, \ldots, x_i, \ldots, x_n)} = 0 \quad \forall i = 1, \ldots, n \tag{21}$$

Examples of value functions that meet this condition are:

1) Linear additive value functions: $V(x_1, \ldots, x_i, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i, \quad a_i = \text{constant}$.
2) Multiplicative value functions: $V(x_1, \ldots, x_i, \ldots, x_n) = \prod_{i=1}^{n} x_i$
3) Multilinear value functions: $V(x, y) = k_x x + k_y y + k_{xy} xy$, where k_x, k_y, k_{xy} are constants, or its multidimensional form.

In all these cases, the value function is a linear function in each attribute. For risk neutral value functions, the chain rule formula for risk aversion coefficients reduces to:

$$\boxed{\gamma_{x_i}^{U} = \gamma_{V}^{U} \frac{\partial V}{\partial x_i}}, \qquad (22)$$

and the transversality condition of equation (17) reduces to the simple expression

$$\boxed{\gamma_{x}^{U} = \gamma_{y}^{U} t}. \qquad (23)$$

Equation (23) relates the ratio of the two risk aversion coefficients to the trade-off coefficient between the two attributes. The trade-off coefficient between the attributes and knowledge of a risk aversion coefficient for one attribute defines the risk aversion coefficient for the other. Again, this result shows how the specification of one dimension of risk attitude determines the risk attitude of the other dimension through the value function trade-offs. We will call this condition *value transversality*.

For the special case of linear additive value functions we have

$$\gamma_{x_i}^{U} = a_i \gamma_{V}^{U} , \qquad (24)$$

giving transversality conditions of

$$\gamma_{x_i}^{U} = \gamma_{x_j}^{U} a_i / a_j . \qquad (25)$$

Note that when the constants a_i are equal for additive value functions, then the chain rule formula shows that the risk aversion coefficients for all attributes are also equal.

$$\gamma_{x_i}^{U} = \gamma_{x_j}^{U} = a\gamma_{V}^{U} \qquad \forall i, j \qquad (26)$$

7 – Applications of Utility Transversality

Having shown that there is only one dimension of risk attitude in multiattribute problems, we will now present several applications of the previous transversality results.

Example: On Fates comparable to death

The following example is adapted from (Howard, 1980) on making trade-offs concerning situations where a decision maker is exposed to fates comparable to death (such as outcomes of medical surgery). The decision maker provides a value function over two attributes: consumption, x, and health state, y. The health state is a disability level normalized from 0 (instant painless death) to 1 (Current health with no disability). The value function over consumption and health states is given as

$$V(x, y) = xy^{\eta}, \tag{27}$$

where x is in dollars, y is the health state, and η is the trade-off coefficient between x and y.

In this model, a utility function, $U(V)$, is assigned over the value function to represent the decision maker's risk attitude towards value

$$U(V) = U(V(x, y)) = 1 - e^{-\gamma V(x,y)} = 1 - e^{-\gamma xy^{\eta}} \tag{28}$$

Where $\gamma = \gamma_V^U$ is the risk aversion coefficient tolerance towards value.

Using the chain rule for risk aversion of equation (9), we can now deduce the risk aversion coefficient towards each of the attribute of the decision problem. For example, in the case of attribute x, we have $\frac{\partial V}{\partial x} = y^{\eta}$ and $\gamma_x^V(x, y) = 0$. Equation (9) reduces to

$$\gamma_x^U(x, y) = \gamma_V^U(V)\frac{\partial V}{\partial x} + \gamma_x^V(x, y) = \gamma y^{\eta} \tag{29}$$

For attribute y we have $\frac{\partial V}{\partial y} = \eta xy^{\eta-1}$ and $\gamma_y^V(x, y) = -\frac{\eta - 1}{y}$

$$\gamma_y^U(x, y) = \gamma_V^U(V)\frac{\partial V}{\partial y} + \gamma_y^V(x, y) = \gamma\eta xy^{\eta-1} - \frac{\eta - 1}{y}. \tag{30}$$

Here we see that characterization of the attribute risk aversion is strongly influenced by the "risk attitude" of the value function.

As a further illustration, if we instead assume that the utility function on attribute x is exponential with risk aversion coefficient γ_0, we apply the utility transversality conditions to arrive at

$$\gamma_y^U(x,y) = \gamma_0 \frac{\eta x + \eta - 1}{y} \ , \tag{31}$$

so the utility on attribute y must be logarithmic.

For the special case where $\eta = 1$, the value function is multiplicative

$$V(x,y) = xy. \tag{32}$$

This value function is a risk neutral value function that satisfies equation (21) where

$$\gamma_x^V(x,y) = 0\,. \tag{33}$$

$$\gamma_y^V(x,y) = 0\,. \tag{34}$$

The value transversality condition then applies, and the ratio of the two risk aversion coefficients is given as

$$\frac{\gamma_x^U(x,y)}{\gamma_y^U(x,y)} = t = -\frac{dy}{dx} = \frac{y}{x} \tag{35}$$

The ratio of the two risk aversion coefficients at any point (x,y) is the negative slope of the isopreference contour at that point. As shown in figure 2, it is also equal to the slope of the line from the origin to the point (x,y). This result is independent of the utility function that is assigned over the value function. Equation (35) shows, again, that assigning arbitrary risk aversion coefficients over the attributes can lead to inconsistencies in decision making since the risk aversion coefficients are related by the indifference curve trade-offs.

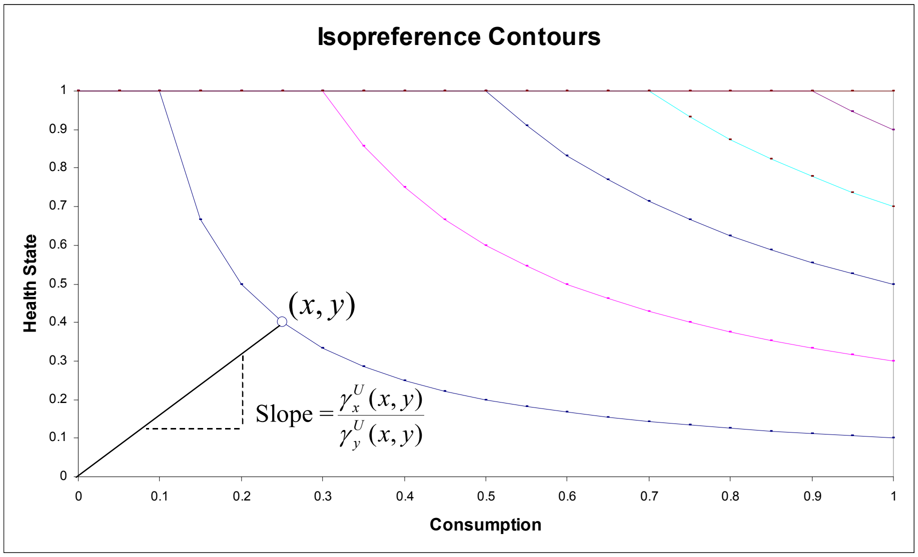

Figure 2 - Slope of line from origin is the ratio of the two risk aversion coefficients

Example: Time Preference

Now let us consider a decision situation with two attributes: $x =$ money received today, and $y =$ money received a year from today. The value function for this situation is chosen to be the net present value and is given as

$$V(x, y) = x + \beta y \tag{36}$$

The term β is the time preference or personal discount rate for one year.

The isopreference curves representing the trade-off between the pay-offs from the two years are shown below

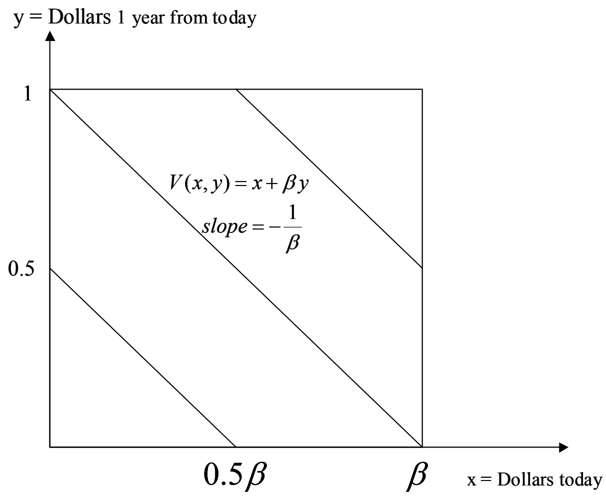

Figure 3 - Value function representing trade-off for 1-year discount.

Note again, that this value function is a risk neutral value function where $\gamma_x^V(x,y) = 0$ and $\gamma_y^V(x,y) = 0$. Using the value transversality condition, we have

$$\frac{\gamma_x^U(x,y)}{\gamma_y^U(x,y)} = -\frac{dy}{dx} = \frac{1}{\beta} \tag{37}$$

Re-arranging gives

$$\gamma_y^U = \beta \gamma_x^U \tag{38}$$

Equation (38) provides us with a relation between the risk aversion coefficient today and the risk aversion coefficient in one year. It shows that the risk tolerance should be compounded at the time preference rate and, again, that this relationship is independent of the utility function that is assigned to the value function. For the standard NPV function, if the risk aversion coefficient over one attribute is constant (exponential utility function, then the risk aversion coefficient over any other attribute is also a constant (exponential utility function). Exponential utility functions are thus conjugate to the additive value function.

If the risk aversion coefficient a year from today is not the same as that calculated from this formula, then the decision maker will exhibit inconsistencies in decision-making behavior and may become a money pump. This is shown by the argument that follows.

Suppose a decision maker has a risk tolerance of $\rho = \$100,000$ and his time preference for money is represented by a 10% compounding rate. His certain equivalent for a lottery to take place one year from now can be calculated by discounting the outcomes of the lottery by his discount rate, 10%. If the decision maker had an exponential utility curve, for example, then his expected utility equation will include the values of the lottery discounted by 10% and his current risk tolerance today. So in the denominator of the exponential term, there is 1.1ρ. Alternatively, we may consider the expected utility of the same deal a year from today using the outcomes of the lottery, (without discounting), and using the decision maker's risk tolerance for money received a year from today. The denominator in the exponential term is $\rho^{'}$. Consistency requires that $\rho^{'} = 1.1\rho$, otherwise the decision maker needs to wait 1 year and his CE for the same deal will be different. The risk tolerance thus compounds with time, and this is the result obtained from equation (38).

Example: Risk Preference of a partnership

In this example we will demonstrate how the previous results can also be applied to the risk aversion coefficient of a partnership in terms of the risk aversion coefficient of an individual. Let us assume the share of individual i is α_i times the share of the partnership. I.e.

$$V_i = \alpha_i V_P \tag{39}$$

Where, V_i is the revenue of individual i from the deal, α_i = individual i's percentage of the deal, and V_P is the revenue of the whole partnership.

The value function for individual i is a linear scaling of the value of the partnership. This is a risk neutral value function and so the value transversality conditions imply

$$\gamma_P = \alpha_i \gamma_i \tag{40}$$

Taking the reciprocal of equation (40), we get

$$\rho_i = \alpha_i \rho_P \tag{41}$$

This result indicates that the fraction of the deal taken by an individual in a partnership should be equal to the ratio of his risk tolerance to the sum of risk tolerances of the partners. This is a classic result in risk sharing literature (Wilson, 1968) and has been derived here using the value transversality conditions. Note also that the transversality conditions imply that if the individual utility function is exponential (has a constant risk tolerance), then the group utility function must also be exponential.

Example: Relating corporate to divisional risk tolerance

Now let us apply our results to the value function of a company, V_C, derived from the value functions of different divisions within a company. The value function of the company is the sum of the value of different divisions

$$V_C = \sum_{i=1}^{n} V_i \tag{42}$$

Once again, the value transversality conditions apply and in this special case we have

$$\gamma_i^U = \gamma_j^U = \gamma_{V_C}^U, \quad \forall i, j \tag{43}$$

This result implies that all divisions should act with the same risk tolerance. The corporate risk tolerance should be used for all corporate decisions.

However, if one of the divisions, k, is owned in partnership with another company, with ownership fraction, α, then using equation (40) we get

$$\gamma_k^U = \alpha \gamma_{V_C}^U \tag{44}$$

This implies that division, k, should have a lower risk aversion coefficient than the other divisions by a factor of α.

8 – Value-Based Implications

In the previous discussions we have shown that a value-based utility function is sufficient to derive the utility functions and risk aversion coefficients over all other attributes using the transversality conditions. This result implies that we can reduce any complicated multiattribute utility assessment into a one-dimensional utility assessment over the value function, or over any single attribute. In this section we will introduce the notion of value-based probability, which reduces a joint probability distribution function into a one-dimensional probability function over the value function. We will also present several other value-based concepts such as the value certain equivalent and value aspiration equivalent. Key to these concepts is the fact that the value is constant along isopreference curves (or surfaces) and that the isopreference curves are nested within each other as they increase in value.

Value-Based Probability

We define the value-based cumulative probability function, $F(V)$, as the probability that the outcome of a joint distribution (lottery) is less than or equal to a certain value for the value function. For any joint probability density function over the attributes x and y, we can define a value-based cumulative probability as

$$F(V_0) \triangleq \iint\limits_{\substack{\text{Area under} \\ V(x,y)=V_0}} f(x,y)\,dxdy \tag{45}$$

We can also define a value based probability density function as

$$f(V) \triangleq F'(V) \text{ (if the derivative exists)} \tag{46}$$

Note that the value based probability can, in many cases, be assessed directly, (such as cumulative probability on NPV) in which case the probability assessment is simplified by dispensing with the need to assess a joint probability distribution on both variables x and y.

Value-Certain Equivalent

Having defined a value-based utility and a value-based probability, the expected utility over value for a given deal is given by the integral

$$\textit{Value Expected Utility} = \int U(V)dF(V) = \int f(V)U(V)dV \tag{47}$$

Now we can define the value certain equivalent as the value-based utility inverse of the expected utility. I.e.

$$Value\ Certain\ Equivalent = \tilde{V} = U^{-1}(Value\ Expected\ Utility) \quad (48)$$

The value certain equivalent defines a whole isopreference contour and does not define specific values of the attributes on that contour. (Figure 4)

Value-Aspiration Equivalent

Using our definition of aspiration equivalent from a recent paper (Abbas and Matheson, 2003), we define the value aspiration equivalent as

$$Value\ Aspiration\ Equivalent = \hat{V} = F^{-1}\left(\int_{-\infty}^{\infty} F(V)dU(V)\right) \quad (49)$$

where $U(V)$ has been normalized to range from 0 to 1.

The value-aspiration equivalent defines an equivalent step utility function that provides the same expected utility as the original utility function over value, and jumps from 0 to 1 at a certain level of the value function (at a certain isopreference contour), which we call the value aspiration equivalent. In our previous paper we have shown that choosing the lottery that maximizes the probability of meeting its aspiration equivalent is identical to choosing the lottery, which has the highest expected utility. Furthermore, the aspiration equivalent makes a sound target for delegation.

Like the value certain equivalent, the value aspiration equivalent defines a whole isopreference contour and does not define specific values of the attributes on that contour (Figure 4). This defines a trade-offs among targets with multiple attributes.

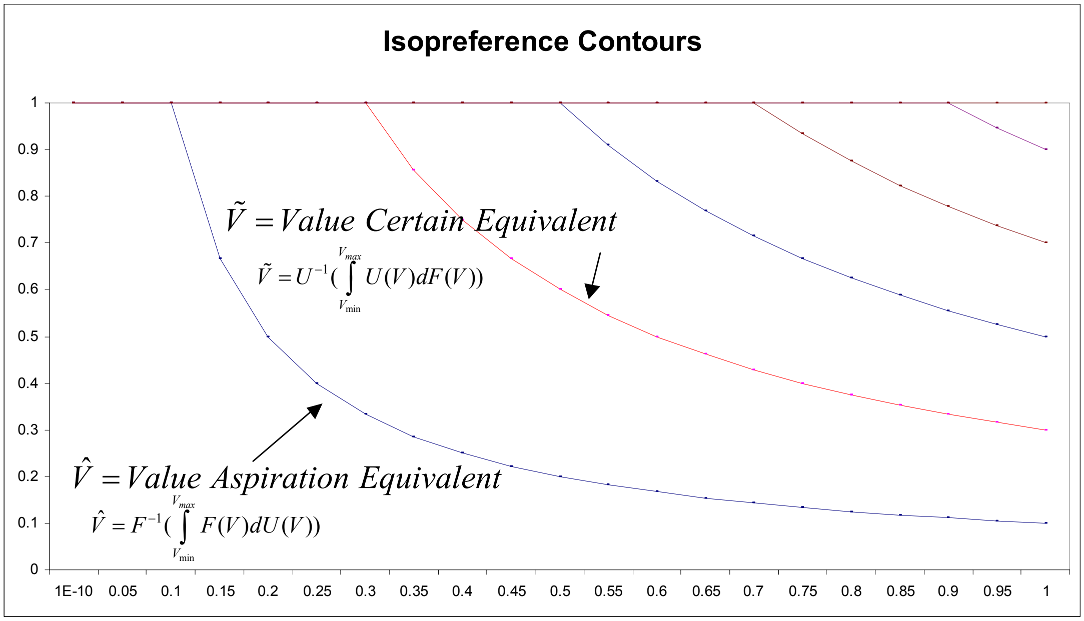

Figure 4 – Value aspiration equivalents and certain equivalents

In business situations, executives often delegate by setting targets. In the value-based multiattribute setting, a value target is a selection of an isopreference curve. This means that if the executive further specifies the point on the isopreference curve, she needs to further specify the trade-off coefficients to help the manager trade-off among the targets of the different attributes. There can be many ways of selecting the value target. In the paper above, we suggest using the aspiration equivalent as a target and refer the reader there for further discussion.

Another way to set targets would be to set individual targets for each attribute. Bordley and Kirkwood (2003) interpret individual utility functions on each attribute as uncertain targets, and a multiattribute utility function as the joint probability of meeting the targets. This approach, however, does not provide trade-offs among the attributes. In contrast, our value-based approach allows trade-offs, which provide a higher probability of reaching the value target (isopreference curve). Figure 5 shows the area of integration of a joint probability density function (represented by the shaded area) for reaching the individual targets for each attribute and can be compared to the area above the corresponding isopreference curve for reaching the value target.

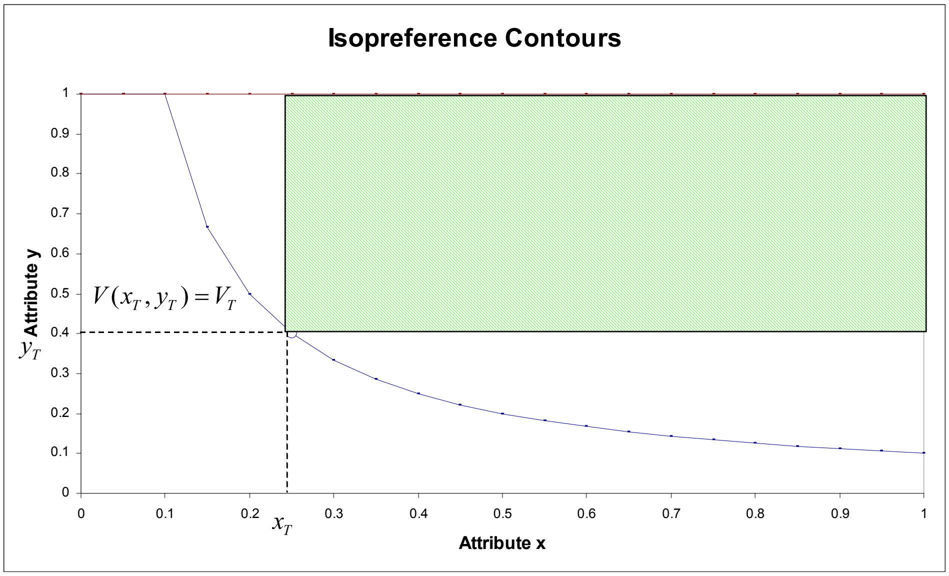

Figure 5. Individual attribute targets vs. value-based targets

Figure (5) shows the effects of incorporating trade-offs between attributes in setting targets. The difference between the area above the isopreference contour and the shaded region contains many outcomes that may be satisfactory jointly, but would be rejected by independent targets. For example, if the attributes represented the performance results of several task managers working on different features of a new product, setting targets using isopreference contours may result in more collaboration between different tasks because the value trade-offs can be made.

Example: Price and profit margin

In this example we will demonstrate how to calculate a value certain equivalent and value aspiration equivalent for prospects with multiple attributes.

Let us assume that a company is interested in two attributes: selling price of a product, *p*, and profit margin*, m*. The value function is the product of the two attributes

$$V(p,m) = pm \quad 0 \le p \le 1, 0 \le m \le 1 \tag{50}$$

Where *p* is the selling price normalized by dividing by the maximum contemplated price, $p_{\max}$, and *m* is the profit margin (fraction from 0 to 1). The value function has the following extreme values

$$V_{\max} = 1, \quad V_{\min} = 0 \tag{51}$$

Let if the company has an exponential utility function over value, the value-based utility function normalized to have values from 0 to 1 is given as

$$U(V) = \frac{1 - e^{-\gamma V}}{1 - e^{-\gamma}} \tag{52}$$

where γ is the risk aversion coefficient. For our example, let us assume the value of γ is 2.

For simplicity we will assume the joint probability density function is uniformly distributed over the domain $0 \le p \le 1, 0 \le m \le 1$ and given as

$$f(p,m) = 1, \qquad 0 \le p \le 1, 0 \le m \le 1 \tag{53}$$

In order to facilitate the expressions, let p_1 be the value of x where the indifference curves intersect the boundary y=1.

$$\begin{aligned} F(V(p,m)) &= \int_{p_1}^{1} \int_{0}^{\frac{V}{p}} 1 dm dp + p_1 \\ &= \int_{p_1}^{1} \frac{V}{p} dp + p_1 = V(1 - \ln(V)) = pm(1 - \ln(pm)) \end{aligned} \tag{54}$$

Therefore

$$F(V) = V(1 - \ln(V)) \tag{55}$$

Figure 6 shows the value-based probability and utility functions for this example.

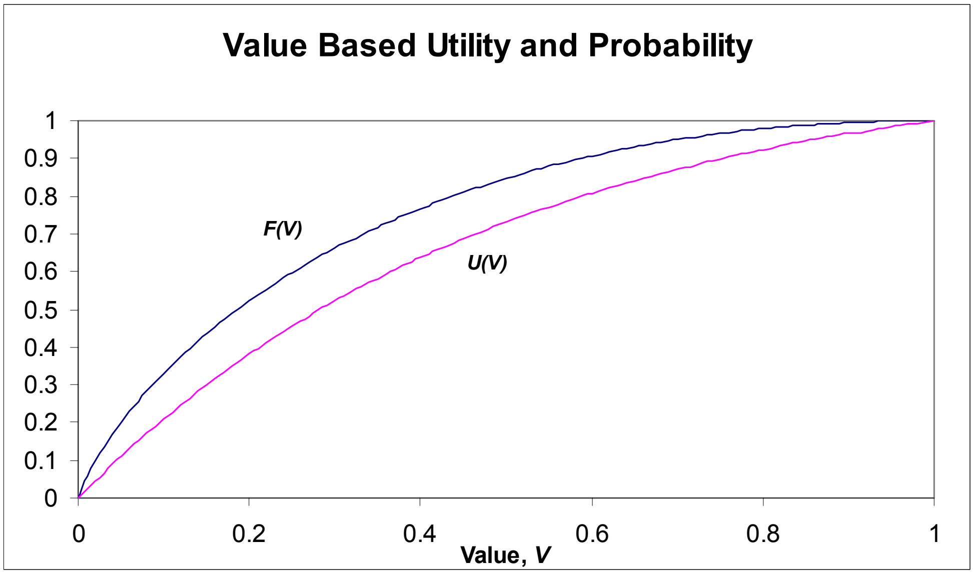

Figure 6- Value Based Utility and Probability

We can solve equations (48) and (49) using numerical integration to get

Value Certain Equivalent = 0.195 and *Value Aspiration Equivalent = 0.25*.

If the company used the value aspiration equivalent as a target, then based on the specific risk aversion and lottery; the targets should be on the isopreference curve

$$\frac{p}{p_{\max}} m = 0.25 . \tag{56}$$

Equation (56) also determines the trade-offs between the attributes at the target values.

9 – Scaled Attributes

Sometimes it is natural to scale attributes before combining them in a value function. The scaling function may be a linear or a non-linear transformation of the attribute. For example, one might measure or assess the speed of an object but place value on the kinetic energy, which is proportional to speed squared. In a consumption example, one might assess the amount of food available but have diminishing increase in satisfaction as the quantity increases. Cases like these can be treated with scaling functions, where we replace the attribute, x_i, with the scaled attribute, $S(x_i)$. Thus the value function becomes $V(S(x_1), S(x_2), ..., S(x_N))$.

Taking derivatives as we did in deriving the chain rule we arrive at

$$\gamma_{x_i}^{V} = \gamma_{s_i}^{V} \frac{\partial V}{\partial S_i} + \gamma_{x_i}^{S_i} \tag{57}$$

Equation (57) can be combined with the chain rule for risk aversion of equation (9) or the transversality condition of equation (17) to generalize these equations to the case of scaled attributes.

Risk neutral value functions present an interesting case. If the value function is risk neutral, the general case being the multilinear form, then the $\gamma_{s_i}^{V}$ are all zero so equation (57) becomes

$$\gamma_{x_i}^{V} = \gamma_{x_i}^{S_i} \tag{58}$$

In words, because the value function is risk neutral, the risk aversion coefficient of the value function for each x_i is identical to the risk aversion coefficient of its scaling function $S_i(x_i)$.

Relation to multilinear utility functions

The multilinear utility function (Keeney & Raiffa, 1976) can be interpreted as a risk neutral value function whose attributes are scaled by the individual utility functions. Since there is no additional utility function on the resulting value measure, their individual risk aversion coefficients are only scaled by the values of the other dimensions, but the shape of the utility functions are specified independently in each dimension. In our formulation, mutual utility independence is equivalent to a risk neutral value function with an identity utility function on this value. Allowing this identity utility function to take on other forms is an opportunity to extend and enrich the multiattribute risk behavior of this important class of functions.

10 – Conclusions

We have demonstrated that for a wide class of multiattribute utility problems, there is only one dimension of risk attitude, in the sense that the risk attitude of one attribute combined with the value functions determines the risk attitude of all of the other attributes. In many N attribute situations, a deterministic value function may be used to specify isopreference contours (N-1 independent tradeoff pairs) while the utility function over value specifies the one additional dimension of risk attitude. Only the latter requires probabilistic questions to assess. This value-based utility structure provides transparency and simplicity while yielding a very rich set of multiattribute utility functions. Introducing the value function explicitly allows the calculation of lotteries on value, and provides additional value-focused insights through certain equivalents, aspiration equivalents, value of information and value of control. We have shown several examples of using the value-based utility formulation in practice.

We introduced transversality conditions to relate risk attitude across the attribute dimensions. They show how value functions may introduce risk attitude and specialize to the important case of risk neutral value functions. Transversality often allows determination of risk tolerance by inspection, particularly when conjugate value functions are present. Transversality also allows us to derive several classical results in simple and quick ways.

Because this formulation is value based, extensions to the concepts of stochastic dominance and utility dominance (Abbas & Matheson, 2003) are straightforward. We look forward to new applications using value-based utility formulations and increased use of the traditional concepts and tools of decision analysis in multiattribute situations.

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