Dynamic Debt Maturity*

Zhiguo He        Konstantin Milbradt

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Abstract

We study a dynamic setting in which a firm chooses its debt maturity structure and default timing endogenously, both without commitment. The firm, who is waiting for the arrival of an upside event, commits to keep its outstanding bond face-values constant, but controls its debt maturity structure via the fraction of newly issued short-term bonds when refinancing its maturing long- and short-term bonds. As a baseline, we show that when the firm’s fundamental is time-invariant, it is impossible to have a shortening equilibrium in which the firm keeps issuing short-term bonds and consequently defaults inefficiently. However, when cash-flows deteriorate over time so that the debt recovery value is affected by the endogenous default timing, then a shortening equilibrium with accelerated default can emerge. In this situation, self-enforcing shortening and lengthening equilibria exist, and the shortening equilibrium may be Pareto-dominated by the lengthening one.

Keywords: Maturity Structure, Dynamic Structural Models, Endogenous Default, No Commitment, Debt Rollover.

*He: University of Chicago, Booth School of Business; and NBER. Email: zhiguo.he@chicagobooth.edu. Milbradt: Northwestern University, Kellogg School of Management; and NBER. milbradt@gmail.com. We thank Guido Lorenzoni, seminar audiences at Kellogg Lunch workshop, Lausanne, SITE Stanford, LSE, INSEAD, and Toulouse for helpful comments.
1 Introduction

The 2007/08 financial crisis has put debt maturity structure and its implications squarely in the focus of both policy discussions as well as the popular press. However, dynamic models of debt maturity structure are difficult to analyze, and hence academics are lagging behind in offering tractable frameworks in which the firm’s debt maturity structure follows some endogenous dynamics. In fact, a widely used framework for analyzing debt maturity structure is based on Leland [1994b, 1998] and Leland and Toft [1996] who, for tractability’s sake, take the frequency of refinancing/rollover as a fixed parameter. In that framework, equity holders are able to commit to a policy of constant debt maturity structure, which equals the inverse of the debt rollover frequency, until default.

This stringent assumption is at odds with mounting empirical evidence that non-financial firms in aggregate tend to have pro-cyclical debt maturity structure (Chen et al. [2013]). What is more relevant to our paper is the evidence of active management of the firm’s debt maturity structure. A recent paper by Xu [2014] shows that speculative-grade firms are actively lengthening their debt maturity structure—especially in good times—via early refinancing. On the other hand, Krishnamurthy [2010] documents that financial firms were shortening their debt maturity structure during the 2007/08 crisis. Our paper not only provides the first dynamic model to investigate this question, but also delivers predictions that are consistent with these empirical patterns.

In short, this paper removes the equity holders’ ability to commit to a debt maturity structure, allowing us to analyze how equity holders adjust the firm’s debt maturity structure facing time-varying firm fundamentals and endogenous bond prices. In our model, the firm has two kinds of debt, long and short term bonds, that mature with constant but different Poisson intensities. Assuming a flat term structure of the risk-free rate, absent default concerns the firm is indifferent between both bonds. However, potential default risk implies that the yield (price) for short-term bonds is endogenously lower (higher) than that of long-term bonds. In response to such an endogenous price wedge, equity holders can influence the firm’s debt maturity structure by changing the maturity composition of new debt issuance: if just-matured long-term bonds are replaced by
short-term bonds, then the firm’s debt maturity structure shortens. To focus on endogenous debt maturity dynamics only, we fix the firm’s book leverage policy, by following the Leland-type model assumptions that the firm commits to maintaining a constant aggregate face-value of outstanding debt. This treatment is consistent with the fact that in practice, most bond covenants put restrictions on the firm’s future leverage policies, but rarely on the firm’s future debt maturity structure.

In refinancing their maturing bonds, equity holders absorb the cash-flow gap between the face value of matured bonds and the proceeds from selling newly issued bonds at market price. When default is imminent, bond prices are low and equity holders are incurring rollover losses. This so-called rollover risk may feed back to earlier default, an effect that emerged in a variant of the classic Leland model that involved finite maturity debt (Leland and Toft [1996] and Leland [1994b]). More importantly, as shown by He and Xiong [2012b] and Diamond and He [2014], all else equal, equity holders are more likely to default if the firm has a shorter debt maturity structure and thus needs to refinance more maturing bonds. The more debt has to be repriced, the heavier the rollover losses are for the firm when fundamentals deteriorate, thereby pushing the firm closer to default.

What is the equity holders’ trade-off involved in shortening the maturity structure by issuing more short-term bonds today? The presence of default risk, which bond investors price, implies that “going short” offers higher issuance proceeds today. This is because short-term bonds fetch higher valuations relative to long-term bonds, as the former have a higher likelihood of maturing before default than the latter. Thus, the benefit of maturity shortening is to reduce the firm’s rollover losses today.

However, as short-term debt comes due faster, shortening increases the future rollover frequency and hence rollover losses, leading to earlier default and thus to lower equity value. This negative long-term effect is the cost of shortening maturity when equity holders decide the optimal issuance policy. Combining both the benefit and cost gives rise to the equity holders’ incentive compatibility condition for issuing short-term bonds. Our main research question is: Can situations arise in which this trade-off favors maturity shortening, so that, even though going short hastens inefficient default, in equilibrium equity holders keep issuing short-term bonds due to an inability to commit?
Note that, because of the presence of long-term bond holders and the lack of commitment, the Modigliani-Miller logic has little guidance to the answer of this question.

As a benchmark, we first consider the case in which a firm produces constant cash flows but is waiting for an upside event (at which point the model ends). We show that there is never any slow drift towards inefficient default via shortening the firm’s maturity structure. Either the firm defaults immediately, or the firm lengthens its debt maturity structure by issuing long-term bonds and never defaults.

We establish the result of “no shortening equilibrium” in the benchmark by analyzing the equity holders’ incentive compatibility condition in the vicinity of the (hypothetical) default boundary. The incentive compatibility condition turns out to be solely determined by the sign of the marginal impact of maturity shortening on the value of short-term bonds. More specifically, equity holders would like to issue more short-term bonds, if shortening the firm’s debt maturity structure marginally raises the market value of short-term bonds. Intuitively, right before default, the savings on today’s rollover losses by issuing more short-term bonds just offset the increase of tomorrow’s rollover losses. The only effect left at work is then that the shorter maturity structure edges the firm closer to default and hence affects the market value of bonds; but a lower distance-to-default drives down the market value of short-term bonds under the assumption of a positive (and constant) loss-given-default. As a result, the equity holders’ incentive compatibility constraint is always violated in the vicinity of default, and the “no shortening equilibrium” result emerges.

This “no shortening equilibrium” is in sharp contrast to Brunnermeier and Oehmke [2013] who show that equity holders might want to privately renegotiate the bond maturity down (toward zero) with each individual bond investor. The key difference is on who bears the rollover losses when there is arrival of unfavorable news in an interim period. In Brunnermeier and Oehmke [2013], there is no covenants about the firm’s aggregate face value of outstanding bonds, and after negative interim news the rollover losses of short-term bonds are absorbed by promising a sufficiently high new face-value to refinance the maturing short-term bonds. This increase in face-value dilutes the (non-renegotiating) existing long-term bond holders. In contrast, in our model equity holders are
absorbing rollover losses through their own deep pockets (or equivalently through equity issuance), as increasing face value to dilute existing bond holders is prohibited by the assumption of a constant aggregate face value. By shutting down the interim dilution channel in Brunnermeier and Oehmke [2013], we identify a new economic force that impacts maturity choice through the dynamic rollover trade-off.

We then move on to show that for firms whose cash flows are deteriorating over time, it is possible to construct a shortening equilibrium in which the firm drifts slowly towards inefficient early default while keeping issuing short-term debt. As in the constant cash-flow case, we require that maturity shortening increases the value of short-term bonds right before default. However, there is a crucial difference between the setting with deteriorating cash flows and that with constant cash flows. For firms whose cash flows are deteriorating over time, all else equal debt values may be higher under an earlier default time. This is because bond holders who take over the firm earlier will receive a higher debt recovery given a better firm fundamental. This force, which is absent in the setting with constant cash flows (and thus constant recovery value), can entice equity holders to shorten the firm’s debt maturity structure ex post, although it would be more efficient to commit to a long debt maturity.

In the case with deteriorating cash flows, starting at some initial state—i.e., today’s cash flows and maturity structure—that is sufficiently far away from bankruptcy, there exist two equilibrium paths toward default, one with maturity shortening and the other with lengthening. In the lengthening equilibrium, the firm keep issuing long-term bonds so its debt maturity structure grows longer and longer over time. In our example, the firm survives longer in the lengthening equilibrium, resulting in a higher overall welfare and even Pareto dominance over the shortening equilibrium. In a way, the multiplicity of equilibria emerges in our model without much surprise. If bond investors expect equity holders to keep shortening the firm’s maturity structure in the future, then bond prices reflect this expectation, self-enforcing the optimality of issuing short-term bonds only. Similarly, the belief of issuing long-term bonds always can be self-enforcing as well. However, we show that when the firm is sufficiently close to default then the model has a unique equilibrium; intuitively,
any future benign (malign) expectation of lengthening (shortening) maturity happen “too late” to be self-enforcing.

The qualitatively different results from these two cases imply the following empirical predictions. First, one is more likely to observe debt maturity shortening in response to deteriorating economic conditions. Second, conditional on deteriorating economic conditions, debt maturity shortening is more likely to occur in firms with already short maturity structures. The first prediction is consistent with the empirical findings cited at the beginning of the Introduction: Xu [2014] shows that speculative-grade firms are actively lengthening their debt maturity structure in good times, and Krishnamurthy [2010] shows that financial firms are shortening their debt maturity shortening right before 2007/08 crisis. We await future research to test the second empirical prediction of our model.

We make two key simplifying assumptions which render the tractability of our model. First, unlike typical Leland-type models, we rule out Brownian cash-flow shocks. In Section 4.4 we discuss how cash-flow volatility affects the connection between the equity’s incentive compatibility of going short and the marginal impact of maturity shortening on the value of short-term bonds. Second, in our model the firm commits to a constant aggregate amount of face-value outstanding, which rules out diluting existing bond holders by promising higher face value to new incoming bond holders following unfavorable news. As we discussed, this dilution effect is what is behind the maturity shortening in Brunnermeier and Oehmke [2013], and we rule out dilution to purposefully contrast our effect to that of Brunnermeier and Oehmke [2013]. Perhaps counter to the common wisdom, in our model, we are able to show that deleveraging—either forced or voluntarily—right before default gives equity holders a strong incentive to engage in maturity lengthening. It is interesting for future research to study endogenous dynamic maturity structure and dynamic leverage simultaneously in a Leland-type model.1

1Dynamic models of endogenous leverage decisions over time are challenging by themselves. The literature usually take the tractable framework of Fischer et al. [1989], Goldstein et al. [2001] so that the firm needs to buy back all outstanding debt if it decides to adjust aggregate debt face value. This assumption requires a strong commitment ability on the side of equity holders. Recently, Dangl and Zechner [2006] study the setting where firms can freely adjust debt face value downwards by issuing less bonds than the amount of bonds that are maturing. DeMarzo and He [2014] study the setting without any commitment on outstanding debt face value so that equity holders may
Debt maturity is an active research area in corporate finance. The repricing of short-term debt given interim news in Flannery [1986], Diamond [1991] and Flannery [1994] is related to the endogenous rollover losses of our paper. For dynamic corporate finance models with finite debt maturity, almost the entire existing literature is based on a Leland-type framework in which a firm commits to a constant debt maturity structure. To the best of our knowledge, our model is the first that investigates the endogenous debt maturity dynamics.

We abstract from various mechanisms that may favor short-term debt. For instance, Calomiris and Kahn [1991] and Diamond and Rajan [2001] emphasize the disciplinary role played by short-term debt, a force not present in our model. At a higher level, this economic force originates from the firm side—rather than the investor side—just like our model. This is because our analysis is based on the underlying equity-debt conflict of endogenous default when absorbing the firm’s rollover losses. In practice, debt maturity shortening can also originate from the concern of the investor side, which is another highly relevant economic force. The best example is Diamond and Dybvig [1983] in which debt investors who suffer idiosyncratic liquidity shocks demand early consumption; He and Milbradt [2014] study its implications in a Leland framework with over-the-counter secondary bond markets.

Our paper is also related to the study of debt maturity and multiplicity of equilibria in the sovereign debt literature, in which models are typically cast in a dynamic setting (e.g., Cole and Kehoe [2000]; Arellano and Ramanarayanan [2012]; Dovis [2012]; Lorenzoni and Werning [2014]).

Often, these models with investors’ liquidity needs only establish the advantage of short-term debt unconditionally, while our model emphasizes the endogenous preference of short-term debt when closer to default.

Arellano and Ramanarayanan [2012] provide a quantitative model where the sovereign country can actively...
Like us, Aguiar and Amador [2013] provide a transparent and tractable framework for analyzing maturity choice in a dynamic framework without commitment. They study a drastically different economic question, however: there, a sovereign needs to reduce its debt and the debt maturity choices matter for the endogenous speed of deleveraging. In contrast, in our model the total face value of debt is fixed at a constant, and the maturity choice trades off rollover losses today versus higher rollover frequencies tomorrow.

After laying out our model in Section 2, we solve the base model with constant cash flows and the setting with deteriorating cash flows in Section 3 and Section 4, respectively. Section 5 provides a numerical example to illustrate the nature of multiple equilibria in our model. We analyze interior equilibria in Section 6, and Section 7 concludes and gives empirical predictions. All proofs are in Appendix A.

2 The Setting

2.1 Firm and Asset

All agents in the economy, equity and debt-holders, are risk-neutral with a constant discount rate \( r \geq 0 \). The firm has assets-in-place generating cash flows at a rate of \( y_t \), whose evolution will be specified later. There is a Poisson event arriving with a constant intensity \( \zeta > 0 \); at this event, assets-in-place pay off a sufficiently large constant \( X > 0 \) and the model ends. This “upside event” gives a terminal date for the model can also be interpreted as the realization of growth options.

We allow the cash-flow rate \( y_t \) to be negative (e.g., operating losses). As \( y_t \) can take negative values, it might be optimal to abandon the asset at some finite time, denoted by \( T_a \). We assume that abandonment is irreversible and costless. Given the cash-flow process \( y_t \) and denote the arrival manage its debt maturity structure and leverage, and show that maturities shorten as the probability of default increases; a similar pattern emerges in Dovis [2012]. As standard in sovereign debt literature, one key motive for the risk-averse sovereign to borrow is for risk-sharing purposes in an incomplete market. Because debt maturity plays a role in how the available assets span shocks, the equilibrium risk-sharing outcomes are affected by debt maturity. This force is absent in most corporate finance models—including this paper—that are typically cast in a risk-neutral setting with some deep-pocketed equity holders (a la Leland framework).
time of upside event by $T_\zeta$, the unlevered firm value (or asset value) is

$$A(y) = \mathbb{E} \left[ \int_0^{\min(T_a, T_\zeta)} e^{-rt}y_t dt + 1_{\{T_\zeta < T_a\}} e^{-rT_\zeta X} \right]. \quad (1)$$

The firm is financed by debt and equity. When equity holders default, debt holders take over the firm with some bankruptcy cost (to be specified later), so that the asset’s recovery value from bankruptcy is $B(y) < A(y)$. We assume that $B'(y) > 0$, i.e., the firm’s liquidation value is increasing in the current state of cash-flows.

### 2.2 Dynamic Maturity Structure and Debt Rollover

#### 2.2.1 Assumptions

We study the dynamic maturity structure of the firm. To this end, we assume that the firm has two kinds of bonds outstanding: long-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_L$, and short-term bonds whose time-to-maturity follows an exponential distribution with mean $1/\delta_S$, where $\delta_i$’s are positive constants with $i \in \{S, L\}$ and $\delta_S > \delta_L$. Thus, bonds mature in an i.i.d. fashion with Poisson intensity $\delta_i > 0$. An equivalent interpretation is that of a sinking-fund bond as discussed in Leland [1994b, 1998].

Maturity is the only characteristic that differs across these two bonds. Both bonds have the same after-tax coupon rate $c$ and the same principal normalized to 1. To avoid arbitrary valuation difference between two bonds, we set the before-tax coupon rate equal to the discount rate, i.e. $\rho c = r$ where $\rho \geq 1$ stands for a tax benefit per unit of coupon. This way, without default both bonds have a unit value, i.e., $D_i^{rf} = D_i^{rf} = 1$. We also assume both bonds have the same seniority to rule out trivial dilution motives. In bankruptcy, both bond holders receive, per unit of face-value, $B(y)$ as the asset’s liquidation value. Throughout, we assume that

$$B(y) < D_i^{rf} = 1, \text{ for } i \in \{S, L\}. \quad (2)$$
This empirically relevant condition simply says that the loss-given-default for bond investors is strictly positive (and equity holder recover nothing in default).

To focus on maturity structure only, we assume that the firm commits to a constant “book leverage” policy. Specifically, following the canonical assumption in Leland [1998], the firm rolls over its bonds in such a way that the total promised face-value is kept at a constant normalized to 1 (hence, the total measure of these two bonds is 1). This assumption can be motivated by bond covenants on future leverage policies taken by the firm. Essentially, it rules out the dilution effect caused by future net debt issuance in response to the firm’s fundamental news. As explained later in Section 3.4.2, the dilution effect is the economic force behind Brunnermeier and Oehmke [2013], and by shutting this off we are highlighting a different channel from Brunnermeier and Oehmke [2013]. Nevertheless, we discuss the robustness of our results to potentially endogenous deleveraging in Section 4.4.

In sum, we implicitly assume that debt covenants, while restricting the firm’s future leverage policies, do not impose restrictions on a firm’s future maturity. This assumption is realistic, as debt covenants often specify restrictions on firm leverage but rarely on debt maturity.

### 2.2.2 Maturity structure and its dynamics

The face value of short-term bonds at time $t$, denoted by $\phi_t \in [0, 1]$, gives the fraction of short-term bonds outstanding. We call $\phi_t$ the current maturity structure of the firm. Given the current maturity structure $\phi_t$, during $[t, t + dt]$ there are $m(\phi_t) dt$ dollars of bonds maturing, where

$$m(\phi_t) \equiv \phi_t \delta_S + (1 - \phi_t) \delta_L. \quad (3)$$

The more short-term the current maturity structure is, the more the debt is rolled over each instant, as we have $m'(\phi) = \delta_S - \delta_L > 0$. This observation is important for later analysis.\footnote{The assumption of random exponentially distributed debt maturities rules out any “lumpiness” in debt maturing, which is termed “granularity” in Choi et al. [2014]. As another interesting dimension of corporate debt structure, debt granularity is related to but different from debt maturity structure.}
Under the constant debt face value assumption, the firm is issuing \( m(\phi_t) \, dt \) units of new bonds to replace its maturing bonds. The main innovation of the paper is to allow equity holders to endogenously choose the proportion of newly issued short-term bonds, which we denote by \( f_t \in [0,1] \).\(^7\) Hence, the dynamics of maturity structure \( \phi_t \) are given by

\[
\frac{d\phi_t}{dt} = -\phi_t \cdot \delta_S + m(\phi_t) \cdot f_t. \tag{4}
\]

Most of our analysis focuses on constant issuance policies that take corner values 0 or 1, i.e. \( f \in \{0,1\} \). Suppose that \( f = 1 \) always, so that the maturity structure is shortened; then \( d\phi_t = \delta_L (1 - \phi_t) \, dt > 0 \), i.e., the maturity structure \( \phi_t \) increases at the fraction of long-term debt multiplied by its maturing speed. Over time, the firm’s maturity structure \( \phi_t \) monotonically rises toward 100% of short-term debt. Similarly, if the firm keeps issuing long-term bonds so that \( f = 0 \), then \( d\phi_t = -\phi_t \delta_S \, dt < 0 \) and the maturity structure \( \phi_t \) monotonically falls toward 0% of short-term debt.

### 2.3 Rollover Losses and Default

#### 2.3.1 Bond market prices

Given the equilibrium default time \( T_b \) (if \( T_b = \infty \) then the firm never defaults), competitive bond investors price long-term and short-term bonds at \( D_S(y_t, \phi_t) \) and \( D_L(y_t, \phi_t) \) respectively.\(^8\) Since we set \( \rho_c = r \), and the recovery value \( B(\cdot) \) is below the face value 1, in general we have \( D_L \leq D_S \leq 1 \) (for the exact argument, see Section 3.2.1). This implies the firm is incurring certain rollover losses, a topic we turn to now.

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\(^7\)We assume that there is no debt buybacks, call provisions do not exist, and maturity of debt contracts cannot be changed once issued. We discuss the robustness of our result with respect to these assumptions in Section A.5.3.

\(^8\)Even if \( T_b \) is deterministic, since we model bond maturity as a Poisson shock, bond holders are still exposed to the risk of default.
2.3.2 Rollover losses and default boundary

In Leland [1994b, 1998], equity holders commit to roll over (refinance) the firm’s maturing bonds by re-issuing bonds of the same type. In our model, the firm can choose the fraction of short-term bonds \( f \) amongst newly issued bonds. Per unit of face value, by issuing an \( f_t \) fraction of short-term bonds, the equity’s net rollover cash-flows are

\[
\frac{f_t D_S(y_t, \phi_t) + (1 - f_t) D_L(y_t, \phi_t) - 1}{\text{proceeds of newly issued bonds} \quad \text{payment to maturing bonds}}.
\]

We call this term “rollover losses.”

Each instant there are \( m(\phi_t) \, dt \) units of face value to be rolled over, hence the instantaneous expected cash flows to equity holders are

\[
y_t \frac{c}{\text{operating CF}} - \frac{c_{\text{coupon}}}{\text{coupon}} + \frac{\zeta E^f}{\text{upside event}} + m(\phi_t) \left[ f_t D_S(y_t, \phi_t) + (1 - f_t) D_L(y_t, \phi_t) - 1 \right].
\]

(5)

Here, the third term “upside event” is the expected equity payoff of this event, \( E^f \), multiplied by its instantaneous probability, \( \zeta \), where we defined \( E^f \equiv X - D^f = X - 1 > 0 \).

When the above cash flows in (5) are negative, these losses are covered by issuing additional equity, which dilutes the value of existing shares.\(^9\) Equity holders are willing to buy more shares and bail out the maturing bond holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing these losses). When equity holders—protected by limited liability—declare default, equity value drops to zero, and bond holders receive the firm’s liquidation value \( B(y_{T_b}) \).

There are two distinct channels that expose equity holders to heavier losses, leading to default. The first, the cash-flow channel, has been studied extensively in the literature. When \( y_t \) deteriorates

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\(^9\)Equity holders are always facing rollover losses as long as \( \rho c = r \) and \( B(y_{T_b}) < 1 \), which imply that \( D_t < 1 \). When \( \rho c > r \), rollover gains occur for safe firms who are far from default. As emphasized in He and Xiong [2012b], since rollover risk kicks in only when the firm is close to default, it is without loss of generality to focus on rollover losses only.

\(^{10}\)This assumption highlights the so-called “endogenous” default in that equity holders default when the are unwilling rather than unable to absorb the loss. The underlying assumption is that either equity holders have deep pockets or the firm faces a frictionless equity market.
(say, \(y_t\) turns negative), equity holders are absorbing operating losses (the first term in (5)). Also, because a lower \(y_t\) leads to more imminent default (say, default occurs once \(y_t\) hits some lower boundary), bond prices \(D_S\) and \(D_L\) drop as well, leading to heavier rollover losses in the third term in (5) for any given \(m(\phi)\).

The second channel, which is novel, is through the endogenous maturity structure \(\phi_t\). Fixing the issuance policy \(f\), the greater \(\phi_t\), the higher the rollover frequency \(m(\phi_t)\). Later we show that bond valuations \(D_i\)'s are decreasing in \(\phi\) as well, leading to heavier rollover losses. Both effects imply that given a shorter maturity structure \(\phi\), equity holders face worse rollover losses in (5) and are thus more prone to default, all else equal. Importantly, equity holders pick the path of the future maturity structure \(\{\phi_s: s > t\}\) via (4) by choosing \(f_t\) endogenously each \(t\), subject to a dynamic incentive compatibility condition to be discussed shortly.

The above discussion suggests that there exists a default curve \(\Phi(y,\cdot)\), where the increasing function \(\Phi(\cdot)\) gives the threshold maturity structure given cash-flow \(y\). In equilibrium, the firm defaults whenever the state lies in

\[
\mathcal{B} = \{(\phi, y) \text{ such that } \phi \geq \Phi(y)\}.
\]

Consistent with this observation, throughout we make the following assumption on off-equilibrium beliefs regarding default. When the firm stays alive at time \(t\) even though creditors expected it to be in default, new bond investors expect the firm to default as long as the \((\phi_s, y_s) \in \mathcal{B}\) for \(s > t\). This implies that if in the next instant \((\phi_{t+dt}, y_{t+dt}) \in \mathcal{B}\), either because cash flow \(y_t\) is decreasing over time or the firm keeps issuing short-term debt so that \(\phi_{t+dt} > \phi_t\), then bond investors apply the lowest possible bond value given by \(D_L = D_S = B(y_{t+dt})\).

3 Baseline Model: Constant Cash-Flows

We first show a negative result for the constant cash-flow setting: There does not exist an equilibrium path in which equity holders keep shortening the firm's debt maturity structure and eventually and
predictably default in the face of larger and larger rollover losses.

3.1 Setting

Consider the simplest setting with constant cash-flows, i.e., \( y_t = y \). We denote by \( D_S (\phi_t; y) \), \( D_L (\phi_t; y) \), and \( E (\phi_t; y) \) the value of short-term bond, long-term bond, and equity, respectively. We explicitly write the cash-flow \( y \) into valuations to emphasize their dependence on the constant \( y \).

Given maturity structure \( \phi_t \) and issuance policy \( f_t \), the expected cash-flows of equity is

\[
y - c + \zeta E^r f + m (\phi_t) [f_t D_S (\phi_t; y) + (1 - f_t) D_L (\phi_t; y) - 1].
\]

The following lemma characterizes two polar cases.

**Lemma 1** Default occurs immediately if \( y - c + \zeta E^r f < 0 \), and equity never defaults if \( y - c + \zeta E^r f + \delta S [B (y) - 1] \geq 0 \).

Intuitively, the rollover term in (6) at best is bounded above by zero, but at worst is \( \delta S [B (y) - 1] \) under the shortest maturity structure (\( \phi = 1 \)) and the lowest debt price \( B (y) \). Hence if \( y - c + \zeta E^r f < 0 \) then the equity’s cash flows in (6) are always negative, leading to immediate default. On the other hand, if \( y - c + \zeta E^r f + \delta S [B (y) - 1] > 0 \), then even under the most pessimistic beliefs equity holders never make losses and thus never default.

3.2 Shortening Equilibrium

When \( 0 \leq y - c + \zeta E^r f < \delta S [1 - B (y)] \), there exist some nontrivial equilibria. We are interested in what we term “shortening” equilibria. Specifically, do there exist equilibria in which equity holders always set \( f = 1 \) (i.e., issuing short-term debt), so that \( \phi \) increases over time and the firm eventually defaults in the face of larger and larger rollover losses?
3.2.1 Debt valuations

Bond holders are taking equity holders’ policy \( f = 1 \) as given. We treat the maturity structure \( \phi \) as the state variable, which follows \( d\phi_t = (1 - \phi_t) \delta_L dt \) where we use (4) with \( f = 1 \). Hence, the bond valuation equation with \( i \in \{ S, L \} \) is\(^{11}\)

\[
\frac{\alpha D_i(\phi; y)}{\text{required return}} = \rho c + \delta_i \left[ 1 - D_i(\phi; y) \right] + \zeta \left[ 1 - D_i(\phi; y) \right] + (1 - \phi) \delta_L D_i'(\phi; y), \tag{7}
\]

and by equal seniority we have the boundary condition

\[
D_i(\Phi(y); y) = B(y). \tag{8}
\]

Later analysis involves the price wedge between short-term and long-term bonds, which is defined as

\[
\Delta(\phi; y) \equiv D_S(\phi; y) - D_L(\phi; y).
\]

Applying \( \delta_S \) and \( \delta_L \) to (7) and taking differences, we obtain

\[
(r + \delta_L + \zeta) \Delta(\phi) = (\delta_S - \delta_L) \left[ 1 - D_S(\phi) \right] + (1 - \phi) \delta_L \Delta'(\phi), \quad \text{and} \quad \Delta(\Phi(y); y) = 0 \tag{9}
\]

As \( 1 - D_S(\phi; y) > 0 \) if default is ever possible, we have

\[
\Delta(\phi) > 0 \quad \text{for} \quad \phi < \Phi(y), \tag{10}
\]

i.e., short-term bonds have a higher price than long-term bonds. Intuitively, short-term bonds are paid back sooner and hence less likely to suffer default losses. Hence, short-term bonds are preferred if equity holders try to minimize the firm’s current rollover losses.

\(^{11}\)Bond holders get paid \( D^\tau f = 1 \) in both the bond maturing event (occurring with intensity \( \delta \)) and upside option event (occurring with intensity \( \zeta \)).
3.2.2 Equity valuation and optimal issuance policy

Equity holders are not only minimizing the firm’s current rollover losses; they also take into account any long-run effect brought on by issuing more short-term bonds. By issuing more short-term bonds today, it shortens the firm’s future maturity structure going forward, aggravating future rollover losses and thus affecting possible default decisions.

Formally, equity holders are controlling the firm’s dynamic maturity structure as in (4). The standard Hamilton-Jacobi-Bellman (HJB) equation for equity, with the choice variable $f$, can be written as (we suppress the dependence on $y$):

$$rE(\phi) = y - c + \zeta \left[ E'f - E(\phi) \right] + \max_{f\in[0,1]} \left\{ \begin{array}{l} m(\phi) \left[ f D_S(\phi) + (1-f) D_L(\phi) - 1 \right] \\
\text{rollover losses} \\
+ \left[ -\phi \delta_S + m(\phi) f \right] E'(\phi) \\
\text{impact of maturity shortening} \end{array} \right\}. \quad (11)$$

By choosing the fraction $f$ of the newly issued short-term bonds, equity holders are balancing today’s “rollover losses” against the “impact of maturity shortening” on their future value.

Due to linearity, the optimal incentive compatible issuance policy $f$ is given by

$$f = \begin{cases} 
1 & \text{if } \Delta(\phi; y) + E'(\phi; y) > 0, \\
0 & \text{if } \Delta(\phi; y) + E'(\phi; y) < 0, \\
[0,1] & \text{if } \Delta(\phi; y) + E'(\phi; y) = 0. 
\end{cases} \quad (12)$$

We call $\Delta(\phi; y) + E'(\phi; y) > 0$ the incentive compatibility condition for equity issuing short-term debt, later $IC$ for short. Issuing more short-term bonds lowers the firm’s rollover losses today, as short-term bonds have higher prices ($\Delta(\phi; y) > 0$). However, issuing more short-term bonds today (higher $f$) makes the firm’s future maturity structure more short-term (higher $\phi$) and thus increase the rollover flow (higher $m(\phi)$). As we show next, this brings the firm closer to default and hurts equity holders’ continuation value, leading to $E'(\phi; y) < 0$. The optimal issuance policy in (12) illustrates this trade-off faced by equity holders.
3.2.3 Endogenous default

Equity holders also choose when to default optimally. At the default boundary $\Phi$ we have these two standard value-matching and smooth-pasting conditions:

$$ E(\Phi; y) = 0, \quad \text{and} \quad E'(\Phi; y) = 0. \quad (13) $$

The second smooth-pasting condition in (13) reflects the optimality of the default decision: The optimal default must occur when the change in equity value is zero.\(^\text{12}\) Applying conditions in (13) to (11), the equity’s expected flow payoff at $\phi = \Phi$ equals to zero:

$$ y - c + \zeta E^{rf} + \max_{f \in [0,1]} m(\Phi) [f D_S(\Phi; y) + (1 - f) D_L(\Phi; y) - 1] = 0. \quad (14) $$

In other words, in our model without volatility, equity holders default exactly at the point when expected cash-flows become zero. Absent the upside event, the default time is thus perfectly predictable.

Equation (14) pins down the default boundary $\Phi(y)$ as a function of the constant cash-flow $y$. At default, both bond values are given by $D_S(\Phi(y); y) = D_L(\Phi(y); y) = B(y)$, leading to a rollover term $m(\Phi) [B(y) - 1] < 0$ in (14) independent of the optimal issuance policy $f$. Plugging $m(\Phi)$ in (3), we have

$$ \Phi(y) = \frac{1}{\delta_S - \delta_L} \left[ \frac{y - c + \zeta E^{rf}}{1 - B(y)} - \delta_L \right]. \quad (15) $$

Because the recovery value $B(y)$ is increasing in $y$, one can verify that $\Phi(y)$ is increasing in $y$, as conjectured in Section 2.3.2.

3.3 Impossibility of Shortening Equilibria

We now give the formal definition for a shortening equilibrium.

\(^{12}\)Rigorously, we should have the change of equity value with respect to time to be zero. Because $\phi$ and time have a one-to-one mapping given by $d\phi = (1 - \phi_t) \delta_L dt$, the smooth-pasting condition in (13) follows.
Definition 1 The equilibrium concept is that of subgame perfect equilibrium. Given an initial maturity structure $\phi_{t=0}$, a shortening equilibrium is a path of $\{\phi_{t=0} \rightarrow \Phi (y)\}$ with $f_t = 1$, so that (11) holds with boundary conditions (13); (7) holds with boundary conditions (8); and, the equity holders’ incentive compatibility condition (12) holds with $f_t = 1$. Off-equilibrium beliefs are assumed to treat any deviations by the equity holders as mistakes, and continue to believe in the closest equilibrium in terms of default time to the one before deviation.

Off-equilibrium beliefs here treat deviations as mistakes. For example, if everyone expected the firm to shorten the maturity structure and then default at a certain time, a deviation today of lengthening does not alter the belief of investors that in the future the firm will keep shortening and default. Another example would be that if the firm was supposed to default today, but did not, then investors assume it will default in the next instant. Essentially, sub-game perfection requires that after a deviation investor beliefs for future play have to be an equilibrium. To select amongst the possible multiple equilibria present after deviation, we impose the additional refinement that they “pick” the equilibrium that is closest to the one they had before in terms of ultimate default time. This is very much akin to a trembling-hand refinement.

To rule out any shortening equilibria, it is sufficient to analyze the equilibrium behavior immediately before default, i.e., $\phi = \Phi - \epsilon$ for a sufficiently small $\epsilon > 0$, where we simply denote the default boundary by $\Phi$. In light of (12), we need to show that $\Delta (\Phi - \epsilon; y) + E' (\Phi - \epsilon; y) < 0$. Since at default we have $\Delta (\Phi; y) = 0$ in (9) and $E' (\Phi; y) = 0$ in (13), the IC condition $\Delta + E'$ is identically zero at $\Phi$. Going one order higher, it is never optimal to choose $f = 1$ right before default at $\Phi - \epsilon$ if

$$\Delta' (\Phi; y) + E'' (\Phi; y) > 0.$$ (16)

We first analyze the benefit of shortening $\Delta' (\Phi; y)$ in (16). From (9) we know that

$$\Delta' (\Phi; y) = -\frac{(\delta_S - \delta_L)}{(1 - \Phi) \delta_L} \left[1 - B (y)\right] < 0,$$ (17)
which says \( \Delta (\Phi - \epsilon; y) > 0 \). When the firm is a bit away from default, short-term bonds have the advantage of maturing before default, leading to a strictly higher price than long-term bonds. This is the benefit of issuing short-term bonds.

Equity holders have to balance this benefit with the cost of more imminent default; the latter is captured by the second term \( E''(\Phi; y) \) in (16). This term is always positive, establishing the optimality of equity holders’ endogenous default decision. The proof of Proposition 1 shows that

\[
E''(\Phi) = \frac{(\delta_S - \delta_L)(1 - B(y))}{(1 - \Phi) \delta_L} - \frac{[\Phi \delta_S + (1 - \Phi) \delta_L]}{(1 - \Phi) \delta_L} D'_S(\Phi; y). \tag{18}
\]

Combining (17) and (18), we have

\[
\Delta'(\Phi; y) + E''(\Phi; y) = -\frac{[\Phi \delta_S + (1 - \Phi) \delta_L]}{(1 - \Phi) \delta_L} D'_S(\Phi; y).
\]

Since \( \Phi \in [0, 1] \), the sign of IC condition \( \Delta'(\Phi; y) + E''(\Phi; y) \) is the opposite of the sign of \( D'_S(\Phi; y) \).

**Proposition 1**  Consider the constant cash-flows setting. Right before default, for \( f = 1 \), the equity holders’ incentive compatibility condition \( \Delta'(\Phi; y) + E''(\Phi; y) \leq 0 \) holds if and only if

\[
D'_S(\Phi; y) \geq 0. \tag{19}
\]

Now we show that when \( y_t \) is constant at \( y \), the sign of \( D'_S(\Phi; y) \) is fully determined by the (opposite) sign of loss-given-default for bond investors. Recall that we assume that \( B(y) < 1 \), i.e., default leads to value losses for bond holders. From (7) with \( \rho_c = r \), we derive that\(^\text{13}\)

\[
D'_S(\Phi; y) = -\frac{(r + \delta_S + \zeta)(1 - B(y))}{(1 - \Phi) \delta_L} < 0.
\]

\(^{13}\)For the general case with \( \rho_c \neq r \), for default being losses to bond values we require \( B(y) < D'' f = \frac{\rho_c + \delta_S + \zeta}{\rho_c + \delta_S + \zeta} \).
In words, the shorter the firm’s maturity structure, the closer the default, and hence the lower the bond value. The next corollary naturally follows from (16) and Proposition 1.

**Corollary 1** There do not exist shortening equilibria where equity holders keep issuing short-term bonds and then default at some finite future time in the constant cash-flow setting.

### 3.4 Discussions

Before we explain the intuition behind Corollary 1, we point out that this result is robust to several natural extensions. In Appendix A.5, we show that shortening equilibria cannot occur when equity holders with liquidity problems are forced to default; when the firm faces some exogenous Poisson default event; and/or when equity holders are endowed with a more relaxed but still bounded reissuance strategy space (instead of \( f \in [0, 1] \)).

#### 3.4.1 Intuition behind Corollary 1

When choosing the fraction of newly issued short-term bonds, equity holders are weighing the benefit of reducing today’s rollover losses against the cost of increasing future rollover losses. Corollary 1 shows that the future cost always dominates the gain from today. What is the intuition?

Suppose that \( r = c = 0 \), and we are at \( 2dt \) before default. We need \( 2dt \) in this thought experiment as we want to compare today’s reduced rollover losses against tomorrow’s heavier rollover losses. At the end of \( dt \), the firm rolls over the maturing bonds successfully. At the end of \( 2dt \), remaining holders receive the recovery value given default.\(^{14}\)

At today, the short-term (long-term) bond will get a full payment of 1 with a probability of \( \delta_S \cdot dt \) (\( \delta_L \cdot dt \)) over \([0, dt]\); otherwise both get the bankruptcy payout \( B(y) \). This value difference \( (\delta_S - \delta_L)[1 - B(y)] \cdot dt \) is reflected in today’s price wedge \( \Delta \) set by competitive bond investors. Hence, for equity holders who are refinancing \( m(\phi) \cdot dt \) units of maturing bonds, the relative benefit

\(^{14}\)For illustration purpose, we can think of the coupon payment and upper side event occurs right after the equity holders’ default decision.
of issuing short-term bonds instead of long-term bonds (by setting \( f = 1 \) instead of \( f = 0 \)) is

\[
m(\phi) \, dt \cdot (\delta_S - \delta_L) \, [1 - B(y)] \, dt > 0. \tag{20}
\]

Cost wise, as short-term bonds have a higher intensity \( \delta_S \) of coming due, equity holders know that the next instant (at the end of \( dt \)) they are facing heavier rollover losses. There are a total of \( (\phi \delta_S + (1 - \phi) \delta_L) \, dt \) units of bonds to be refinanced, each with a rollover loss of \( B(y) - 1 \) as at \( dt \) all bonds have the same price \( B(y) \). The impact of today’s issuance policy on tomorrow’s rollover loss is therefore:

\[
\frac{\partial}{\partial f} \left[ (\phi \delta_S + (1 - \phi) \delta_L) \, dt \cdot (B(y) - 1) \right] = \frac{\partial \phi}{\partial f} \cdot \frac{\partial}{\partial \phi} \left[ (\phi \delta_S + (1 - \phi) \delta_L) \, (B(y) - 1) \, dt \right] = m(\phi) \, dt \cdot (\delta_S - \delta_L) \, (B(y) - 1) \, dt, \tag{21}
\]

where \( \frac{\partial \phi}{\partial f} = m(\phi) \, dt \) from (4) captures how today’s issuance policy \( f \) affects tomorrow’s maturity structure \( \phi \). As a result, right before default so that only today and tomorrow count, the benefit from saving today’s rollover losses in (20) exactly offsets the cost of having higher rollover losses (21) in the next instant! Indeed, this is reflected in (18): the first term in \( E''(\Phi) \)—which captures the future losses caused by maturity shortening—equals the gain from reducing today’s rollover loss (17).

What is behind the second term in (18)? In the above thought experiment we have kept bond prices unchanged, i.e. \( D_S = D_L = B(y) \). Because \( \frac{\partial \phi}{\partial f} = m(\phi) \, dt > 0 \), issuing short-term bonds pushes the maturity structure \( \phi_t \) toward the default threshold \( \Phi \). This in turn pushes the firm closer to default, bringing about a first-order negative impact on bond prices and hence future rollover losses. Equity holders internalize this negative effect, which is captured by the second term in (18).\(^{15}\) Consequently, Proposition 1 holds due to this additional negative effect on bond prices when shortening the firm’s maturity structure.

\(^{15}\)The reason that only the short-term bond price \( D_S \) shows up is that equity is only issuing short-term bonds in the hypothetical shortening equilibrium. When we focus on lengthening equilibrium, only the long-term bond price \( D_L \) shows up; see Corollary 3.
3.4.2 Comparison to Brunnermeier and Oehmke (2013)

Our analysis highlights an economic mechanism that is different from Brunnermeier and Oehmke [2013]. In that paper, the firm with a long-term asset is borrowing from a continuum of identical creditors. Only standard debt contracts are considered with promised face value and maturity, and covenants are not allowed. News about the long-term asset arrives at interim periods, so that a debt contract maturing on that date will be repriced accordingly, as in Diamond [1991]. For certain types of interim uncertainty resolutions (e.g., whether it is about profitability or recovery value), Brunnermeier and Oehmke [2013] show that, given other creditors’ debt contracts, equity holders find it optimal to deviate by offering any individual creditor a debt contract that matures one period earlier, so that it gets repriced sooner. In equilibrium, equity holders offer the same deal to every creditor, and the firm’s maturity will be “rat raced” to zero.

The repricing mechanism constitutes the key difference between Brunnermeier and Oehmke [2013] and our model. In their model, after negative interim news, a relative short-term bond gets repriced by adjusting up the promised face value to renegotiating bond holders. Because all bonds have the same seniority, including the repriced ones, repricing causes dilution of those relative long-term bonds without repricing opportunities. Put differently, the rollover losses are absorbed by the promised higher face values, which dilutes existing long-term bond holders.

As emphasized in Section 2.2.1, in our model the firm commits to maintain a constant total outstanding face value when refinancing its maturing bonds. This amounts to a bond covenant about the firm’s “book leverage,” so that equity holders cannot simply issue more bonds to cover the firm’s rollover losses as in Brunnermeier and Oehmke [2013]. Instead, equity holders are absorbing these losses through their own deep pockets (or through equity issuance), and existing long-term bonds remain undiluted. Interestingly, once we shut down the interim dilution channel that drives the result in Brunnermeier and Oehmke [2013], we identify a new economic force not present in their paper.

We make the constant face-value assumption for two reasons. First, as it is a standard assum-
tion in the dynamic structural corporate finance models starting from Leland and Toft [1996], our analysis represents the minimum departure from the literature. More importantly, the full commitment on the firm’s book leverage policies isolates the standard dilution issues (via promised face values) from the firm’s endogenous maturity decisions, which is the focus of our paper. Besides, in practice, most of bond covenants have some restrictions regarding the firm’s future leverage policies, but rarely on the firm’s future maturity structures. This empirical observation lends support to our premise of a full commitment on the firm’s book leverage policy but no commitment on its debt maturity structure policy. For more discussions on the robustness of our results to deleveraging processes, see Section 4.4.

4 Maturity Shortening with Time-Decreasing Cash-Flows

In contrast to Corollary 1, shortening equilibria can exist when the firm’s cash-flows are deteriorating slowly over time. We show that the general intuition discussed in Section 3.4.1 yields a similar necessary condition for shortening equilibria as in (19); time-varying cash-flows, however, have profound implications which may overturn the negative result in Corollary 1.

In this section, illustration is more straightforward in terms of the dynamics of the firm’s time-to-default $\tau \equiv T_b - t$; recall $T_b$ is the firm’s endogenous default time. Naturally, $d\tau = -dt$, and $y_\tau$ and $\phi_\tau$ are the cash-flow and the maturity structure with $\tau$ periods left until default. We call the cash-flow when the firm defaults, i.e., $y_b = y_{\tau=0}$, defaulting or ultimate cash-flow; it plays an important role in later analysis.

4.1 Deterministic and Cornered Equilibria

In this section we focus on equilibria where equity holders are taking “deterministic” and “cornered” issuance strategies. Section 6 considers deterministic equilibria with “deterministic” interior issuance policies.
**Definition 2** Equilibria are considered “deterministic” if the firm’s issuance policy $f_\tau$ is a deterministic function of time-to-default. Equilibria are “deterministic” and “cornered” if the firm’s deterministic issuance policy takes a corner solution $f_\tau \in \{0, 1\}$.

As an example, suppose that we are in the constant cash-flows case studied in Section 3. Proposition 1 and Lemma 1 together imply that there are two possible deterministic and cornered equilibria: either the firm defaults immediately, or the firm keeps issuing long-term bonds and never defaults. In contrast, we will show the equilibrium structure is much richer in the time-varying cash-flow case.

Because cash-flows depend on time-to-default deterministically and there are no other payoff-relevant shocks in the model (other than the upside event shock), focusing on “deterministic” issuance policies essentially rules out sun-spot type equilibria. Cornered strategies are in general optimal for risk-neutral equity holders who are solving a linear problem, and note that the class of “deterministic” and “cornered” equilibria have not ruled out time-varying issuance polices.\footnote{For instance, we could have some issuance policy that jumps from $f_\tau = 0$ to $f_{\tau+} = 1$ at certain pre-specified time-to-default $\tau$. However, Lemma 4 in the Appendix shows that this never holds on equilibrium paths.} However, cornered strategies indeed impose restrictions on the set of equilibria. Section 6 considers all possible equilibria, including $f_\tau \in (0, 1)$ for some $\tau$.\footnote{For instance, an interior issuance policy say $f \in (0, 1)$ which affects bond valuations can make equity holders indifferent between shortening ($f = 1$) or lengthening ($f = 0$), which in turn implies the optimality of an interior $f$.}

### 4.2 Setting and Valuations

Let us introduce a time-dependent cash-flow $y_\tau$ with drift

$$dy_\tau = \mu_y (y_\tau) d\tau,$$

with $\mu_y (y) > 0$. Here, $y_\tau$ is increasing with time-to-maturity or $y_t$ is decreasing over time.
4.2.1 Incentive compatibility and endogenous default

We now have both current cash-flow $y$ and debt maturity $\phi$ as state variables. Bond values solve the following Partial Differential Equation (PDE) where $i \in \{S, L\}$:

$$ r D_i(\phi, y) = \rho c + \delta_i [1 - D_i(\phi, y)] + \zeta [1 - D_i(\phi, y)] $$

$$ + [-\phi \delta S + m(\phi) f] \frac{\partial}{\partial \phi} D_i(\phi, y) + \mu_y(y) \frac{\partial}{\partial y} D_i(\phi, y), $$

(23)

and equity value solves the following PDE

$$ r E(\phi, y) = \text{CF net coupon} + \zeta \left[ E^f - E(\phi, y) \right] + \mu_y(y) \frac{\partial}{\partial y} E(\phi, y) $$

$$ + \max_{f \in [0, 1]} \left\{ \begin{array}{l}
  m(\phi) [f D_S(\phi, y) + (1 - f) D_L(\phi, y) - 1] \\
  \text{rollover losses}
  + [-\phi \delta S + m(\phi) f] \frac{\partial}{\partial \phi} E(\phi, y)
  \text{maturity shortening}
\end{array} \right\}. $$

(24)

The same argument as Section 3.2.2 leads to the same IC condition (12) for equity holders, with a necessary modification to a partial derivative with respect to $\phi$, i.e., $E_\phi(\phi, y) \equiv \frac{\partial}{\partial \phi} E(\phi, y)$, due to the two-dimensional state space:

$$ f = \begin{cases} 
1 & \text{if } E_\phi(\phi, y) + \Delta(\phi, y) > 0 \\
[0, 1] & \text{if } E_\phi(\phi, y) + \Delta(\phi, y) = 0 \\
0 & \text{if } E_\phi(\phi, y) + \Delta(\phi, y) < 0
\end{cases} $$

(25)

Define $IC(\phi, y) \equiv \Delta(\phi, y) + E_\phi(\phi, y)$, so $IC(\phi, y) > 0$ implies $f = 1$.

Similar to the discussion in Section 3.2.3, at the optimal default boundary equity holders’ instantaneous expected flow payoff equals zero. This implies the same default boundary given in (15),
which is reproduced here (recall \( y_b = y_{\tau=0} \) denotes the defaulting or \textit{ultimate} cash-flow)

\[
\Phi (y_b) = \frac{1}{\delta_S - \delta_L} \left[ \frac{y_b - c + \zeta E^{rf}}{1 - B(y_b)} - \delta_L \right], \text{ with } \Phi' (y_b) > 0.
\]

This gives the endogenous default boundary in the \((y, \phi)\) space. Lemma 2 gives the smooth pasting property of \( E(\cdot, \cdot) \) at the default boundary on the state space of \((\phi, y)\).

**Lemma 2** \textit{At the endogenous default boundary we have a value matching condition} \( E(\Phi (y_b), y_b) = 0 \), \textit{and two smooth-pasting conditions on each dimension} \( E_\phi (\Phi (y_b), y_b) = 0 \) \textit{and} \( E_y (\Phi (y_b), y_b) = 0 \).

4.2.2 **Time-to-default and valuations**

Given the ultimate bankruptcy state \((\phi_{\tau=0} = \Phi (y_b), y_{\tau=0} = y_b)\), the equilibrium path \((\phi_{\tau}, y_{\tau})\) is essentially a one-dimensional object indexed by time-to-default \( \tau \), working our way back from the boundary. Hence given any equilibrium path we can rewrite the bond and equity values by \( D_i (\tau, y_b) \) and \( E (\tau, y_b) \) respectively as a function of \( \tau \) only, while treating the defaulting cash-flow state \( y_b \) as a parameter. Thus, (23) and (24) become:

\[
r D_i (\tau, y_b) = \rho c + \delta_i [1 - D_i (\tau, y_b)] + \zeta [1 - D_i (\tau, y_b)] - \frac{\partial}{\partial \tau} D_i (\tau, y_b), \text{ for } i \in \{S, L\}. \quad (26)
\]

\[
r E (\tau, y_b) = y (\tau, y_b) - c + \zeta \left[ E^{rf} - E (\tau, y_b) \right]
\]

\[
+ m (\phi (\tau, \phi_b)) [f_S D_S (\tau, y_b) + (1 - f_L) D_L (\tau, y_b) - 1] - \frac{\partial}{\partial \tau} E (\tau, y_b). \quad (27)
\]

where \( y (\tau, y_b) \) is the cash-flow \( y_{\tau} \) given ultimate (defaulting) cash-flow \( y_b \), and \( \phi (\tau, \phi_b) \) is the maturity structure \( \phi_{\tau} \) given ultimate (defaulting) maturity structure \( \phi_b \). The closed-form solutions for bond values are (recall \( \rho c = r \))

\[
D_i (\tau, y_b) = 1 - e^{-(r+\delta_i+\zeta)\tau} [1 - B(y_b)], \text{ for } i \in \{S, L\}. \quad (28)
\]

For the solution to equity \( E (\tau, y_b) \), see Appendix A.1.3.
Effectively, we are working with the state space of \((\tau, y_b)\) instead of the state space of \((\phi, y)\). Given any (deterministic) equilibrium issuance policy \(\{f_\tau\}\), there is a deterministic mapping between these two state spaces.\(^{18}\) Consequently, via changing coordinates, one can translate \(D_i(\tau, y_b)\) and \(E(\tau, y_b)\) back to the form of \(D_i(\phi, y)\) and \(E(\phi, y)\) by solving for \(\tau\) as a function of \((\phi, y)\).

4.3 Can Shortening Equilibria Exist?

As one of our most important results, we revisit the possibility of shortening equilibria in this section. In contrast to Section 3.3, we show a positive result: shortening equilibria can occur in a setting with deteriorating cash flows.

4.3.1 Incentive compatibility condition right before default

As before, we postulate a shortening equilibrium, and evaluate the \(IC\) condition (25) right before default. Again, we have a zero \(IC\) condition at default: \(E_\phi(\Phi(y_b), y_b) = 0\) from Lemma 2, and equal seniority implies a zero short-long price wedge \(\Delta(\Phi(y_b), y_b) = 0\). Hence we analyze the sign of \(E_\phi(\phi, y) + \Delta(\phi, y)\) slightly away from \(\tau = 0\) along the path of \((\phi_\tau, y_\tau)\), i.e., the path which originates at the default state \((\Phi(y_b), y_b)\). Differentiating the \(IC\) condition with respect to \(\tau\), we need to evaluate the sign of

\[
IC_\tau(\tau, y_b)|_{\tau=0} = \frac{\partial}{\partial \tau} \left[ E_\phi(\tau, y_b) + \Delta(\tau, y_b) \right]_{\tau=0} .
\]

If (29) is strictly positive, then \(IC(\phi, y) = E_\phi(\phi, y) + \Delta(\phi, y) > 0\) for \(\tau > 0\) right before default, implying that issuing short-term bonds right before default is incentive compatible. The next proposition shows that a necessary condition for a shortening equilibrium to exist is that shortening debt maturity has a strictly positive partial impact on the value of short-term debt around the vicinity of default.

\(^{18}\)For the technical details on this change of variables, see Appendix A.1.1.
Proposition 2 The unique cornered shortening equilibrium, \( f = 1 \), occurs in the vicinity of \( \tau = 0 \) if and only if

\[
\frac{\partial}{\partial \phi} D_S(\Phi(y_b), y_b) \bigg|_{f=1} \geq 0. \tag{30}
\]

Recall that Corollary 1 states that in the constant cash-flow case, we have \( D'_S(\Phi; y) < 0 \) given a positive loss-given-default \( (B(y) < 1) \), which rules out the possibility of any shortening equilibria. However, for deteriorating cash-flows the shortening equilibrium exists even with a positive loss-given-default. The next section explains the difference.

4.3.2 How condition (30) differs from condition (19)?

The condition (30) in Proposition 2 and the condition (19) in Corollary 1 are similar; but they differ in one crucial aspect. Although both involve taking a derivative with respect to \( \phi \), \( D'_S(\Phi; y) > 0 \) in (19) has a “total” derivative while \( \frac{\partial}{\partial \phi} D_S(\Phi(y_b), y_b) > 0 \) in (30) has a “partial” derivative. This difference is highlighted when cash-flows are deteriorating over time. In short, when \( y_\tau \) is time-varying, the cash-flows at the time of default, \( y_{\tau=0} = y_b \), and hence the bond recovery value \( B(y_b) \), become endogenous. The partial derivative in (30) exactly reflects this important effect.

We investigate the marginal impact of maturity shortening on bond values around the default boundary. Taking the partial derivative of \( D_S(\phi, y) \) at \( (\Phi(y_b), y_b) \) with respect to \( \phi \), i.e., \( \frac{\partial D_S(\Phi(y_b), y_b)}{\partial \phi} \), and translating everything into the \((\tau, y_b)\) space, we have:

\[
\frac{\partial D_S(\tau(\phi, y_b), y_b(\phi, y))}{\partial \phi} \bigg|_{\tau=0, f=1} = \frac{\partial D_S(\tau, y_b)}{\partial \tau} \bigg|_{\tau=0, f=1} \frac{\partial \tau}{\partial \phi} \bigg|_{\tau=0, f=1} + \frac{\partial D_S(\tau, y_b)}{\partial y_b} \bigg|_{\tau=0, f=1} \frac{\partial y_b}{\partial \phi} \bigg|_{\tau=0, f=1}. \tag{31}
\]

The first term captures how maturity shortening affects the firm’s time-to-default, which is present in the constant cash-flow case. The novel second term captures the resulting change of default cash-flow level \( y_b \), which directly affects the recovery value \( B(y_b) \) received by bond investors. In
Appendix A.1.1 we show the following intuitive results:

\[
\frac{\partial \tau}{\partial \phi} \bigg|_{\tau=0, f=1} < 0, \quad \text{and} \quad \frac{\partial y_b}{\partial \phi} \bigg|_{\tau=0, f=1} > 0. \tag{32}
\]

The first says that fixing the current cash-flow state, shortening maturity worsens rollover losses and hence reduces the time-to-default \(\tau\). For the second, because cash-flows are decreasing over time, the reduction of time-to-maturity increases the cash-flows at default.

We analyze the first term in (31). Using (28), we derive the impact of time-to-default on the bond value as (recall \(B(y_b) < 1\) and \(\rho c = \tau\))

\[
\frac{\partial D_S(\tau, y_b)}{\partial \tau} \bigg|_{\tau=0} = (r + \delta_S + \zeta) [1 - B(y_b)] > 0.
\]

Together with \(\frac{\partial \tau}{\partial \phi} < 0\) in (32), we see that the first term in (31) is negative. Intuitively, shortening the maturity structure edges the firm closer to default, hurting bond values. The same negative force is present in the constant cash-flow case, which goes against condition (30).

In contrast to the constant cash-flow case, there is a second term present when cash-flows are time varying. We derive \(\frac{\partial D_S(\tau, y_b)}{\partial y_b}\) using (28):

\[
\frac{\partial D_S(\tau, y_b)}{\partial y_b} \bigg|_{\tau=0} = B'(y_b) > 0.
\]

as the firm’s liquidation value increases with its profitability. Because \(\frac{\partial y}{\partial \phi} > 0\) in (32), the second term in (31) is positive. Intuitively, by bringing the firm closer to default, shortening the maturity structure allows the bond holders to take over the firm at an earlier time with a better fundamental \(y_b\), raising bond values. When the positive second term dominates the negative first term, condition (30) holds and hence shortening equilibria may exist. Section 5 gives a numerical example in which the firm follows the path of a shortening equilibrium.

For better illustration, Figure 1 schematically depicts potential paths of a shortening equilibrium for both the case of constant cash-flows and that of time-decreasing cash-flows. In the left panel with
constant cash-flows, when the firm issues more short-term bonds, the firm moves closer to default; however, the equilibrium path, as well as the bond recovery value, are unchanged. In contrast, in the right panel with time-decreasing cash-flows, issuing more short-term bonds shortens the firm’s survival time, but the firm lands on a path sitting above the equilibrium one. As the second term in (31) captures, this deviating path features greater cash-flows at default and hence a higher bond recovery value.

4.4 Summary and Discussions

Our analysis suggests the following key incentive compatibility condition for shortening equilibria: right before default, the short-term debt value is (locally) increasing when the firm’s debt maturity structure gets shorter. In our model with refinancing/rollover losses, the firm defaults earlier when its debt maturity structure is shorter. Earlier default could be a good news to debt holders if the recovery value drops over time, especially when the firm’s fundamentals are deteriorating. Section 5 illustrates a numerical example which has interesting welfare implications.

We adopt a fairly stylized setting in order to deliver the main economic mechanism in a transparent way. However, this simplification may come at some cost, as there are at least two important
ingredients that may be missing in our framework: cash-flow volatility, and potentially endogenous (de)leveraging policies. We discuss each in turn.

**Stochastic cash flows** We work in a setting without cash-flow volatility. Mathematically, adding volatility on $y$ leads to some second-order derivative terms in (23) and (24), and we are unable to recover the clean expression (30) for equity holders’ IC condition right before default. We cannot think of any obvious link between fundamental volatility and endogenous debt maturity structure.\(^{19}\) Nevertheless, there is one interesting observation hinting that volatility might help the existence of shortening equilibria. Note that (30) requires debt values to go up if the firm defaults earlier (when equity holders shorten the firm’s debt maturity structure); or debt value goes down if the firm survives longer. With positive cash-flow volatility, if the firm survives longer, the additional volatility is likely to cause the debt value to go down even further, because the debt value is concave in fundamental as their upside is capped. We await future research to explore this possibility.

**Potential deleveraging** To isolate debt maturity from leverage decisions, we follow the Leland tradition in assuming that the firm commits to a constant aggregate face value (normalized to 1). Dynamic leverage decisions without commitment are a challenging research question itself,\(^{20}\) and no doubt in practice firms have certain flexibilities in simultaneously adjusting their leverage and debt maturity structure.

There is one seemingly sensible argument suggesting that a firm might use more short-term debt if it ever wants to delever, i.e., cutting its debt face value (denoted by $F_t$) over time. Interestingly, although we cannot offer a thorough analysis, *in our benchmark setting with constant cash flows*, we

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\(^{19}\)Except that a higher volatility can be interpreted as more uncertainty resolution, and hence all else equal debt maturity becomes effective longer (as debt becomes riskier).

\(^{20}\)The literature usually take the tractable framework of Fischer et al. [1989], Goldstein et al. [2001] so that the firm needs to buy back all outstanding debt if it decides to adjust aggregate debt face value. Apparently, this assumption requires a strong commitment ability on the side of equity holders. Recently, Dangl and Zechner [2006] study the setting where a firm can freely adjust its aggregate debt fact value downwards by issuing less bonds than the amount of bonds that are maturing. DeMarzo and He [2014] study the setting without any commitment on outstanding debt face value so that equity holders may either repurchase or issue more at any point of time; it is shown that equity holders always would like to issue more. In sharp contrast to our paper in which firms who commit to a constant aggregate face value but can freely adjust debt maturity structure over time, Dangl and Zechner [2006] and DeMarzo and He [2014] instead assume that the firm can change its book leverage, but is able to commit to certain debt maturity structure (parameterized by some exogenous rollover frequency).
are able to rule out the possibility of a shortening equilibrium while the firm is deleveraging (either voluntarily or involuntarily). Appendix A.5.4 gives the argument in details, but the intuition turns out to be quite powerful. With a changing debt burden $F_t$, as a standard technique in this literature (Goldstein et al. [2001], Fischer et al. [1989], Dangl and Zechner [2006], DeMarzo and He [2014]), the *effective* firm fundamental becomes its cash flows scaled by the debt face value, i.e., $\hat{y}_t = y_t / F_t$. Also, the debt price in (30) is generally increasing in $\hat{y}_t$. A time-decreasing $F_t$ translates to a time-increasing $\hat{y}_t$, which is opposite the intuitive requirement that shortening equilibria require deteriorating cash flows as established in Section 4.3. Taken together, our analysis reveals a strong force pushing equity holders to issue long-term debt when firms are cutting their debt burden over time. Conversely, firms would like to shorten their debt maturity structure if they are leveraging up toward bankruptcy (in fact, this observation is consistent with Brunnermeier and Oehmke [2013]).

4.5 Lengthening Equilibrium and Multiple Equilibria

We now study the equilibria in which the firm’s debt maturity structure is lengthening. Because of deteriorating cash-flows, equity will default eventually even if the firm keeps lengthening its debt maturity, i.e., $f_\tau = 0$. Almost exactly the same analysis as in Proposition 2 applies in this case. In words, we have a lengthening equilibrium if, at the default boundary, the value of long-term bond gets hurt by maturity shortening.

**Proposition 3** The unique cornered lengthening equilibrium, $f = 0$, occurs in the vicinity of $\tau = 0$ if and only if

$$\frac{\partial}{\partial \phi} D_L (\Phi (y_b), y_b) \leq 0.$$  \(33\)

Hence, either a unique shortening equilibrium or a unique lengthening equilibrium might exist given the right *ultimate* bankruptcy state. Further, Lemma 4 in the Appendix shows that, within the class of deterministic equilibria, there cannot be any jumps in equilibrium issuance strategies—either from shortening to lengthening or vice versa. This property helps greatly reduce the dimensionality of the multiplicity of equilibria. An immediate implication is that given any initial state $(\phi, y)$
(different from the *ultimate* bankruptcy state), within the class of deterministic cornered equilibria, there are at most two unique cornered paths leading to default, either always shortening with $f = 1$ or always lengthening with $f = 0$.

This multiplicity of either shortening equilibrium or lengthening equilibrium emerges without too much surprise, echoing the intuition of self-enforcing default in the sovereign debt literature (e.g., Cole and Kehoe [2000]). If bond investors expect equity holders to keep shortening the firm’s maturity structure in the future, then bond investors price this expectation in the bond’s market valuation, which can self-enforce the optimality of issuing short-term bonds only. Similarly, the belief of issuing long-term bonds always can be self-enforcing as well.

The next proposition proves the optimality of a cornered issuance strategies along the whole path, *if indeed such a strategy is optimal at the time of default*. In other words, working backwards from the default boundary, if $f_0 \in \{0, 1\}$ is incentive compatible, then equity holders find it optimal to set $f_s = f_0$ for the whole path traced out by $s \in [0, \tau]$. Combined with Lemma 4 in the Appendix, we also establish that there exist at most two deterministic cornered equilibria.

**Proposition 4** Given the initial starting value $(\phi, y)$, there exist (at most) two deterministic cornered equilibria: one with shortening always $f_s = 1$ for $s \in [0, \tau^S]$ such that $y(\tau^S, y^S_s) = y$ and $\phi(\tau^S, y^S_s) = \phi$, and the other with lengthening always $f_s = 0$ for $s \in [0, \tau^L]$ with $y(\tau^L, y^L_s) = y$ and $\phi(\tau^L, y^L_s) = \phi$. Moreover, for the continuous IC condition of either $f_s = 1$ or $f_s = 0$ along the whole path $s \in [0, \tau^i]$, it is sufficient to check the IC condition on the default boundary given by either (30) or (33), respectively.

The dynamics embedded in our model allow us to say more. The existence of multiple equilibria is not guaranteed, and for some initial state, either the shortening equilibrium or the lengthening one becomes the unique equilibrium. Intuitively, if the firm starts off extremely close to the default boundary satisfying (30) in Proposition 2, then the only equilibrium path is indeed the shortening equilibrium, as a benign expectation of lengthening maturity in the future happens “too late” to save the firm. This intuition can also be expressed in a geometric way, because the respective regions on
the boundary for lengthening and shortening equilibria are non-overlapping. Hence, for points close to the boundary, even if we change the issuance strategy arbitrarily, we cannot change the path fast enough as \( \left| \frac{d\phi}{dt} \right| < \infty \) to avoid hitting the specific region, due to the bounded issuance strategy space (here, \( f \in [0, 1] \)).\(^{21}\) The following proposition summarizes this observation:

**Proposition 5** *There exists a no-return region with positive measure, in which starting from there either shortening equilibrium or lengthening equilibrium is the unique equilibrium.*

5 An Example with Constant Negative Drift

We now consider the case in which the cash-flow drift is a negative constant, i.e., \( dy_t = -\mu dt \) where \( \mu > 0 \) is a positive constant.

5.1 Liquidation Value \( B(y) \)

We first derive the firm’s liquidation value \( B(y) \). Motivated by bankruptcy cost, we assume that debt holders are less efficient in running the liquidated firm, relative to equity holders. Specifically, we assume that, post-default, the upside payoff \( X \) becomes \( \alpha_X X \) with \( \alpha_X \in (0, 1) \); and, given cash-flow \( y_\tau \), the cash-flow post-default becomes \( \alpha_y y_\tau \). Since in our numerical examples the defaulting cash-flows \( y_b < 0 \), to capture the inefficiency we set \( \alpha_y > 1 \). This specification is similar to Mella-Barral and Perraudin [1997].

For simplicity the liquidated firm is assumed to be unlevered. Also, debt holders will optimally terminate the firm when the expected flow payoff \( \alpha_y y_t + \zeta \alpha_X X \) hits zero from above, which implies \( B(y) = 0 \) at \( y = -\frac{\alpha_X}{\alpha_y} \zeta X \). Given this boundary condition, the liquidation value \( B(y) \) which satisfies \( rB(y) = \alpha_y y + \zeta [\alpha_X X - B(y)] - \mu y B'(y) \) can be solved as:

\[
B(y) = \begin{cases} 
\frac{\zeta \alpha_X X + \alpha_y y}{r+\zeta} + \left( \frac{\exp \left[ -\frac{(r+\zeta)}{\alpha_y} (\zeta \alpha_X X + \alpha_y y) \right] - 1}{(r+\zeta)^2} \right) \mu y & \text{for } y > -\frac{\alpha_X}{\alpha_y} \zeta X, \\
0 & \text{otherwise}
\end{cases}
\]

\(^{21}\)If the issuance strategy space is unbounded, then the firm can change its maturity structure instantaneously so that \( \left| \frac{d\phi}{dt} \right| = \infty \), and hence this argument fails.
By setting $\alpha_X = \alpha_y = 1$ we recover the unlevered asset value $A(y)$ defined in (1). The difference $A(y) - B(y) > 0$ is due to the inefficient management of debt holders and thus can be interpreted as a bankruptcy cost.

5.2 Shortening and Lengthening Equilibria

Figure 2 graphs the two unique cornered equilibrium paths starting from the same initial state $(\phi, y) = (0,.99)$, one a shortening equilibrium and the other a lengthening equilibrium, together with the default boundary $\Phi(y)$. In the shortening equilibrium, the firm keeps issuing short-term bonds and defaults at $(\phi_b^S = \Phi(y_b^S), y_b^S)$ if the upside event fails to realize along the path. Since the defaulting cash-flow $y_b^S$ is negative, $\alpha_y > 1$ says that the firm is experiencing even worse (negative) cash-flows under the debt holders’ management. From (31) and the discussion afterward, a relatively high $\alpha_y$—which implies a greater $B'(y_b)$—helps satisfy the IC condition in the shortening equilibrium.

As shown in Figure 2, there is another lengthening equilibrium given the same initial state, in which equity holders find it optimal to keep issuing long-term bonds and default at $(\phi_b^L = \Phi(y_b^L), y_b^L)$. The times of default, $T_b$, differ greatly across these two equilibria: $T_b^S = 0.43$ for the shortening equilibrium while $T_b^L = 1.55$ for the lengthening equilibrium. In the next section we analyze the welfare of these two equilibria in detail.

Suppose that we are in the shortening equilibrium, i.e., bond investors believe that equity holders will keep shortening the firm’s maturity structure. As we mentioned in Section 4.5, if the belief of bond investors switches to “equity holders will keep issuing long-term bonds” in an unanticipated way, then we can switch to a lengthening equilibrium, provided that we are sufficiently far away from default. Once we are too close to the default boundary, however, there cannot be such a switch of belief any more, because the lengthening path would hit $\Phi(y_b)$ in a shortening region. In other words, lengthening beliefs are inconsistent by backward induction. This is what Proposition 5 says: in the no-return “black hole” region, the firm is absorbed into the shortening equilibrium without any hope of returning.
Figure 2: Example with $\rho = 1$, $c = r = 10\%$, $D^{rf} = 1$, $E^{rf} = 12$, $\mu = 2$, $\zeta = .35$, $\delta_S = 10$, $\delta_L = 1$, $\alpha_y = 3$, $\alpha_X = .95$. The initial state is $(y, \phi) = (0, .99)$. **Left panel:** Default boundary (solid line), shortening equilibrium path (dashed line), lengthening equilibrium path (dot-dashed line), lowest shortening equilibrium path (red dotted line). **Right panel:** Total firm value as a function of default time $T$. Here, the default time is $T_S^b = 0.43$ ($T_L^b = 1.55$) for the shortening (lengthening) equilibrium, and the first-best $T_a = T_{FB} = 2.28$ (without tax benefit so $\rho = 1$).

The left panel of Figure 2 also shows the region in which shortening equilibria exist in our model, which is the shaded area towards the north-east corner of the $(\phi, y)$ space. This implies that shortening equilibria are more likely for firms that already have a short debt maturity structure in place, a point we return to when we discuss our model’s empirical predictions in Section 7.

5.3 Welfare Analysis

In the setting with time-deteriorating cash-flows, there is a natural optimal stopping time even for unlevered firms. The welfare analysis becomes interesting when we layer this optimal stopping problem on top of a standard equity-debt agency frictions, in which equity is choosing the optimal debt maturity structure to maximize equity value only. We base our analysis on the example in Section 5, but we will comment on the generality of our results.

5.3.1 Time of default and firm value

Take any arbitrary, not necessarily equilibrium, time of default denoted by $T$; we investigate the levered firm value as a function of $T$ in general. Each instant, in expectation the firm generates
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Initial \((y, \phi) = (0,.99)\) & \(T_b\) & \(V (T_b)\) & \(E (\phi, y)\) & \(D_S (\phi, y)\) & \(D_L (\phi, y)\) \\
\hline
Lengthening equilibrium & \(T^L_b = 1.55\) & 3.55 & 2.55 & 0.99 & 0.89 \\
\hline
Shortening equilibrium & \(T^S_b = 0.43\) & 2.17 & 1.17 & 0.99 & 0.82 \\
\hline
\end{tabular}

Table 1: Firm, equity, long-term bond, and short-term bond values for \(\rho = 1, c = r = 10\%,  D^rJ = 1,\) \(E^rJ = 12, \mu = 2, \zeta = .35, \delta_S = 10, \delta_L = 1, \alpha_y = 3,\) \(\alpha_X = .95,\) and initial point \((y, \phi) = (0,.99)\).

a surplus of \([y_t + \zeta X + (\rho - 1) c] dt\). In default, the firm recovers \(B (y_T)\). The cash-flows are discounted at a rate \(r + \zeta > r\) due to the upside event. Hence, the levered firm value, given the default time \(T\) (we omit the dependence on the initial cash-flow \(y\)), is

\[
V (T) \equiv \int_0^T e^{-(r+\zeta) t} [y_t + \zeta X + (\rho - 1) c] dt + e^{-(r+\zeta) T} B (y_T). \tag{35}
\]

There are two differences when we compare the levered firm value (35) to the asset’s unlevered value \(A (y)\) in (1). First, the levered firm receives a tax subsidy \((\rho - 1) c\). Second, the levered firm defaults to induce bankruptcy costs, i.e., \(B (y) < A (y)\). Note \(B (\cdot)\) has taken into account the potential optimal abandonment time after default in the setting of time-decreasing cash-flows.

We focus on the inefficient default timing caused by the equilibrium debt maturity dynamics. For clarity, in this subsection we eliminate the debt tax subsidy by setting \(\rho = 1\). This way, the only difference between (35) and (1) is the bankruptcy cost embedded in the difference between \(B (y)\) and \(A (y)\), and the optimal stopping time which maximizes (35) is simple. Let \(T_{FB} \equiv \arg \max_T V (T)\); clearly \(T_{FB} = T_a = \inf \{ t : y_t < -\zeta X \}\), as the optimal abandonment time of the unlevered firm (recall Section 2.1) maximizes the firm value and minimize the bankruptcy cost (to zero).\(^{22}\)

\subsection{Potential inefficiency of shortening equilibrium}

We use the welfare function \(V (T)\) to evaluate the welfare across two equilibria, one with \(T^S_b\) and the other with \(T^L_b\); see vertical lines in the right panel of Figure 2. By \(\Phi' (y_b) > 0\), shortening equilibria always have a smaller default time than lengthening equilibria, i.e., \(T^S_b < T^L_b\). We also indicate the

\(^{22}\) At \(y = -\zeta X\), both \(B (y) = A (y) = 0\) because both equity and debt holders will terminate the firm immediately. This implies a zero bankruptcy cost.
first-best stopping time \( T_{FB} = T_a \) in Figure 2. Further, in terms of total surplus \( V \), the shortening equilibrium is inferior to the lengthening equilibrium, i.e., \( V(T_b^S) < V(T_b^L) \). We caution that we pick the numerical example so that \( V(T_b^S) < V(T_b^L) \) holds; as explained shortly, in general one cannot rank these two equilibria.

What is more interesting for this numerical example is that the lengthening equilibrium in fact Pareto dominates the shortening equilibrium, as shown by Table 1.\(^{23}\) This implies that, all parties, short-term and long-term debt holders together with equity holders, should be better off by taking the lengthening equilibrium. Because we do not allow for ex ante transfers in our model, this Pareto ranking is much stronger than the notion of relative inefficiency of shortening equilibrium in terms of firm value, i.e., \( V(T_b^S) < V(T_b^L) \). For instance, the Pareto ranking is stronger than a typical inefficiency outcome in which short-term debt holders or equity holders gain slightly—say risk shifting or dilution—at some significant expense of long-term debt holders.

**Local versus global inefficiency** In our model, it is quite general to have \( V(T) \) be nonmonotone in the firm survival time \( T \) (see Appendix A.3.1). Figure 2 also shows that the welfare \( V(T) \) is downward sloping at the default time \( T_b^S \) in the shortening equilibrium. This is intriguing, as it indicates that equity holders are maximizing the whole firm value by shortening the debt maturity structure, if only local deviations were allowed. It turns out that this is not a coincidence. Take the derivative of the firm value \( V = E + \phi D_S + (1 - \phi) D_L \) with respect to the maturity structure \( \phi \):

\[
V_{\phi}(\phi, y) = \Delta(\phi, y) + E_{\phi}(\phi, y) + \phi \frac{\partial}{\partial \phi} D_S(\phi, y) + (1 - \phi) \frac{\partial}{\partial \phi} D_L(\phi, y). \tag{36}
\]

The next paragraph shows that \( V_{\phi}(\phi, y) > 0 \) in any shortening equilibrium around \( (\phi = \Phi(y), y) \). Given this, since maturity shortening implies earlier default \( \frac{\partial T_b^S}{\partial \phi} < 0 \), we have \( V'(T_b^S) = V_{\phi}/\frac{\partial T_b^S}{\partial \phi} < 0 \) in the vicinity of the default boundary for shortening equilibria. This says that the firm value is

\(^{23}\)Again, this property of Pareto dominance may not holds generally, and we find other numerical examples in which relative to the shortening equilibrium, equity and short-term bond holders gain in the lengthening equilibrium while long-term bond holders lose strictly.
indeed higher by defaulting earlier in a local sense.\footnote{The local-efficiency property of the shortening equilibrium, while intriguing, is less general. For instance, the result that $\frac{\partial}{\partial \phi} D_L (\phi, y) > \frac{\partial}{\partial \phi} D_S (\phi, y)$ in the vicinity of the bankruptcy boundary might change if $\rho c \neq r$. Perhaps more empirically relevant situations are that there are other stakeholders who may suffer from earlier default. In Appendix A.5.5 we imagine that the firm has another group of debt holders holding consol bonds whose valuation does not enter the equity holders’ rollover decisions at all. As earlier default leads to value losses to consol bonds, the maturity-shortening equilibrium may become locally inefficient, in the sense that right before default the firm value is improved by marginally lengthening the firm’s maturity structure.}

In (36), the first part is the equity’s IC condition, which is non-negative in any shortening equilibrium. The second term is positive due to condition (30). For the third term regarding long-term bonds, under $\rho c = r$ one can show that $\frac{\partial}{\partial \phi} D_L (\phi, y) > \frac{\partial}{\partial \phi} D_S (\phi, y)$ at the default boundary, i.e., long-term bond holders also gain from earlier default caused by maturity shortening. This implies that all parties in the firm—right before default—will vote against marginally lengthening the firm’s maturity structure! This holds despite the fact that, when the firm is far away from the default boundary, all parties will vote for the globally more efficient lengthening equilibrium.

6 Equilibria with Interior Issuance Policies

Our analysis has so far focused on deterministic cornered issuance policies, i.e., $f \in \{0, 1\}$. This section extends our analysis to the class of all deterministic equilibria by allowing for the possibility of interior issuance policies so that $f \in [0, 1]$.

6.1 Unique Equilibrium around Default Boundary

We work on the state space of $(\tau, y_b)$, and denote the equilibrium issuance strategy by $f (\tau, y_b) \in [0, 1]$. Recall the equity’s IC condition $IC (\tau, y_b) \equiv \Delta (\tau, y_b) + E_\phi (\tau, y_b)$ with $IC (0, y_b) = 0$. We investigate $IC_\tau (\tau, y_b)$, which is the partial derivative of IC condition along the direction of time-to-default. Evaluating on the default boundary, i.e., $\tau = 0$, we show in the Appendix that

$$IC_\tau (0, y_b) = m (\phi_b) \left[ f (0, y_b) \frac{\partial}{\partial \phi} \Delta (0, y_b) + \frac{\partial}{\partial \phi} D_L (0, y_b) \right].$$

(37)
Denote by $f_{\tau=0}$ the equilibrium issuance policy $f(0, y_b)$ at $\tau = 0$. We know that, if $f_{\tau=0}$ takes cornered values, then $IC_{\tau}(0, y_b)|_{f_{\tau=0}=1} > 0 \iff \frac{\partial D_S(\Phi(y_b), y_b)}{\partial \phi} > 0$ or $IC_{\tau}(0, y_b)|_{f_{\tau=0}=0} < 0 \iff \frac{\partial D_L(\Phi(y_b), y_b)}{\partial \phi} < 0$ so that equity finds it optimal to issue short-term or long-term debt right before default, respectively. If $f \in (0, 1)$ takes an interior value, then we must have $IC_{\tau}(0, y_b) = 0$ in (37), i.e.,\footnote{We show that the right hand side of (38) is independent of $f$ in the proof of Proposition 6 in the Appendix.}

$$f = -\frac{\frac{\partial}{\partial \phi} D_L(0, y_b)}{\frac{\partial}{\partial \phi} \Delta(0, y_b)}. \quad (38)$$

We now show generally that on the default boundary the equilibrium is unique, either interior or cornered. Thus, there is no-return regions in which the multiplicity of equilibria vanishes to yield a unique one when sufficiently close to the default boundary, as shown by Proposition 5.

**Proposition 6** Focus on the class of deterministic equilibria. Then, if admissible as given by (A.16), there exists a unique equilibrium $f_{\tau=0}$ on the default boundary. Further, if $f_{\tau=0} \in \{0, 1\}$ and we are at the point where the constraint $f \in [0, 1]$ is strictly binding, we have a unique equilibrium also in the vicinity of the default boundary ($\Phi(y_b), y_b$).

The intuition is similar to the discussion at the end of Section 4.5. The source of multiplicity comes from the self-enforcing expectations of future (issuance) policies. However, if the firm is close to the default boundary, then there will be not enough room for future expectations to be self-enforcing, and a unique equilibrium arises. Once we move further away from the boundary, there might be enough time for self-enforcing expectations to introduce multiplicity. Geometrically, this implies that sufficiently far away from the default boundary, paths can cross each other, and hence multiple equilibria emerge (see Figure 2).

### 6.2 An Example of Equilibrium with Interior Issuance Policy

When we are away from default, the analysis becomes more complicated when allowing for interior issuance polices. We show in the proof of Lemma 6 in the Appendix that along any path with $IC(\tau, y_b) = 0$, there exists a unique $f_{\tau} \in [0, 1]$ exists for any $\tau \in [0, s]$. Further, we can derive
the unique interior issuance policy \( f_\tau \) at \( \tau \) explicitly given the forward-looking (in natural time \( t \)) endogenous equilibrium objects.\(^{26}\) These endogenous equilibrium objects are essentially functions of the equilibrium path \( \{ f_s \}_{s=0}^\tau \), and together with \( f_{\tau=0} \) given in (38) we can solve for \( f_\tau \) by backward induction (in time-to-default \( \tau \)). Thus, in conjunction with Proposition 6 this implies that there exists a unique path to any admissible bankruptcy point \( (y_b(y), y_b) \).\(^{27}\)

Let us pick an ultimate point on the boundary at which the firm defaults that lies in the region of interior equilibria, and work our way backward to trace out the path. Figure 3 maps one such path, first in the left panel in relation to the previous analyzed corner equilibrium paths, and then in a zoomed-out fashion in the right panel. In the left panel, sufficiently far away from the default boundary our interior path crosses with our previous lengthening equilibrium path, leading to multiple equilibria at that intersection point: one lengthening, and (at least) one interior equilibrium. The right panel reveals that the firm’s maturity structure is no longer monotone in time (as implied by the cornered equilibria) in this interior issuance equilibrium, i.e., \( \frac{d\phi(t)}{dt} \) switches signs. For large \( y \)'s the firm is shortening its maturity structure (high \( f \), although not cornered), leading to a slow rise in \( \phi \). However, once the firm is getting close enough to \( \Phi(y) \), it will reverse course and start lengthening its maturity structure (low \( f \), although not cornered).

7 Empirical Predictions and Conclusion

Our model is based on a Leland framework where the basic agency conflict is the equity holders' endogenous default at the expense of debt holders. We study a dynamic setting in which a firm can commit to keeping the overall face-value of debt outstanding constant, but cannot commit to its future maturity structure. Instead, the firm chooses its debt maturity structure endogenously over time in response to observable firm fundamentals. It controls its maturity structure via choosing the fraction of newly issued short-term bonds when refinancing its matured bonds. As a baseline, we show that when the firm’s cash-flows are constant then it is impossible to have the “shortening-

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\(^{26}\)That is, incorporating all times from today until the default time.

\(^{27}\)Essentially, the bankruptcy point is admissible if the path is pointing away from the bankruptcy set \( \mathcal{B} \), not into \( \mathcal{B} \), with respect to \( \tau \).
to-death” equilibrium where the firm keeps issuing short-term bonds and default consequently. In this setting in which recovery in default is constant, the shortening debt maturity structure just imposes faster default which hurts bond-holders.

In contrast, when cash-flows deteriorate over time so that the debt recovery value is affected by the endogenous default timing, then a shortening equilibrium can emerge. For a shortening equilibrium to arise, from the perspective of bond holders, the benefit of a more favorable recovery value by taking the firm over earlier must outweigh the increased expected default risk due to earlier default. The shortening equilibrium can be locally efficient while being globally inefficient (in fact, could be Pareto dominated), relative to the lengthening equilibrium.

Though highly stylized, our model yields the following empirical predictions. First, one is more likely to observe debt maturity shortening in response to deteriorating economic conditions. This is consistent with the empirical findings cited at the beginning of the Introduction: that speculative-grade firms are actively lengthening their debt maturity structure in good times as shown by Xu [2014], and that financial firms are shortening their debt maturity shortening right before 2007/08 crisis as documented by Krishnamurthy [2010]. Second, as illustrated by Figure 2, shortening equilibria exists only when the existing debt maturity is sufficiently short ($\phi$ is sufficiently high). Hence, our model suggests that conditional on deteriorating economic conditions, debt maturity

Figure 3: **Equilibrium with interior issuance policy.** **Left panel:** Interior issuance policies while also showing cornered strategy equilibria. **Right panel:** extended graph of interior issuance equilibrium path with non-monotone path (first shortening, then lengthening when close to default).
shortening is more likely to be observed in firms with already short maturity structures. We are unaware of any existing empirical research on this prediction.

We obtain great tractability and hence sharp analytical results by assuming that the firm commits to a constant debt face value over time, and there is no volatility in cash-flows. As we discussed in Section 4.4, relaxing either of them is a nontrivial task, and will be an interesting direction for future research.
References


A Appendix

A.1 Change of variables, value functions, recovery value and admissibility

A.1.1 Change of variables

We will solve the model in terms of \((\tau, y_0)\), that is time to maturity and cash-flow at time of bankruptcy. This makes the equations all ODEs that we have to consider on the equilibrium path. We then calculate separately the IC conditions via the derivatives of \(E_\phi\) under different assumptions of the issuance strategies. For the moment, fix the issuance strategy \(f(\tau)\).

The proportion of short-term debt \(\phi(\tau, y_0)\). Recall that we have \(\phi \equiv \frac{\dot{S}}{S}\) is the proportion of short-term debt. Consider an arbitrary path \(f(\tau) \in [0, 1]\) for the issuance strategy. Then, we have

\[
\phi'(\tau) = \phi(\tau) \left[ \delta_{S} (1 - f(\tau)) + f(\tau) \delta_{L} \right] - \delta_{L} f(\tau)
\]

Integrating up, imposing \(\phi(0) = \phi_0 = \Phi(y_0)\), we have

\[
\phi(\tau, y_0) = e^{\int_0^\tau \left[ \delta_{S} (1 - f(\tau)) + f(\tau) \delta_{L} \right] d\tau} \left[ \Phi(y_0) - \delta_{L} \int_0^\tau e^{-\int_0^u \left[ \delta_{S} (1 - f(u)) + f(u) \delta_{L} \right] du} f(u) du \right]
\]

(A.1)

Taking derivatives, while keeping \(f(\tau)\) fixed, we have

\[
\frac{\partial \phi}{\partial \tau} = \frac{\partial \phi(\tau, y_0)}{\partial \tau} = \phi(\tau, y_0) \left[ \delta_{S} (1 - f(\tau)) + f(\tau) \delta_{L} \right] - \delta_{L} f(\tau)
\]

(A.2)

\[
\frac{\partial \phi}{\partial y_0} = \frac{\partial \phi(\tau, y_0)}{\partial y_0} = \Phi'(y_0) e^{\int_0^\tau \left[ \delta_{S} (1 - f(\tau)) + f(\tau) \delta_{L} \right] d\tau}
\]

(A.3)

The current cash-flow state \(y(\tau, y_0)\). Next, let us assume there exists \(h_0\) so that \(h_0(y_0) = h_0(y_0) + \mu\tau\). In the linear growth specification, we have \(h_0(x) = x\), whereas in the exponential growth specification we have \(h_0(x) = \log(x)\).

Derivatives w.r.t. \(\phi\). The ODEs are solved in terms of

\[
z = (\tau, y_0)
\]

(A.4)

However, the incentives of the equity holders are derived from the Markov system

\[
x = (\phi, y_0)
\]

(A.5)

as the optimal \(f\) requires the derivative \(E_\phi\). We are looking for points \(z = g(x)\) such that \(h(x, z) = h(x, g(x)) = 0\) where

\[
h(x, z) = \begin{bmatrix}
    h_1(x, z) \\
    h_2(x, z)
\end{bmatrix} = \begin{bmatrix}
    -\phi + \phi(\tau, y_0) \\
    -h_0(y_0) + h_0(y(\tau, y_0))
\end{bmatrix} = 0
\]

(A.6)

and where

\[
g(x) = \begin{bmatrix}
    \tau(\phi, y_0) \\
    y_0(\phi, y_0)
\end{bmatrix}
\]

(A.7)

To calculate the derivative of \(E\) for example \(E(\tau, y_0) = E(z)\) w.r.t. \(\phi\), we have to use

\[
\frac{\partial}{\partial \phi} E(\tau, y_0) = E_\tau(\tau, y_0) \frac{\partial \tau}{\partial \phi} + E_{y_0}(\tau, y_0) \frac{\partial y_0}{\partial \phi} = \left[ \frac{\partial}{\partial z} E(z) \right] \cdot \left[ \frac{\partial z}{\partial \phi} \right]
\]

(A.8)

The Jacobian matrix is given by

\[
J = \frac{\partial h(x, z)}{\partial z} = \begin{bmatrix}
    \frac{\partial h_1}{\partial \tau} & \frac{\partial h_1}{\partial y_0} \\
    \frac{\partial h_2}{\partial \tau} & \frac{\partial h_2}{\partial y_0}
\end{bmatrix}
\]

(A.9)

Then, applying the chain rule when taking the derivative w.r.t. \(x\),

\[
\frac{\partial z}{\partial \phi} = \frac{\partial g(x)}{\partial \phi} = \frac{\partial}{\partial \phi} \left[ \begin{bmatrix}
    \tau(\phi, y_0) \\
    y_0(\phi, y)
\end{bmatrix} = -J^{-1} \frac{\partial}{\partial \phi} h(x, z)
\right]
\]

(A.10)
Let us calculate the different derivatives. First, we have
\[
\frac{\partial h_1}{\partial \phi} = -1 \\
\frac{\partial h_2}{\partial \phi} = 0
\]  
so that \( \frac{\partial}{\partial \phi} h(x, z) = -[1, 0]^{\top} \). Then, we have
\[
\frac{\partial z}{\partial \phi} = \left[ \frac{\partial \tau \phi}{\partial \phi} \right]^{-1} = \frac{1}{\partial \tau \phi} \left[ \begin{array}{c} \frac{\partial h_2}{\partial \phi} \\
\frac{\partial h_1}{\partial \phi} \\
\frac{\partial h_2}{\partial \phi} \\
\frac{\partial h_1}{\partial \phi} 
\end{array} \right]^{-1} \left[ \begin{array}{c} \frac{\partial h_2}{\partial \phi} \\
\frac{\partial h_1}{\partial \phi} \\
\frac{\partial h_2}{\partial \phi} \\
\frac{\partial h_1}{\partial \phi} 
\end{array} \right] = \frac{1}{h'_0(y_b) \frac{\partial \tau \phi}{\partial \phi} - \mu \frac{\partial \phi}{\partial \phi}} \left[ \begin{array}{c} h'_0(y_b) \\
-h \end{array} \right]
\]  
Thus, we ultimately have
\[
\left[ \frac{\partial \tau \phi}{\partial \phi} \right] = h'_0(y_b) \left\{ \phi(\tau, y_b) [\delta_S (1 - f(\tau)) + f(\tau) \delta_L] - \delta_L f \right\} - \mu \Phi'(y_b) e^{\tau \phi (1 - f) + f \delta_L \tau} \delta_L ds
\]  
\[\text{(A.14)}\]

A.1.2 Admissible paths

The bankruptcy boundary and the change of variables interact in a specific way. Essentially, we cannot allow such \( f \)'s that will point inside the bankruptcy region \( B \) when increasing \( \tau \). To this end, we need to impose
\[
\Phi'(y_b) \left( \frac{\partial \phi(\tau, y_b)}{\partial \tau} \right)_{\tau=0} = \Phi(y_b) [\delta_S (1 - f) + f \delta_L] - \delta_L f
\]
\[\text{(A.15)}\]
at \( \tau = 0 \). Multiplying through by \( \mu/h'_0(y_b) > 0 \), and rearranging, we have the following inequality that defines admissible \( f \):
\[
0 > h'_0(y_b) \left( \Phi(y_b) [\delta_S (1 - f) + f \delta_L] - \delta_L f \right) - \mu \Phi'(y_b)
\]
\[\text{(A.16)}\]

A.1.3 Debt and Equity solutions for Section 4.2

Next, let us derive debt and equity values for given paths of \( f \) for \( (\tau, y_b) \).

**Debt.** Debt has an ODE
\[
(r + \delta_i + \zeta) D_I(\tau, y_b) = (\rho c + \delta_i + \zeta) - \frac{\partial}{\partial \tau} D_I(\tau, y_b)
\]
that is solved by
\[
D_S(\tau, y_b) = \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta} e^{-(r+\delta_S+\zeta)\tau} \left[ B(y_b) - \frac{\rho c + \delta_S + \zeta}{r + \delta_S + \zeta} \right]
\]
\[\text{(A.17)}\]
\[
D_L(\tau, y_b) = \frac{\rho c + \delta_L + \zeta}{r + \delta_L + \zeta} e^{-(r+\delta_L+\zeta)\tau} \left[ B(y_b) - \frac{\rho c + \delta_L + \zeta}{r + \delta_L + \zeta} \right]
\]
\[\text{(A.18)}\]
Importantly, for a given \( (\tau, y_b) \) debt values are independent of the path of \( f \). Imposing \( \rho c = 1 \) we get the result in the main text.

**Equity.** Equity solves the ODE where \( y = y(\tau, y_b) \) and \( \phi = \phi(\tau, y_b) \)
\[
(r + \zeta) E(\tau, y_b) = y + \zeta E'f + c + m(\phi) [f D_S(\tau, y_b) + (1 - f) D_L(\tau, y_b) - 1] - \frac{\partial}{\partial \tau} E(\tau, y_b)
\]
\[\text{(A.19)}\]
with boundary condition \( \frac{\partial}{\partial \tau} E(\tau, y_b)|_{\tau=0} = 0 \). For future reference, we will differentiate w.r.t. \( \phi \) to get

\[
E_{\tau \phi} (\tau, y_b) = m' (\phi) [f D_S (\tau, y_b) + (1 - f) D_L (\tau, y_b) - 1] + m (\phi) \left[ f \frac{\partial D_S (\tau, y_b)}{\partial \phi} + (1 - f) \frac{\partial D_L (\tau, y_b)}{\partial \phi} \right] - (r + \zeta) E_{\phi} (\tau, y_b)
\]

where we abused notation for \( E_{\tau \phi} \). Differentiating w.r.t. \( y_b \), we have

\[
(r + \zeta) \frac{\partial E (\tau, y_b)}{\partial y_b} = \frac{\partial y (\tau, y_b)}{\partial y_b} + (\delta_S - \delta_L) [f D_S (\tau, y_b) + (1 - f) D_L (\tau, y_b) - 1] \frac{\partial \phi (\tau, y_b)}{\partial y_b} + m (\phi) \left[ f \frac{\partial D_S (\tau, y_b)}{\partial y_b} + (1 - f) \frac{\partial D_L (\tau, y_b)}{\partial y_b} \right] - \frac{\partial}{\partial \tau} \left( \frac{\partial E (\tau, y_b)}{\partial y_b} \right)
\]

with boundary condition \( \frac{\partial}{\partial \tau} \left( \frac{\partial E (\tau, y_b)}{\partial y_b} \right)|_{\tau=0} = 0 \) and where we used \( m' (\phi) = \delta_S - \delta_L \). Integrating up for a given path of \( f \), we have

\[
E (\tau, y_b) = \int_{\tau}^{T} e^{(r + \zeta)(u - \tau)} \left\{ y (u, y_b) + \zeta E^r f - c + m (\phi (u, y_b)) [f D_S (u, y_b) + (1 - f) D_L (u, y_b) - 1] \right\} du
\]

Here, we can see how \( f \) affects the value of equity even for a given \((\tau, y_b)\). Integrating up \( \frac{\partial E (\tau, y_b)}{\partial y_b} \), we have

\[
\frac{\partial E (\tau, y_b)}{\partial y_b} = \int_{\tau}^{T} e^{(r + \zeta)(u - \tau)} \left\{ \frac{\partial y (u, y_b)}{\partial y_b} + (\delta_S - \delta_L) [f D_S (u, y_b) + (1 - f) D_L (u, y_b) - 1] \frac{\partial \phi (u, y_b)}{\partial y_b} + m (\phi (u, y_b)) \left[ f \frac{\partial D_S (u, y_b)}{\partial y_b} + (1 - f) \frac{\partial D_L (u, y_b)}{\partial y_b} \right] \right\} du
\]

### A.2 Proofs of Section 3

**Proof of Lemma 1.** We use the fact that \( B (y) \leq D_S \leq D^f = 1 \) and \( B (y) \leq D_L \leq D^f = 1 \) to bound the rollover term in (6):

\[
0 \geq m (\phi_t) [f_t D_S (\phi_t; y) + (1 - f_t) D_L (\phi_t; y) - 1] \geq m (\phi) [f B (y) + (1 - f_t) B (y) - 1] = m (\phi) [B (y) - 1] \geq \delta_S [B (y) - 1],
\]

where we used \( \delta_S = \max_{\phi \in [0, 1]} m (\phi) \). Hence if \( y - c + \zeta E^r f < 0 \) then the cash flows to equity are always negative, leading to immediate default. On the other hand, if \( y - c + \zeta E^r f + \delta_S [B (y) - 1] > 0 \), then even under the most pessimistic beliefs equity holders never make losses and thus never default.

**Proof of Proposition 1.** The equation of (11) for \( f = 1 \) is:

\[
r E (\phi; y) = y - c + \zeta E^r f - E (\phi; y) + [\phi \delta_S + (1 - \phi) \delta_L] [D_S (\phi; y) - 1] + (1 - \phi) \delta_L E' (\phi; y)
\]

We then take the derivative with respect to \( \phi \) of (A.24):

\[
(r + \zeta) E' (\phi; y) = (\delta_S - \delta_L) [D_S (\phi; y) - 1] + [\phi \delta_S + (1 - \phi) \delta_L] D'_S (\phi; y) - \delta_L E' (\phi; y) + (1 - \phi) \delta_L E'' (\phi; y).
\]

Evaluating this equation at the default boundary \( \Phi \), together with \( E' (\Phi; y) = 0 \) and \( D_S (\Phi; y) = B (y) \), we have equation (18) in the main text.

### A.3 Proofs of Section 4

**Proof of Lemma 2.** \( E (\Phi (y_b), y_b) = 0 \) at default is obvious as equity defaults when their cash-flows turn exactly zero in our deterministic setting. Plugging in \( E (\Phi (y_b), y_b) = 0 \) into the ODE for equity valuation, we see that \( E_{\tau} (\tau, y_b)|_{\tau=0} = 0 \). Change the coordinates of the state space to \( (\phi, y) \), we have

\[
E_{\phi} (\Phi (y_b), y_b)|_{\tau=0} = E_{\tau} (\tau, y_b) \frac{\partial \tau}{\partial \phi}|_{\tau=0} + E_{y_b} (\tau, y_b) \frac{\partial y_b}{\partial \phi}|_{\tau=0} = E_{y_b} (\tau, y_b) \frac{\partial y_b}{\partial \phi}|_{\tau=0},
\]

where we use \( E_{\tau} (\tau, y_b)|_{\tau=0} = 0 \). However, we have \( E_{y_b} (\tau, y_b)|_{\tau=0} \), because equity defaults at \( \tau = 0 \) and \( y_b \) only...
and we can thus define

However, incentives are not invariant to the path taken, as we will show below. Consider now

We know that the default time is

Suppose we consider an arbitrary equilibrium path

Next, for cornered shortening, we need \( \frac{\partial \IC}{\partial \tau} \big|_{\tau=0,f=1} > 0 \). Thus, plugging in \( f = 1 \) and assuming \( \rho c = r \), we have

As \( m (\Phi (y)) \geq \delta_L > 0 \), we have proved the result. For uniqueness, we refer the reader to the full proof of Proposition 6.

Proof of Proposition 3. See proof of Proposition 6

Proof of Proposition 4. Combine proof of Proposition 6 with Lemma 3 below.

Lemma 3 For cornered equilibria, the IC condition on the default boundary is sufficient for the IC along the whole path.

Proof of Lemma 3. We start with the following observation. Suppose the current state of the system is given by \((\phi, y)\). Firm-value is then given by

\[
V (\phi, y) = E (\phi, y) + \phi D_S (\phi, y) + (1 - \phi) D_L (\phi, y)
\]

Suppose we consider an arbitrary equilibrium path \((\phi, y) \rightarrow (\Phi (y_b) , y_b)\) where default occurs at the point \((\Phi (y_b) , y_b)\).

We know that the default time is

\[
\tau = \frac{y - y_b}{\mu}
\]

by the linear growth specification of \( y \). That is, we fix the starting point and the end-point of the path, and thereby the time to default, but leave the actual issuance strategy \( \{ f_t \} \) and thus the actual path taken by \( \phi \) undefined. Let us sum up all the cash-flows to get an alternate expression for firm value,

\[
V (\tau, y_b) = \int_0^\tau e^{(r + \zeta)(\phi - \phi)} \left[ y_b + \mu s + \zeta X + (\rho - 1) c \right] ds + e^{-(r + \zeta)\tau} B (y_b)
\]

and we can thus define

\[
E (\tau, y_b) = V (\tau, y_b) - \phi (\tau, y_b) D_S (\tau, y_b) - [1 - \phi (\tau, y_b)] D_L (\tau, y_b)
\]

Importantly, we see that equity value is invariant to the specific path of \( \phi \) taken as long as \( y_b \) and thus \( \tau \) is held fixed. However, incentives are not invariant to the path taken, as we will show below. Consider now

\[
E_{\phi} = \frac{\partial}{\partial \phi} [V (\phi, y) - \phi D_S (\phi, y) - (1 - \phi) D_L (\phi, y)]
\]

\[
= \left\{ \frac{\partial}{\partial \tau} V (\tau, y) - \phi \frac{\partial}{\partial \tau} D_S (\tau, y) - (1 - \phi) \frac{\partial}{\partial \tau} D_L (\tau, y) \right\} \frac{\partial \tau}{\partial \phi}
\]

\[
+ \left\{ \frac{\partial}{\partial y_b} V (\tau, y) - \phi \frac{\partial}{\partial y_b} D_S (\tau, y) - (1 - \phi) \frac{\partial}{\partial y_b} D_L (\tau, y) \right\} \frac{\partial y_b}{\partial \phi}
\]

\[
- D_S (\phi, y) + D_L (\phi, y)
\]

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so that we have, after rearranging

\[ IC(\tau, y_b) = \Delta(\tau, y_b) + E_\phi(\tau, y_b) \]

\[ \frac{\partial}{\partial \tau} V(\tau, y_b) - \phi(\tau, y_b) \frac{\partial}{\partial \tau} DS(\tau, y_b) - [1 - \phi(\tau, y_b)] \frac{\partial}{\partial \tau} DL(\tau, y_b) \] 

\[ + \left\{ \frac{\partial}{\partial y_b} V(\tau, y_b) - \phi(\tau, y_b) \frac{\partial}{\partial y_b} DS(\tau, y_b) - [1 - \phi(\tau, y_b)] \frac{\partial}{\partial y_b} DL(\tau, y_b) \right\} \frac{\partial y_b}{\partial \phi} \]

Thus, importantly, we see that the path of \( f \) for a given \( \phi(\tau, y_b) \) only is reflected in \( e^{\int_0^\tau [\delta_S(1-f) + \delta_L f] ds} \) that enters through the change-of-variables. Note that

\[ V(\tau, y_b) = [y_b + \zeta X + (\rho - 1) c] \frac{1 - e^{-(r+\zeta)\tau}}{r + \zeta} + \mu \frac{1 - e^{-(r+\zeta)\tau}}{r + \zeta} B(y_b) \]

\[ \frac{\partial}{\partial \tau} V(\tau, y_b) = e^{-(r+\zeta)\tau} [y_b + \zeta X + (\rho - 1) c - (r + \zeta) B(y_b)] + \mu \frac{1 - e^{-(r+\zeta)\tau}}{r + \zeta} \]

\[ \frac{\partial}{\partial y_b} V(\tau, y_b) = \frac{1 - e^{-(r+\zeta)\tau}}{r + \zeta} + e^{-(r+\zeta)\tau} B'(y_b) \]

And thus, plugging in, we have

\[ \left\{ \frac{\partial}{\partial \tau} \right\} = e^{-(r+\zeta)\tau} [y_b + \zeta X + (\rho - 1) c + B(y_b)] + \mu \frac{1 - e^{-(r+\zeta)\tau}}{r + \zeta} \]

\[ + \phi(\tau, y_b) (r + \delta_S + \zeta) e^{-(r+\delta_S + \zeta)\tau} \left[ B(y_b) - \frac{pc + \delta_S + \zeta}{r + \delta_S + \zeta} \right] \]

\[ + [1 - \phi(\tau, y_b)] (r + \delta_L + \zeta) e^{-(r+\delta_L + \zeta)\tau} \left[ B(y_b) - \frac{pc + \delta_L + \zeta}{r + \delta_L + \zeta} \right] \]

\[ \left\{ \frac{\partial}{\partial y_b} \right\} = \frac{1 - e^{-(r+\zeta)\tau}}{r + \zeta} + B'(y_b) \left\{ e^{-(r+\zeta)\tau} - \phi(\tau, y_b) e^{-(r+\delta_S + \zeta)\tau} - [1 - \phi(\tau, y_b)] e^{-(r+\delta_L + \zeta)\tau} \right\} \]

We note that \( \left\{ \frac{\partial}{\partial \tau} \right\}_{\tau=0} = 0 \) and \( \left\{ \frac{\partial}{\partial y_b} \right\}_{\tau=0} = 0 \), so that indeed we have \( IC(0, y_b) = 0 \).

In the linear specification, we note that

\[ \frac{\partial y_b}{\partial \phi} = -\mu \frac{\partial \tau}{\partial \phi} = -\mu \frac{\partial \tau}{\partial \phi} \]

\[ = -\mu \frac{\phi(\tau, y_b)}{\phi(\tau, y_b)} - \mu \delta y_b(\tau, y_b) \]

\[ = -\mu \left\{ \phi(\tau, y_b) [\delta_S (1-f) + f \delta_L] - f \delta_L \right\} - \mu \Phi(y_b) e^{\int_0^\tau [\delta_S (1-f) + f \delta_L] ds} > 0 \]

as for \( f \in \{0, 1\} \) paths do not cross and we must have \( \frac{\partial \tau}{\partial \phi} < 0 \). As the \( IC(\tau, y_b) \) condition is not monotone in \( \tau \), we use a scaled up version \( e^{\delta \tau} IC(\tau, y_b) \) for a specific \( k \). We can show that for shortening equilibria (i.e. \( f = 1 \))

\[ \frac{\partial}{\partial \tau} \left[ e^{(r+\zeta+\delta_S)\tau} IC(\tau, y_b) \right] = -e^{\delta_S \tau} \left\{ \delta_S - (\delta_S - \delta_L) \left[ 1 - \phi(y_b) \right] e^{\delta_L \tau} \right\} \left\{ (\delta_S + r + \zeta) \left[ 1 - B(y_b) \right] - \mu B'(y_b) \right\} \]

\[ \frac{\delta_S \Phi(y_b) - \delta_L}{\delta_S \Phi(y_b) - \delta_L - \mu \Phi(y_b)} \]

\[ = -e^{\delta_S \tau} \left\{ \delta_S - (\delta_S - \delta_L) \left[ 1 - \phi(y_b) \right] \right\} \left\{ (\delta_S + r + \zeta) \left[ 1 - B(y_b) \right] - \mu B'(y_b) \right\} \]

\[ \frac{\delta_S \Phi(y_b) - \delta_L}{\delta_S \Phi(y_b) - \delta_L - \mu \Phi(y_b)} \]

and for lengthening equilibria (i.e. \( f = 0 \)) we have

\[ \frac{\partial}{\partial \tau} \left[ e^{(r+\zeta+\delta_S)\tau} IC(\tau, y_b) \right] = -e^{\delta_S \tau} \left\{ \delta_S + (\delta_S - \delta_L) \Phi(y_b) e^{\delta_L \tau} \right\} \left\{ (\delta_S + r + \zeta) \left[ 1 - B(y_b) \right] - \mu B'(y_b) \right\} \]

\[ \frac{\delta_S \Phi(y_b) - \delta_L}{\delta_S \Phi(y_b) - \delta_L - \mu \Phi(y_b)} \]

\[ = -e^{\delta_S \tau} \left\{ \delta_S + (\delta_S - \delta_L) \phi(\tau, y_b) \right\} \left\{ (\delta_S + r + \zeta) \left[ 1 - B(y_b) \right] - \mu B'(y_b) \right\} \]

\[ \frac{\delta_S \Phi(y_b) - \delta_L}{\delta_S \Phi(y_b) - \delta_L - \mu \Phi(y_b)} \]
Here, by \( \phi_\tau (\tau, y_\delta) = \phi (\tau, y_\delta) - m (\phi (\tau, y_\delta)) f \) in the the shortening equilibrium we have \( [1 - \phi (\tau, y_\delta)] = [1 - \Phi (y_\delta)] e^{\delta T} \) and in the lengthening we have \( \phi (\tau, y_\delta) = \Phi (y_\delta) e^{\delta T} \). We see that at \( \tau = 0 \) these collapse to our boundary conditions for shortening and lengthening equilibria, respectively. Note that \( m (\phi) \in [\delta L, \delta S] \) as \( \phi \) is bounded, we know there is a maximal \( \tau \) corresponding to any ultimate bankruptcy cash-flow \( y_\delta \) (beyond this \( \tau \) we would have \( \phi (\tau, y_\delta) \notin [0, 1] \); recall we are reversing time and there is divergence!). Thus, the IC condition at \( T \) is sufficient for all paths.

**Proof of Lemma 5.** Let us first discuss admissibility: on the boundary, the path has to point away from the default region \( B \) in terms of \( \tau \). Suppose then that we consider a point in the vicinity of \( \tau = 0 \), say \( \tau = 0 + \varepsilon \). We know that there is a maximum adjustment speed of \( \frac{dy}{dt} \) and that \( \phi \) is continuous in \( \tau \). This implies that a point \( (\phi, y_\delta) \) that is very close to the boundary where only a shortening equilibrium exists cannot change \( \phi \) quickly enough to reach either the interior equilibrium or lengthening equilibrium. Similar reasoning applies for lengthening equilibria.

**A.3.1 Appendix for Section 5.3.2**

Although the global optimum of \( V (T) \) is trivial, the local behavior of \( V (T) \) can be more intriguing for the relevant region \( T < T_\alpha = \max_T V (T) \) in which \( T_\delta \) and \( T_\delta \) lie. The right panel of Figure 2 plots \( V (T) \), which is non-monotone in \( T \) for \( T \) far less than \( T_\alpha \). This implies that there is a region in which faster default, potentially due to shortening of debt maturity, can be welfare enhancing locally!

To better understand the mechanism, we investigate \( V' (T) \) which is the marginal impact of delaying default on firm value (multiplying both sides by \( e^{(r+\zeta)T} \)):

\[
e^{(r+\zeta)T} V' (T) = \frac{dy_T}{dT} + \frac{dy_T}{dT} + \frac{dy_T}{dT} = \frac{dy_T}{dT} + \frac{dy_T}{dT}.
\]

As in any frictionless optimal stopping problem, the first term is zero.\(^{28}\) The second term captures the positive bankruptcy cost. The third term captures the impact of delaying default on the firm’s liquidation value. In our example with \( \alpha_\delta > 1 \), it is possible that \( B' (y_T) - A' (y_T) > 0 \) because of worsening cash-flows in default. Together with deteriorating cash-flows \( \frac{dy_T}{dT} < 0 \), this force can make the third term negative. As a result, \( V (T) \) may not be always increasing in \( T \), and the right panel of Figure 2 shows that indeed at the shortening equilibrium we have \( V' (T_\delta) < 0 \).

Here is another intuition. Recall that \( A (y) - B (y) \) captures the (endogenous) bankruptcy cost in our model. When \( A' (y_T) - B' (y_T) < 0 \), we have an endogenous bankruptcy cost that is decreasing in the defaulting cash-flows. The earlier the default, the higher the defaulting cash-flows, and the smaller the bankruptcy cost. This force contributes to the non-monotonicity of the levered firm value as a function of the default time.

**A.4 Proofs of Section 6**

We will prove the main result, Proposition 6, in a sequence of lemmas. Lemma 4 establishes that there is continuity of the issuance strategy \( f \) w.r.t. time-to-maturity \( \tau \). Lemma 5 establishes the uniqueness of \( f_{\tau=0} \) if it is admissible on the boundary. Lemma 6 provides the uniqueness of a path leading away from the boundary.

**Lemma 4** There is no discontinuities in \( f \) on any equilibrium path, i.e. \( \left| \frac{f_{\tau=\alpha} - f_{\tau-\alpha}}{\tau} \right| < \infty \) everywhere.

**Proof of Lemma 4.** Note that \( E_\phi (\phi, y) \) can be calculated as

\[
\frac{\partial}{\partial \phi} E (\phi, y) = \frac{\partial}{\partial \phi} E (\tau, y_\delta) = E_r (\tau, y_\delta) \frac{\partial \tau}{\partial \phi} + E_n (\tau, y_\delta) \frac{\partial y_\delta}{\partial \phi}.
\]

As before, the first term captures the effect of time-to-default \( \tau \), while the second term captures the effect of defaulting cash-flows \( y_\delta \). Suppose now there exists a time-to-default \( \tilde{\tau} \) at which there is a jump in \( f \), i.e., \( f_{\tilde{\tau}} = f_{\tilde{\tau}+} \). Equity values and debt values (and thus the bond value wedge \( \Delta \)) are continuous across \( \tilde{\tau} \) along the path \( (\phi_\tau, y_\delta) \) by inspection.

\(^{28}\)This is because the unlevered firm value \( A (y) \) satisfies the differential equation \( rA (y) = y + \zeta [X - A (y)] + A' (y) \frac{dy}{dt} \).

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of (A.17), (A.18) and (A.22). However, equity’s derivative with respect to \(\tau\), i.e., \(E_\tau\), displays a discontinuity at the policy switching point \(\hat{\tau}\). Plugging into (A.19), we have

\[
E_{\tau^\leftarrow} - E_{\tau^+} = m(\phi) \Delta \cdot (f_{\tau^\leftarrow} - f_{\tau^+}) = m(\phi) \Delta.
\]

(A.28)

Since \(m(\phi) \Delta > 0\), it implies that when equity switches to issuing more short-term bonds at \(\hat{\tau}\), i.e., \(f_{\tau^\leftarrow} - f_{\tau^+}\), the equity value’s derivative with respect to \(\tau\) jumps up, i.e., the benefit of surviving longer goes up.

In the original \((\phi, y)\) state space, denote the corresponding switching points to be \((\hat{\phi}_-, \hat{y}_-)\) and \((\hat{\phi}_+, \hat{y}_+)\). Equity’s incentive compatibility condition depends on \(\frac{\partial}{\partial \phi} E_\phi (\phi, y)\) at these two points. By writing out the terms in integral form, and noting that any \(f\) are bounded, we can show that in (A.27), both the \(\frac{\partial}{\partial \phi}\) in the first term, and the entire second term related to \(y\), i.e., \(E_y (\tau, y) \frac{\partial \phi}{\partial \phi}\), are continuous at the switching point. Hence, equation (A.28) implies that

\[
E_\phi \left( \hat{\phi}_-, \hat{y}_- \right) - E_\phi \left( \hat{\phi}_+, \hat{y}_+ \right) = E_{\tau^\leftarrow} - E_{\tau^+} \cdot \frac{\partial \tau}{\partial \phi} = m(\phi) \Delta \left( f_{\tau^\leftarrow} - f_{\tau^+} \right) \cdot \frac{\partial \tau}{\partial \phi}.
\]

Next, note that \(\frac{\partial \tau}{\partial \phi} < 0\), i.e., shortening maturity gives rise to a shorter time-to-default. Following the intuition right after (A.28), when equity switches to issuing short-term bonds, the benefit of surviving longer going up implies that minimal negative impact of shortening maturity is more severe. To make the general point, let us write

\[
IC \left( \hat{\phi}_+, \hat{y}_+ \right) = \Delta \left( \hat{\phi}_+, \hat{y}_+ \right) + [m(\phi) \Delta \left( f_{\tau^\leftarrow} - f_{\tau^+} \right) \frac{\partial \tau}{\partial \phi}]
\]

\[
= \Delta \left( \hat{\phi}_-, \hat{y}_- \right) + [m(\phi) \Delta \left( f_{\tau^\leftarrow} - f_{\tau^+} \right) \frac{\partial \tau}{\partial \phi}]
\]

Consider first the case when \(f_{\tau^\leftarrow} = 1\) and \(f_{\tau^+} < 1\). This implies that \(m(\phi) \Delta \left( f_{\tau^\leftarrow} - f_{\tau^+} \right) \left( \frac{\partial \tau}{\partial \phi} \right) > 0\) and we immediately have a violation: if \(f_{\tau^\leftarrow} = 1\) was optimal, then \(IC \left( \hat{\phi}_+, \hat{y}_+ \right) > IC \left( \hat{\phi}_-, \hat{y}_- \right) \geq 0\) and thus \(f_{\tau^+} < 1\) violates the \(IC\) condition. Next, consider the case when \(f_{\tau^\leftarrow} = 0\) and \(f_{\tau^+} > 0\). This implies that \(m(\phi) \Delta \left( f_{\tau^\leftarrow} - f_{\tau^+} \right) \left( \frac{\partial \tau}{\partial \phi} \right) \leq 0\), which implies \(IC \left( \hat{\phi}_+, \hat{y}_+ \right) < IC \left( \hat{\phi}_-, \hat{y}_- \right) \leq 0\) and thus invalidates \(f_{\tau^+} > 0\). Lastly, consider the case when \(f_{\tau^\leftarrow} \in [0, 1]\) such that \(IC \left( \hat{\phi}_-, \hat{y}_- \right) = 0\). Then we immediately see that any \(f_{\tau^\leftarrow} \neq f_{\tau^+}\) violates \(IC\): (i) if \(f_{\tau^\leftarrow} \in (0, 1)\), then we must have \(IC \left( \hat{\phi}_+, \hat{y}_+ \right) = 0\) as well, which is violated by \(m(\phi) \Delta \left( f_{\tau^\leftarrow} - f_{\tau^+} \right) \left( \frac{\partial \tau}{\partial \phi} \right) \neq 0\). (ii) if \(f_{\tau^\leftarrow} \in \{0, 1\}\), then we are in the above proofs, and see that the violation exactly runs counter to the \(IC\) condition.

\begin{lemma}
If an equilibrium \(f_{\tau^\leftarrow} = 0\) exists on the boundary, it is unique. It might not exists due to the admissibility condition.
\end{lemma}

\textbf{Proof of Lemma 5} First, let us concentrate on \(f\) on the boundary. Taking derivatives of (A.17) and (A.18) w.r.t. \(\phi\) via (A.14), and evaluating at \(\tau = 0\), we have

\[
\frac{\partial D_1}{\partial \phi} = \frac{h_0^\prime (y)}{h_0 (y)} \left\{ (pc - r) + (r + \zeta + \delta_t) [1 - B (y)] \right\} - \mu B^\prime (y)
\]

(A.29)

Differentiating \(IC\) w.r.t. \(\tau\), we have \(IC = \Delta \tau \Phi (y) + E_y \tau (\tau, y)\). Plugging in for \(E_y \tau (\tau, y)\) from (A.20), evaluating at \(\tau = 0\) so that \(IC = \Delta = E_y\Phi\), and noting that \(\Delta = \left( \delta_S - \delta_L \right) [1 - B (y)]\), we have

\[
\frac{\partial IC}{\partial \tau} \bigg|_{\tau=0} = m(\Phi (y)) \left\{ \frac{\partial D_S (\tau, y)}{\partial \phi} + (1 - f) \frac{\partial D_L (\tau, y)}{\partial \phi} \right\} \bigg|_{\tau=0}.
\]

(A.30)

\[
= m(\Phi (y)) \left\{ \frac{h_0^\prime (y)}{h_0 (y)} \left\{ (pc - r) + (r + \zeta + f \delta_S + (1 - f) \delta_L) [1 - B (y)] \right\} - \mu B^\prime (y) \right\} \frac{\partial \phi}{\partial \phi}.
\]

(A.31)

As \(m(\phi) > 0\), we can ignore this term for determining the sign. Next, let us collect all terms in the numerator multiplying \(f\), which are given by \(\{h_0^\prime (y) \left( \delta_S - \delta_L \right) [1 - B (y)]\} \). Further, we know from condition (A.16) that for all admissible \(f\) the denominator has to be negative. Thus, we can concentrate on the numerator to determine the
optimal \( f \). We have

\[
h'_0(y_b) \left\{ \phi c - r + [r + \zeta + f \delta_S + (1-f) \delta_L] \left[ 1 - B(y_b) \right] \right\} - \mu B'(y_b) \\
= \left[ h'_0(y_b) \left\{ \phi c - r + [r + \zeta + \delta_L] \left[ 1 - B(y_b) \right] \right\} - \mu B'(y_b) \right] \\
+ \mu B'(y_b) \left[ \delta_S - \delta_L \right] \left[ 1 - B(y_b) \right]
\]

(A.32)

and we see that we have a linear function in \( f \), which is increasing by

\[
(\delta_S - \delta_L) \left[ 1 - B(y_b) \right] > 0
\]

(A.33)

Thus, we have at most one unique root in (A.32). Importantly, we also know that (A.32) crosses 0 from above if at all. As the numerator is monotone, this implies a unique equilibrium. If the numerator is everywhere negative for \( f \in [0, 1] \), then \( f = 1 \). If the numerator is everywhere positive for \( f \in [0, 1] \), then \( f = 0 \) if this is admissible. Lastly, if there exits an admissible

\[
\hat{f} = \frac{\mu B'(y_b) - h'_0(y_b) \left\{ (\phi c - r) + (r + \zeta + \delta_L) \left[ 1 - B(y_b) \right] \right\}}{h'_0(y_b) \left( \delta_S - \delta_L \right) \left[ 1 - B(y_b) \right]} \in (0, 1)
\]

(A.34)

then this is the unique equilibrium. We thus have

\[
f_{\tau=0} = \min \left[ 1, \max \left\{ \hat{f}, 0 \right\} \right]
\]

(A.35)

as the unique equilibrium subject to admissibility \( () \). ■

**Lemma 6** For a given point \((\Phi(y_b), y_b)\) there exists a unique equilibrium path \(\tau\) leading away from the boundary.

**Proof of Lemma 6.** Writing out \(IC(\tau, y_b)\), we have

\[
IC(\tau, y_b) = \Delta(\tau, y_b) + E_\phi(\tau, y_b) \\
= D_s(\tau, y_b) - D_l(\tau, y_b) + \frac{\partial y_b}{\partial \phi} \left\{ \frac{\partial}{\partial y_b} E(\tau, y_b) \right\} \\
+ \frac{\partial \tau}{\partial \phi} \left\{ y(\tau, y_b) - c + \zeta f E'f - (r + \zeta) E(\tau, y_b) \right\} + m(\phi(\tau, y_b)) \left\{ f D_s(\tau, y_b) + (1-f) D_l(\tau, y_b) - 1 \right\}
\]

(A.36)

Let us move things under the common denominator \(h'_0(y_b)\) \{\phi(\tau, y_b) [\delta_S (1 - f) + f \delta_L] - \delta_L f \} - \mu \frac{\partial}{\partial y_b} \phi(\tau, y_b)

\[
IC(\tau, y_b) = \frac{1}{h'_0(y_b) \left\{ \phi(\tau, y_b) [\delta_S (1 - f) + f \delta_L] - \delta_L f \right\} - \mu \frac{\partial}{\partial y_b} \phi(\tau, y_b)} \left\{ \Delta(\tau, y_b) \left[ h'_0(y_b) \left\{ \phi(\tau, y_b) [\delta_S - f \delta_L] - \delta_L f \right\} - \mu \frac{\partial}{\partial y_b} \phi(\tau, y_b) \right] \\
+ h'_0(y_b) \left\{ y(\tau, y_b) - c + \zeta f E'f - (r + \zeta) E(\tau, y_b) \right\} + m(\phi(\tau, y_b)) \left\{ f D_s(\tau, y_b) + (1-f) D_l(\tau, y_b) - 1 \right\} \right\}
\]

(A.37)

Suppose we have an interior equilibrium. For interior equilibria we have \(IC(\tau, y_b) = 0\), so that for non-zero denominators, we must have

\[
0 = \Delta(\tau, y_b) \left[ h'_0(y_b) \left\{ \phi(\tau, y_b) [\delta_S - f \delta_L] - \delta_L f \right\} - \mu \frac{\partial}{\partial y_b} \phi(\tau, y_b) \right] \\
+ \mu h'_0(y_b) \left\{ y(\tau, y_b) - c + \zeta f E'f - (r + \zeta) E(\tau, y_b) \right\} + m(\phi(\tau, y_b)) \left\{ f D_s(\tau, y_b) + (1-f) D_l(\tau, y_b) - 1 \right\}
\]

(A.38)

Plugging in, we see that \( f \) cancels out:

\[
\frac{-h'_0(y_b) \phi [\delta_S - \delta_L] + \delta_L - m(\phi)}{h'_0(y_b) \left\{ \phi(\tau, y_b) [\delta_S - \delta_L] + \delta_L - h'_0(y_b) \cdot m(\phi(\tau, y_b)) \right\}} \Delta(\tau, y_b) f \\
= \Delta(\tau, y_b) \left[ h'_0(y_b) \left\{ \phi(\tau, y_b) [\delta_S - \delta_L] - \mu \frac{\partial}{\partial y_b} \phi(\tau, y_b) \right\} \\
+ h'_0(y_b) \left\{ y(\tau, y_b) - c + \zeta f E'f - (r + \zeta) E(\tau, y_b) \right\} + m(\phi(\tau, y_b)) \left\{ f D_s(\tau, y_b) + (1-f) D_l(\tau, y_b) - 1 \right\} \right]
\]

(A.39)
Let us take the derivative with respect to $\tau$ of the RHS only, noting that the LHS is identically 0 across $\tau$ as long as we have an interior equilibrium. We then have

$$
0 = \left[ h'_0 (y_b) \delta_S \phi (\tau, y_b) - \mu \frac{\partial \phi (\tau, y_b)}{\partial y_b} \right] \frac{\partial \Delta (\tau, y_b)}{\partial \tau} + \Delta (\tau, y_b) \left[ h'_0 (y_b) \delta_S \frac{\partial \phi (\tau, y_b)}{\partial \tau} - \mu \frac{\partial^2 \phi (\tau, y_b)}{\partial y_b \partial \tau} \right] - \mu \frac{\partial^2 E (\tau, y_b)}{\partial y_b \partial \tau} + h'_0 (y_b) \left\{ \frac{\partial u (\tau, y_b)}{\partial \tau} + m (\phi (\tau, y_b)) \frac{\partial D_L (\tau, y_b)}{\partial \tau} \right\} - (r + \zeta) \frac{\partial E (\tau, y_b)}{\partial \tau} \tag{A.40}
$$

where bold-face functions indicate (linear) functions of \textit{contemporaneous} $f$. Plugging in for the bold-face functions, dropping $(\tau, y)$ for brevity, we have

$$
0 = \left[ h'_0 (y_b) \delta_S \phi - \mu \frac{\partial \phi}{\partial y_b} \right] \frac{\partial \Delta}{\partial \tau} + \Delta \left[ h'_0 (y_b) \delta_S \left[ f \{ -m (\phi) \} + \delta_S \phi \right] - \mu \left[ f \left\{ (\delta_L - \delta_S) \frac{\partial \phi}{\partial y_b} \right\} + \delta_S \frac{\partial \phi}{\partial y_b} \right] \right] - \mu \left[ \frac{\partial u}{\partial y_b} + (\delta_S - \delta_L) [D_L - 1] \frac{\partial \phi}{\partial y_b} \right] + m (\phi) \frac{\partial D_L}{\partial y_b} - (r + \zeta) \frac{\partial E}{\partial y_b} + h'_0 (y_b) \left\{ \frac{\partial u}{\partial \tau} + m' (\phi) [D_L - 1] \left[ f \{ -m (\phi) \} + \delta_S \phi \right] - (r + \zeta) \frac{\partial E}{\partial \tau} \right\} \tag{A.41}
$$

where we left terms multiplying $f$ bold-face. Gathering terms as

$$
0 = \text{(numerator)} - \text{(denominator)} f \iff f = \frac{\text{(numerator)}}{\text{(denominator)}} \tag{A.42}
$$

we have

$$
\text{denominator} = m (\phi) \left[ h'_0 (y_b) \left\{ \delta_S \phi (\tau, y_b) - (\delta_S - \delta_L) (1 - D_L) \right\} + \mu \frac{\partial \Delta}{\partial y_b} \right]
$$

$$
\text{numerator} = \left[ h'_0 (y_b) \delta_S \phi - \mu \frac{\partial \phi}{\partial y_b} \right] \frac{\partial \Delta}{\partial \tau} + \Delta \delta_S \left[ h'_0 (y_b) \delta_S \phi - \mu \frac{\partial \phi}{\partial y_b} \right] - \mu \left[ \frac{\partial u}{\partial y_b} + (\delta_S - \delta_L) [D_L - 1] \frac{\partial \phi}{\partial y_b} \right] + m (\phi) \frac{\partial D_L}{\partial y_b} - (r + \zeta) \frac{\partial E}{\partial y_b} + h'_0 (y_b) \left\{ \frac{\partial u}{\partial \tau} + m' (\phi) [D_L - 1] \delta_S \phi + m (\phi) \frac{\partial D_L}{\partial \tau} \right\} \tag{A.43}
$$
Thus, by linearity we have a unique candidate \( f_r \). The bold terms feature contemporaaneous \( f \) that is linear in all cases:

\[
m(\phi) = \delta_L + \phi(\delta_S - \delta_L) \tag{A.44}
\]

\[
y(\tau, y_b) = \begin{cases} 
y_0 + \mu \tau & \text{linear} \\
y_0 e^{\mu \tau} & \text{exponential} \end{cases} \tag{A.45}
\]

\[
\frac{\partial}{\partial y_b} y(\tau, y_b) = \begin{cases} 
1 & \text{linear} \\
\mu y_b e^{\mu \tau} & \text{exponential} \end{cases} \tag{A.46}
\]

\[
\frac{\partial \phi}{\partial \tau}(\tau, y_b) = e^{\xi [\delta_L (1-f_s) + f_s \delta_L \delta_S]_b} \left[ \Phi(\phi) - \delta_L \int_0^\tau e^{-\xi [\delta_L (1-f_s) + f_s \delta_L \delta_S]_b} f_s ds \right] \tag{A.48}
\]

Note that the interior equilibrium path is unique for any ultimate bankruptcy state \((\Phi(y_b) , y_b)\) as it is stems from a linear equation. Thus, suppose that \( f_{r=0} \in \{0, 1\} \). Then we know that \( IC_r (0, y_b) \geq 0 \) and \( f_{r=0} \) stays cornered until a time \( \tau \) at which \( IC_r (\tau, y_b) = 0 \). Suppose \( f_{r=0} \in (0, 1) \). Then immediately we have, by Lemma 4, as \( f \) is continuous that the above determines the path of \( f \) uniquely as it is a linear equation, until a time \( \tau \) at which \( f \) becomes cornered. In this case, then, \( IC \) starts diverging from 0 and again \( f \) is uniquely determined by the sign of \( IC \). They key step here is to note that \( IC \) is continuous by the functions involved and by the continuity of \( f \).

**Proof of Proposition 6.** Uniqueness of \( f_{r=0} \) on the default boundary follows from Lemma 5. Continuity of \( \phi \) follows from Lemma 4. The existence of unique paths leading away from any admissible boundary point is established by
Lemma 6. Finally, for cornered equilibria, the fact that they stay cornered for some distance away from the boundary implies that paths cannot cross, and additionally we know that \( \phi \) has a bounded rate of change. Thus, as Proposition 5 showed, the equilibrium stays unique for some distance away from the boundary for \( f_{\tau=0} \in \{0, 1\} \) with the restriction on \( f \) strictly binding. ■

### A.4.1 Welfare

The total value of the firm is given by both (35) and

\[
V = E + \phi D_S + (1 - \phi) D_L
\]

(A.63)

as there is no claimants to the cash-flow stream here besides debt and equity. Suppose that for the equilibrium we are investigating, we have \( \tau = T_b(\phi, y) \) as the time-to-default. It is easy to show that for any cornered strategy we have \( \frac{\partial T}{\partial \phi} < 0 \) by \( \Psi'(y_b) > 0 \) (for non-cornered strategies, this does not have to hold as \( f \) is free to adjust). We thus have, by value equivalence,

\[
V(\phi, y) = E(\phi, y) + \phi D_S(\phi, y) + (1 - \phi) D_L(\phi, y) = V(T_b(\phi, y), y)
\]

Whether \( f \) is socially optimal depends on if \( V_\phi \) has the appropriate sign. Taking derivatives w.r.t. \( \phi \), we have

\[
V_\phi(\phi, y) = E_\phi + \Delta + \left[ \phi \frac{\partial D_S}{\partial \phi} + (1 - \phi) \frac{\partial D_L}{\partial \phi} \right] = V_T(T_b(\phi, y), y) \left| \frac{\partial \tau}{\partial \phi} \right|_{\tau=T_b}
\]

(A.64)

There is some caution warranted here – away from \( \tau = 0 \), the derivatives of the debt valuations w.r.t. \( \phi \) have to include changes in the policy functions \( \frac{\partial f(s)}{\partial \phi} \) for \( s \leq \tau \) unless we are looking at corner paths only. Suppose we are looking at a cornered equilibrium with \( f = 1 \). Then we know we must have \( IC \geq 0 \). Next, when evaluating at \( \tau = 0 \), we have \( IC = 0 \) and thus the sign of \( V_\phi \) and social optimality of \( f \) is determined by the sign of

\[
\left[ \phi \frac{\partial D_S}{\partial \phi} + (1 - \phi) \frac{\partial D_L}{\partial \phi} \right]
\]

(A.65)

Suppose that on the boundary \( \frac{\partial D_S}{\partial \phi} > \frac{\partial D_L}{\partial \phi} > 0 \) so \( f_{\tau=0} = 1 \) is both an equilibrium and locally socially optimal. By \( f \) being cornered, we also know that \( f = 1 \) even for slight changes to \( \phi \), and by continuity of the value functions \( \left[ \phi \frac{\partial D_S}{\partial \phi} + (1 - \phi) \frac{\partial D_L}{\partial \phi} \right] > 0 \) for some time even away from the boundary. This of course implies that

\[
V_T(T_b(\phi, y), y) = \frac{IC(\phi, y) + \phi \frac{\partial D_S}{\partial \phi} + (1 - \phi) \frac{\partial D_L}{\partial \phi}}{\left| \frac{\partial \tau}{\partial \phi} \right|_{\tau=T_b}} < 0
\]

Thus, any shortening-only equilibrium has, at least in the vicinity of the default boundary, a local maximum to the left. Note here the divergence of private incentives (which feature the full choice \( f \) of the proportions of short versus long in the infinitesimal period) and the social incentives (which have to deal with aggregate proportions \( \phi \) of short versus long debt when \( f \) is changed).

### A.5 Extensions

#### A.5.1 Exogenous default boundary

In the main text we follow the Leland tradition by assuming that equity holders either have deep pockets or can issue equity in a frictionless fashion, so that the default boundary is determined endogenously once the equity’s option value of keeping the firm alive turns zero. The smooth-pasting condition \( E'(\Phi) = 0 \) ensues, which implies a zero \( IC \) condition \( \Delta(\Phi; y) + E'(\Phi; y) = 0 \) at default. Consequently, we need to go one order of derivative higher to sign the \( IC \) condition.

Suppose instead that equity holders are forced to default before they are willing to; this can happen for liquidity reasons if equity holders do not have deep pockets, or financial markets become illiquid due to information-driven problems (cite some recent liquidity papers). Say that the default boundary is \( \hat{\Phi} \) with \( E\left(\hat{\Phi}\right) = 0 \); and equal seniority implies a zero debt price wedge \( \Delta\left(\hat{\Phi}\right) = 0 \). But, we must have \( E'(\hat{\Phi}) < 0 \) as equity holders always have the option to default earlier than \( \hat{\Phi} \); the fact that they hang on during the process \( \phi \uparrow \hat{\Phi} \) implies that \( E(\phi) > E\left(\hat{\Phi}\right) = 0 \) for \( \phi < \hat{\Phi} \). Therefore \( \Delta\left(\hat{\Phi}\right) + E'(\hat{\Phi}) < 0 \) right before default, and equity holders always want to issue long-term bonds.
becomes irrelevant. As explained in footnote 15, it is because in shortening equilibria the firm is issuing short-term to equity. It turns out that only the price of short-term debt As suggested in (12), the key IC condition compares the pricing wedge to the long-run impact of maturity shortening to equity. It turns out that only the price of short-term debt $D_S$ matters in Corollary 1; the long-term debt price becomes irrelevant. As explained in footnote 15, it is because in shortening equilibria the firm is issuing short-term bonds only, i.e., $f$ is cornered to $f = 1$ given the allowable set of $[0, 1]$.

The assumption of $f \in (0, 1]$ might be violated, as firms can repurchase bonds, or may face certain covenants restricting the firm to reissue certain long-term bonds at minimum. We hence modify the allowable set for the fraction of newly short-term bonds to be $f \in [f_1, f_h]$. Under this assumption, in shortening equilibria the firm takes the highest fraction $f_h$, which can be either below 1 so that the firm is issuing some mixture of short-term and long-term bonds, or above 1 to accommodate repurchases. Now we show that our result in Corollary 1 holds in this relaxed setting. We require the firm’s maturity structure is shortening at the hypothetical default point $\Phi$, i.e., we must have $\phi$ increasing

\[
\frac{d\phi}{dt} \bigg|_{\phi=\Phi} = -\Phi \delta_S + m(\Phi) f_h > 0.
\]

Equity solves

\[
\begin{align*}
    r E (\phi; y) &= y - c + \zeta \left[ E^{y} - E (\phi; y) \right] \\
    &+ \max_{f \in [f_1, f_h]} \left\{ m(\phi) \left[ f D_S (\phi; y) + (1 - f) D_L (\phi; y) - 1 \right] + [-\phi \delta_S + m(\phi) f] E (\phi; y) \right\}.
\end{align*}
\]

Assuming $f = f_h$ is optimal, we have

\[
\begin{align*}
    r E (\phi; y) &= y - c + \zeta \left[ E^{y} - E (\phi; y) \right] \\
    &+ [\phi \delta_S + (1 - \phi) \delta_L] [f_h D_S (\phi; y) + (1 - f_h) D_L (\phi; y) - 1] \\
    &+ [-\phi \delta_S + (\phi \delta_S + (1 - \phi) \delta_L) f_h] E (\phi; y).
\end{align*}
\]

Taking derivatives w.r.t. $\phi$, we have

\[
\begin{align*}
(r + \zeta) E' (\phi; y) &= (\delta_S - \delta_L) [f_h D_S (\phi; y) + (1 - f_h) D_L (\phi; y) - 1] \\
    &+ [\phi \delta_S + (1 - \phi) \delta_L] [f_h D_S' (\phi; y) + (1 - f_h) D_L' (\phi; y)] \\
    &+ [-\phi \delta_S + (\phi \delta_S + (1 - \phi) \delta_L) f_h] E' (\phi; y) \\
    &+ [-\phi \delta_S + (\phi \delta_S + (1 - \phi) \delta_L) f_h] E'' (\phi; y)
\end{align*}
\]

Since $E = 0, E' = 0$ at $\Phi$, and equal seniority $D_S = D_L = B (y)$, we have

\[
0 = (\delta_S - \delta_L) [B (y) - 1] + m(\Phi) [f_h D_S' (\phi; y) + (1 - f_h) D_L' (\phi; y)] + [-\phi \delta_S + m(\Phi) f_h] E'' (\phi; y).
\]
Rearranging, we have
\[ E''(\phi, y) = \frac{(\delta_S - \delta_L) [1 - B(y)] - m(\Phi)}{-\phi \delta_S + m(\Phi)} f_h \left[ f_h D'_S(\phi; y) + (1 - f_h) D'_L(\phi; y) \right] \]
Recall that for shortening we require \(-\phi \delta_S + m(\Phi) f_h > 0\), which rules out \(f_h < 0\). Next, note that under this assumption we have
\[ \Delta'(\Phi; y) = -\frac{(\delta_S - \delta_L) [1 - B(y)]}{-\phi \delta_S + m(\Phi)} < 0 \]
Plugging in, we have
\[ \Delta'(\Phi; y) + E''(\phi; y) = -\frac{m(\Phi)}{-\phi \delta_S + m(\Phi)} f_h \left[ f_h D'_S(\phi; y) + (1 - f_h) D'_L(\phi; y) \right] \]
From the bond ODE we have
\[ D'_L(\Phi; y) = -\frac{(r + \delta_L + \zeta) [1 - B(y)]}{-\phi \delta_S + m(\Phi)} f_h \]
\[ f_h \Delta'(\Phi; y) + D'_L(\Phi; y) = -\frac{[f_h (\delta_S - \delta_L) + r + \delta_L + \zeta) [1 - B(y)]}{-\phi \delta_S + m(\Phi)} f_h \]
so that finally
\[ IC_{\phi}(\Phi) = \Delta'(\Phi; y) + E''(\phi; y) = -\frac{m(\Phi)}{-\phi \delta_S + m(\Phi)} f_h \left[ (f_h \delta_S + (1 - f_h) \delta_L + r + \zeta) [1 - B(y)] \right] \]
We need \(IC_{\phi}(\Phi) < 0\) as we want \(IC(\Phi - \varepsilon) > 0\) as this implies \(f = 1\) at \(\Phi - \varepsilon\). Using the approximation \(IC(\Phi - \varepsilon) \approx IC(\Phi) - IC_{\phi} \varepsilon > 0\), we see that we need so only when \(f_h\) is so negative
\[ [(f_h \delta_S + (1 - f_h) \delta_L + r + \zeta)] < 0 \iff f_h < -\frac{r + \delta_L + \zeta}{\delta_S - \delta_L} \]
the firm might choose the highest \(f_h\) and default slowly. But when \(f_h < 0\), the firm is repurchasing back short-term debt!

A.5.4 Allowing for deleveraging
We know that a proportion \(m(\phi)\) of bond is maturing every instant. Suppose that a portion of \((1 - \alpha)\) of maturing debt is (forcibly) retired, with \(\alpha \in (0, 1)\). Then overall face value \(F_t\) is dynamically changing according to
\[ dF_t = -m(\phi_t) (1 - \alpha) F_t dt \]
whereas the amount of short-term debt changes according to
\[ dS_t = [-\delta_S S_t + f \cdot am(\phi_t) F_t] dt \]
Thus, the maturity structure changes according to
\[ d\phi = \frac{dS}{F} - \frac{S}{F} dF = [-\phi \delta_S + f \cdot am(\phi) - \phi (-m(\phi) (1 - \alpha))] dt \]
\[ = [-\phi \delta_S + f \cdot \alpha + \phi (1 - \alpha)] m(\phi) dt \]
\[ = \mu_{\phi}(f) dt \]
which is not a function of \(F\), but only of \(\phi\) and \(\alpha\).

The dynamic equation for equity value, i.e., the HJB, is given by
\[ (r + \zeta) E = y - (1 - \pi) cF + \zeta E^f(F) + m(\phi) F [\alpha (f D_S + (1 - f) D_L) - 1] + \mu_{\phi}(f) E'(\phi) \]
The FOC w.r.t. \(f\) is then given by
\[ \alpha \cdot m(\phi) \max_{f \in [0, 1]} f \{ F (D_S - D_L) + E_{\phi} \} = \alpha \cdot m(\phi) \max_{f \in [0, 1]} f \{ F \cdot \Delta + E_{\phi} \} \]
Note that the upside event is given by $E^{\uparrow} = X - F$ so that

$$(r + \zeta)E = y - (1 - \pi) cF + \zeta [X - F] + m(\phi) F [\alpha (fD_S + (1 - f) D_L) - 1] + \mu_F (f) E_F (F, \phi) + \mu_F (\phi) E_F (F, \phi)$$

where $(\phi, F)$ are the state variables. Suppose that $y$ is constant.

Then, we change variables in that we take $F_0$ as the default point and $(\tau, F_0)$ as the path towards this default point. Then we have

$$(r + \zeta) E(\tau, F_0) = y - (1 - \pi) cF(\tau, F_0) + \zeta [X - F(\tau, F_0)] + m(\phi) F(\tau, F_0) (\alpha [fD_S (\tau, F_0) + (1 - f) D_L (\tau, F_0)] - 1) - E_r (\tau, F_0)$$

The default boundary $\Phi (F_0)$ is given by $E = E_r = 0$ which yields

$$0 = y - (1 - \pi) cF_0 + \zeta (X - F_0) + m(\Phi) [\alpha B (y) - F_0]$$

which yields

$$\Phi (F_0) = - y - (1 - \pi) cF_0 + \zeta (X - F_0) + \delta_L [\alpha B (y) - F_0]$$

Taking partial derivatives w.r.t. $\phi$ and note that $F_0 (\tau, F_0) = 0$ so that

$$E_{r\phi} (\tau, F_0) = m'(\phi) F(\tau, F_0) (\alpha [fD_S (\tau, F_0) + (1 - f) D_L (\tau, F_0)] - 1) + m(\phi) F(\tau, F_0) \alpha \left[ f \frac{\partial D_S (\tau, F_0)}{\partial \phi} + (1 - f) \frac{\partial D_L (\tau, F_0)}{\partial \phi} \right] - (r + \zeta) E_{\phi} (\tau, F_0)$$

Note that at default $\tau = 0$, we have by equal seniority and smooth pasting $IC (\tau, F_0) = F \cdot 0 + 0 = 0$, so that we need $IC_r (\tau, F_0) > 0$ for a shortening equilibrium. Taking derivatives w.r.t. $\tau$, then

$$IC_r = F_r \Delta + F_{rr} + E_{r\phi}$$

$$= F_r \cdot 0 + F_0 (\delta_S - \delta_L) \left[ 1 - \frac{B(y)}{F_0} \right] + (\delta_S - \delta_L) F_0 \left( \alpha \frac{B(y)}{F_0} - 1 \right) + m(\phi) F_0 \alpha \left[ f \frac{\partial D_S (\tau, F_0)}{\partial \phi} + (1 - f) \frac{\partial D_L (\tau, F_0)}{\partial \phi} \right]$$

Thus, conjecturing $f = 1$, we need $IC_r > 0$ at default $\tau = 0$, so that

$$0 < - (\delta_S - \delta_L) (1 - \alpha) B(y) + m(\phi) F_0 \alpha \left[ \frac{\partial D_S (0, F_0)}{\partial \phi} + \frac{\partial D_L (0, F_0)}{\partial \phi} \right]$$

We immediately see the first term is always negative. We note that bonds per unit of face-value are simply priced according to

$$rD_\iota (\tau, F_0) = c + \delta_\iota [1 - D_\iota (\tau, F_0)] + \zeta [1 - D_\iota (\tau, F_0)] - D_r D_\iota (\tau, F_0)$$

which is solved by

$$D_\iota (\tau, F_0) = \frac{c + \delta_\iota + \zeta}{r + \delta_\iota + \zeta} + e^{-(r + \delta_\iota + \zeta) \tau} \left[ \frac{B(y)}{F_0} - \frac{c + \delta_\iota + \zeta}{r + \delta_\iota + \zeta} \right]$$

which gives, assuming $c = r$

$$\frac{\partial}{\partial \tau} D_\iota (\tau, F_0)_{\tau = 0} = (r + \delta_\iota + \zeta) \left[ 1 - \frac{B(y)}{F_0} \right] > 0$$

$$\frac{\partial}{\partial F_0} D_\iota (\tau, F_0)_{\tau = 0} = - \frac{B(y)}{(F_0)^2} < 0$$

Finally we have, at the default boundary,

$$\frac{\partial \tau}{\partial \phi} = < 0$$

$$\frac{\partial F_0}{\partial \phi} = > 0$$

The explanation is straightforward: as we shorten the maturity structure, i.e. increase $\phi$, default becomes more imminent and the time-to-default $\tau$ decreases (i.e., for the conjectured shortening path we move closer to the default
boundary); but, earlier default also implies that the overall face-value at default, \( F_0 \), increases as we have less time until default to retire debt. Putting these inequalities together, we have \( IC_\tau |_{\tau=0} < 0 \), and a shortening equilibrium is impossible.

We note that nothing in the proof that operates at the default boundary depended on \( \alpha \) being exogenous instead of endogenous.

**A.5.5 Extensions of Section 4**

Suppose that the firm borrows from another group of debt holders holding consol bonds with coupon \( c_{\text{consol}} \), a la Leland (1994), which is absent from rollover concerns. To make the analysis stark and simple, we assume that these consol bonds get zero payment in both the upper and the default events.\(^{29}\) As a result, the valuation formula for the long-term and short-term bonds remain identical. The equity holder’s problem remains almost the same, with the only adjustment of an additional coupon outflow of \( c_{\text{consol}} \). The default boundary becomes

\[
\Phi (y_b) = \frac{1}{\delta_S - \delta_L} \left[ y_b - c - c_{\text{consol}} + \zeta E^r y \right] - \delta_L ,
\]

which affects the endogenous time-to-default \( \tau \). The value of consol bonds, denoted by \( D_{\text{consol}} \), is given by

\[
D_{\text{consol}} (\tau, y_b) = \frac{\rho c_{\text{consol}}}{r + \zeta} \left[ 1 - e^{-(r+\zeta)\tau} \right],
\]

with \( \frac{\partial}{\partial \phi} D_{\text{consol}} (\phi, y) \bigg|_{r=0} = \rho c_{\text{consol}} \frac{\partial \tau}{\partial \phi} < 0 \). Intuitively, shortening maturity structure leads to an earlier default and hence a lower value of consol bonds.

Now the firm value includes the value of consol bonds. As before, we can decompose the local effect of maturity shortening on the firm value, i.e., \( V_\phi (\phi, y) \), into

\[
V_\phi (\phi, y) = E_\phi (\phi, y) + \Delta (\phi, y) + \phi \frac{\partial}{\partial \phi} D_S (\phi, y) + (1 - \phi) \frac{\partial}{\partial \phi} D_L (\phi, y) + \frac{\partial}{\partial \phi} D_{\text{consol}} (\phi, y).
\]

The last negative term is increasing in \( c_{\text{consol}} \) and may dominate the second positive term in a maturity-shortening equilibrium, leading to \( V_\phi (\Phi (y_b), y_b) < 0 \).

\(^{29}\)Zero recovery in the default event can be justified by the assumption that the consol bonds are junior to the term bonds we analyzed so far.