Abstract
This paper characterizes the equilibrium demand and risk premiums in the presence of skewness risk. In a model with a single time period, we extend the classical mean-variance two-fund separation theorem to a three-fund separation theorem. The additional fund is the skewness portfolio, i.e., a portfolio that gives the optimal hedge of the squared market return; it contributes to the skewness risk premium through co-variation with the squared market return and supports a stochastic discount factor that is quadratic in the market return. When the skewness portfolio does not replicate the squared market return, a tracking error appears; this tracking error contributes to risk premiums through kurtosis and pentosis risk. In addition to the common powers of market returns, this tracking error shows up in stochastic discount factors as priced factors that are products of the tracking error and market returns. Extending the one-period to a two-period framework introduces an additional volatility-based risk factor in the stochastic discount factor as well as a hedging demand in individual asset allocations due to volatility risk.

Keywords
skewness, portfolio, aggregation, stochastic discount factor, polynomials of pricing factors, market return

JEL Classification
D52, D53, G11, G12
1 Introduction

Empirical studies documented that securities’ returns are not normally distributed; the seminal work of Harvey and Siddique (2000) showed that skewness risk is an important component in the risk-premium. This renewed interest in the compensation of skewness risks and led to an active literature stream\(^1\). This stream typically assumes that the aggregation of preferences leads to stochastic discount factors that are polynomials in the market return, conditional on the current information set. However, it is well known that the aggregation of preferences in incomplete markets leads to a representative agent where the weight of each individual is stochastic; thus, the stochastic discount factor may actually depend on individual securities due to unspanned powers of the market return. We study in detail the aggregation of preferences with skewed returns.

Our paper proceeds through two steps. Although the focus of our analysis is on aggregation of (heterogeneous) investors, our first step studies the case of a representative agent. We show that the common practice of using polynomials in the market return is warranted as long as the representative investor is myopic, i.e. as long as she only cares about investing over the next time period; in line with this, we link our results to the beta pricing relationships proposed by Harvey and Siddique (2000). However, when the representative investor derives utility from terminal wealth by investing over two time periods, we find that that assets’ risk premiums over the first period also depend on its (intertemporal) co-variation with the (conditional) market variance over the second period; this can be seen as a form of co-market volatility. This additional co-variation extends the usual variance-skewness pricing relationships in the literature and shows up in the stochastic discount factor as a priced factor that is in addition to polynomials of the market return.

Our major second step studies equilibrium risk premiums and demand with heterogeneous investors. With myopic investors, i.e. investors that care only about the current time period, we derive individual demand and prove a three-fund separation theorem: agents hold the risk-free asset, the market portfolio and a new, so-called skewness portfolio, in proportions that reflect

\(^{1}\)For example, Dittmar (2002) and Barone-Adesi et al. (2004) study how skewness risk is priced; Chung et al. (2006) test whether Fama-French factors proxy for skewness and higher moments; Engle (2011) links the skewness that shows up in asymmetric volatility models to systemic risk; Boyer and Vorkink (2013) examine the impact of skewness preference on option prices.
their preferences for variance and skewness risk. The skewness portfolio is the portfolio that provides the optimal hedge to the squared market return. We show that an asset’s skewness risk is priced as long as the co-skewness of this asset with the market portfolio as well as the aggregate skew-tolerance do not vanish. The equilibrium contribution to the risk-premium of individual stocks is driven by their covariance with the squared market return; it matches formally the risk-premium contribution in the single period, representative investor case and supports the common practice of using a stochastic discount factor (SDF) that is quadratic in the market return.

To shed further light on the co-leverage (market volatility) factor that was priced in the two period case with a representative investor, we extend the previous analysis and study the equilibrium allocation with heterogeneous investors that maximize terminal wealth derived from investing over two periods of time. We prove a four-fund separation theorem over the first period: in addition to the risk-free asset, the market and the skewness portfolio, investors hold a new, so-called leverage portfolio, in proportions that reflect their preferences for variance/skewness risk and for wealth induced (intertemporal) changes in the risk tolerance. The leverage portfolio is the portfolio (over the first time period) that provides the optimal hedge to the (conditional) variance of the market return over the second period; investors’ holdings can be interpreted as a form of the well-known intertemporal hedge demand term that is common in portfolio selection models. The equilibrium contribution to the risk-premium depends, among others, on the co-variation of individual stocks with the (conditional) market variance over the next period. It matches formally the risk-premium contribution in the single period, representative investor case; this additional co-variation with market volatility extends the usual variance-skewness pricing relationships in the literature.

The remainder of our analysis focuses on the single period case with heterogeneous investors. A tracking error shows up in hedging the squared market return with the available securities; beyond average skew-tolerance, this leads to an additional lower order contribution to the risk premium through kurtosis due to the so-called cross-sectional variance of investor’s skew-tolerances. Put differently, there may be an additional priced factor that contributes to skewness risk: the product of the market return with the tracking error in the squared market return. Another tracking error shows up in hedging the cubed market return with the available securities; beyond
average skew-tolerance, this leads to an additional lower order contribution to the risk premium through pentosis (the fifth moment) due to the cross-sectional variance and the so-called cross-sectional skewness of investor’s skew-tolerances.

Our paper makes several contributions. First, we provide a structural analysis of individual and aggregate demand functions, as well as equilibrium risk premiums and associated portfolio holdings. We revisit the seminal contribution of Samuelson (1970), see also the small risk analysis in Gollier (2001), but depart from it in two directions: for one, we allow for a richer pattern of risk premiums in order to analyze the pricing of skewness risk; in addition, we do not intend to interpret his small noise expansion parameter as we are interested only in structural properties of risk-premiums and portfolio holdings. Thereby we confirm Samuelson (1970) for myopic investors: the classical CAPM two fund separation theorem continues to hold as a first approximation; in addition we show that agents add the skewness portfolio in a second order approximation and that higher order approximations lead to lower order contributions of skewness risk through tracking errors.

Second, we point out in two-period extension that the intertemporal hedge terms of non-myopic investors will affect demand through the leverage portfolio and risk-premiums through an additional priced factor; the latter shows up as a co-market volatility effect. This fits into a recent literature that considers the feedback effect on pricing of (intertemporal) changes in the investment opportunity set: for example, in presence of long dated assets, i.e. assets with payoffs that are far in the future, Martin (2012) uses the SDF over multiple periods to show that the pricing of a broad class of these assets is driven by the possibility of extraordinarily bad news.

Finally, we clarify the common practice of studying SDF that are polynomials in the market return. Most empirical studies looked at skewness extensions of the CAPM which add the squared market return as a factor; they justify this extension on ad-hoc Taylor-series expansions for agents’ utility functions that are truncated at the third-order term, see, e.g., Kraus and Litzenberger (1976), Barone-Adesi (1985), Dittmar (2002). We provide a rigorous foundation for

---

2 Structural properties of the market demand functions have been studied, e.g. by Hildenbrand (1983) and Grandmont (1992). We focus on a portfolio choice problem.

3 An example in Martin (2012), page 349, Figure 1 illustrates this point when the investment horizon is t=250 years. Our framework consider only implication of heterogeneity in one-period (when t=0,1) and two-period models (when t=0,1,2). To shed light on Martin’s finding within our framework, one has to look at the implication of our model for many more than two time-periods. This is beyond the scope of this paper and merits scrutiny in future research.
this common practice and show that it is warranted with myopic investors in complete markets, that is when investors only care about one single time-period and the squared market return can be hedged perfectly. When investors are not myopic or markets are incomplete, however, we show that several additional priced factors come up that depend on the cross-sectional variance and skewness of investors’ skew-tolerances.

Our structural result in single-period cases unifies several theoretical insights. First of all, our results match previous ones about incomplete markets that investors’ heterogeneity and in particular the cross-sectional variance of investors’ characteristics may matter in equilibrium pricing, see, e.g., Constantinides and Duffie (1996). In addition, we show that aggregation of skewness risk for pricing is far more complicated than thought: the standard approach of studying polynomials of market returns (based on a postulated representative agent) misses several additional, priced factors that are products of market returns with tracking portfolios. Finally, it is well known about incomplete markets that the utility functions aggregate into a single so-called representative agent through stochastic weights, see, e.g. Magill and Quinzii (1996); our structural result therefore has the interpretation that these stochastic weights matter for pricing. Our structural analysis also provides an explanation for recent empirical evidence: Mitton and Vorkink (2007) and Boyer and Vorkink (2013) document for stocks and options, respectively, that more than co-skewness with the market is priced in stock returns; our unspanned factors provide a theoretical foundation for this.

The outline of our paper is as follows. The next section discusses the general framework and relevant concepts. The third section studies the representative investor case, while the following sections study heterogeneous investors. Section 4 discusses demand and the portfolio separation theorem in the presence of skewness risk as well as the pricing of skewness risk. Finally, section 5 shows that the aggregation in incomplete markets leads to several risk factors that are unspanned by polynomial SDFs in the market return. The paper concludes with section 6. All proofs are postponed to the appendix: appendix A considers the proofs in the single-period case; appendix B provides the proofs in the two-period case and is intended as an online appendix.
2 General return framework

This section introduces the general framework and the relevant concepts for understanding the market prices of higher order moments. Throughout, we consider a scale-location model for the vector $R = (R_i)_{1 \leq i \leq n}$ of returns on the $n$ risky assets of interest on a given period of time. More precisely, there exists a random vector $Y = (Y_i)_{1 \leq i \leq n}$ with zero mean vector under some given joint probability distribution, such that for $\sigma > 0$:

$$R = E(R) + \sigma Y,$$

where $E(R)$ is the expectation of $R$. We always assume that the vector $Y$ admits finite moments at any required order and that the variance matrix $Var(Y)$ is positive and symmetric. Note that return volatilities are determined by the scale parameter $\sigma$ through

$$Var(R) = \sigma^2 \Sigma, \text{ with } \Sigma = Var(Y).$$

Since our interest focuses on (the cross section of) risk premiums, we introduce a risk free return $R_f$ and decompose expected returns as follows:

$$E(R_i) = R_f + \pi_i(\sigma), i = 1, \ldots, n;$$

here, $\pi_i(\sigma)$ denotes the risk premium on asset $i$. This risk premium must obviously depend on the scale of risk $\sigma$ and we will always assume that it is a smooth function of $\sigma$. In particular, when $\sigma$ goes to zero, all asset volatilities go to zero and we expect risk premiums doing the same. Extending by continuity the risk premium function, we will assume throughout that:

$$\pi_i(0) = 0, i = 1, \ldots, n.$$

We did not define our financial markets for $\sigma = 0$, since we would end up with $(n+1)$ risk-free assets. However, it makes sense to study the behavior of the risk premium function in a right neighborhood of zero, involving (by continuity extension) the value of the function $\pi_i(\cdot)$ as well as of its derivatives at $\sigma = 0$. Smoothness properties of risk premiums functions will be implied by assumed smoothness properties of investors’ utility functions at stake. Let us assume for the moment that some investor with utility function $u$ chooses her portfolio as solution of the first order conditions

$$E[u'(W(\sigma))(R_i - R_f)] = 0, \text{ for } i = 1, \ldots, n.$$  (2)
These optimality conditions (2) are very general when the random variable \(W(\sigma)\) stands for some wealth level produced by the portfolio choice of interest. These equations may stem from both static or dynamic portfolio optimization. We do not want to make this explicit at this stage, it is only important to assume that all randomness in \(W(\sigma)\) is produced by asset volatilities and that it is erased when the scale of risk \(\sigma\) goes to zero. It is worth realizing that, up to required smoothness and integrability conditions, this very general setting already implies that risk premiums have a zero (right-hand) derivative at \(\sigma = 0\):

**Lemma 1** If \(u'(W(0))\) is a deterministic non-zero variable, we have \(\pi'_i(0) = 0\) for all \(i = 1, ..., n\).

Lemma 1 is consistent with the general result (see e.g. Gollier (2001), p. 56) that the optimal share of wealth to invest in a risky asset is approximately proportional to the ratio of the expectation and variance of the excess return. Lemma 1 implies that risk aversion considered in this paper has a second-order nature, as defined by Segal and Spivak (1990). (First order risk aversion is beyond the scope of this paper.)

Based on Lemma 1, the scale parameter \(\sigma\) allows us to describe the pattern of risk premiums from their series expansions:

\[
\pi_i(\sigma) = \mathbb{E}(R_i) - R_f = \sum_{j=2}^{\infty} a_{ij} \sigma^j.
\]

Samuelson (1970) assumes that the above expansion involves only one term, i.e. he sets \(a_{ij} = 0\) for all \(j > 2\); this, however, appears overly restrictive for a thorough analysis of the price of risk coming from higher order moments beyond variance. In addition, Samuelson (1970) is interested in small scale parameters \(\sigma\); as he intends to motivate the use of mean-variance analysis, he also puts forward an interpretation in continuous time, where the scale parameter \(\sigma\) corresponds to a small interval of time \(\Delta t\). No such restrictions are involved in our setting defined by (1) and (3). This setting is actually general, as it only maintains smoothness assumptions about investors’ preferences that are necessary to ensure that risk premiums are analytical functions of the scale parameter \(\sigma\). We will actually set the focus in this paper on the first four coefficients \(a_{i2}, a_{i3}, a_{i4}\) and \(a_{i5}\), so that we even do not really assume that risk premiums are analytical functions. There are then two interpretations of our results: either we adopt the assumption of analytical functions and describe general properties of the market; or we may consider, under less
restrictive assumptions, that our results are accurate descriptions only for small $\sigma$, see Samuelson (1970) and more recently Judd and Guu (2001).

Since our analysis focuses on the role of investors’ preferences\footnote{Our focus is on the demand side of financial markets and should be completed when the offer of financial assets is also be seen as a function of the scale parameter $\sigma$. However, such an extension would simply complicate notation without modifying the economic interpretation of results.}, we assume throughout that only investor’s demand (may) depend on the scale of risk $\sigma$ but that the supply of financial assets is given independently of $\sigma$. In other words, a market portfolio is given by a vector $\xi$ of shares of total wealth invested in the $n$ risky assets. Since there is also a risk-free asset and no short sale constraints, the vector $\xi$ can be any vector in $\mathbb{R}^n$ with strictly positive entries. It defines the market return

$$R_M = \xi^\top R,$$

where we denoted by $\xi^\top$ the transpose of the vector $\xi$. Throughout this paper we will refer by the superscript $\top$ to the transpose of a vector.

From the extant literature on pricing of higher order moments (see Chung et al. (2006) and references therein), we expect the risk premium on any risky asset $i = 1, \ldots, n$, to be determined by the various contributions of this asset to market variance, skewness, kurtosis, pentosis, etc. To study this, we adopt:

**Definition 1** For asset $i = 1, \ldots, n$, the systematic co-moments are defined as follows:

$$b_{i,M} = \frac{1}{\sigma^2} \text{Cov}[R_i, (R_M - E(R_M))],$$
$$c_{i,M} = \frac{1}{\sigma^3} \text{Cov}[R_i, (R_M - E(R_M))^2],$$
$$d_{i,M} = \frac{1}{\sigma^4} \text{Cov}[R_i, (R_M - E(R_M))^3],$$
$$f_{i,M} = \frac{1}{\sigma^5} \text{Cov}[R_i, (R_M - E(R_M))^4].$$

We refer to $b_{i,M}, c_{i,M}, d_{i,M}$ and $f_{i,M}$ as the market beta, the market co-skewness, the market co-kurtosis and the market pentosis, respectively.

By comparison with standard notations, the definition of market moments and the asset co-moments standardizes by the scale of risk; thus they do not depend on $\sigma$. This standardization simplifies the presentation but does not change the main messages of the paper.
Note also that we set the focus on centered systematic co-moments and do not standardize by variance; this is different from, e.g., Chung et al. (2006). Our convention allows us to interpret the co-moments as shares of asset \( i \) in the aggregate corresponding moment namely respectively market variance \( V_M \), skewness \( S_M \), kurtosis \( K_M \) and pentosis \( P_M \):

\[
\sum_{i=1}^{n} \xi_i b_i, M = \frac{1}{\sigma^2} \text{Var} [R_M] = \text{Var} [(\xi^\perp Y)] = V_M,
\]

\[
\sum_{i=1}^{n} \xi_i c_i, M = \frac{1}{\sigma^3} E [(R_M - E(R_M))^3] = E [(\xi^\perp Y)^3] = S_M,
\]

\[
\sum_{i=1}^{n} \xi_i d_i, M = \frac{1}{\sigma^4} E [(R_M - E(R_M))^4] = E [(\xi^\perp Y)^4] = K_M,
\]

\[
\sum_{i=1}^{n} \xi_i f_i, M = \frac{1}{\sigma^5} E [(R_M - E(R_M))^5] = E [(\xi^\perp Y)^5] = P_M.
\]

From an economic perspective, it is worth pointing out that even if the market skewness or pentosis is zero, some specific asset may have a non-zero systematic component and that this may play a role in the determination of its market price.

Chung et al. (2006) study asset pricing implications of a model that “does not stop at the second moment” and considers up to the fourth moment. They stress that “there is no reason to stop with the fourth moment”, i.e. even higher order moment like pentosis may matter for asset pricing. They base their argument on the remark that “variance, skewness and kurtosis tell us something about the tails of the distribution, but they fall short of specifying the tail precisely”.

In this paper, our main interest is in the price of skewness; we will argue that when considering investors with heterogeneous preferences for skewness, an asset pricing model that does not want “to stop with the second moment” must take also into account some specific pricing factors due to preference heterogeneity. To put it differently, we will argue that the simple fact that investors care for higher order moments implies that they need to track not only the market return, but also the pricing factor \((R_M - E(R_M))^2\) that shows up through market co-skewness. We will see that this has a significant pricing impact in case of market incompleteness (there is no perfect hedge of the squared market return) jointly with investor heterogeneity. To see this, we will study terms in the series expansion (3) as far as \( a_{i4} \) or \( a_{i5} \) and the associated pricing of market co-kurtosis and market co-pentosis.
3 Pricing kernels and moment preferences in representative agent analysis

This section studies the pricing of (higher order) moments without taking into account complications due to preference heterogeneity: throughout this section, we assume that the representative investor has a known preference structure. This assumption will be questioned in the following sections through explicit discussions of aggregation issues.

Ultimately, our analysis with heterogeneous investors implies that the representative investor paradigm falls short of properly characterizing the impact of preferences for skewness on financial asset prices. However, before carrying out this analysis we first recall in this section the classical picture of risk premiums assigned to systematic co-moments in the case of a representative investor. The pricing formulas in the literature that result from a Taylor expansion of the utility function of a representative agent (see e.g. Harvey and Siddique (2000) and Dittmar (2002)) will find a more general interpretation in the context of our series expansions (3).

3.1 Case of a representative investor over one period

This subsection starts our analysis of pricing by studying a single period of time. Let us consider a representative investor who maximizes expected utility $E[u(W)]$ of her terminal wealth $W$ obtained by investing her initial wealth $q$ in a portfolio. Throughout, the portfolio is defined in terms of shares of wealth invested $\theta = (\theta_i)_{1 \leq i \leq n}$; this gives terminal wealth:

$$W(\sigma) = q \left[ R_f + \sum_{i=1}^{n} \theta_i (R_i - R_f) \right]. \quad (5)$$

The demand $\theta$ of the representative investor for risky assets solves the first order conditions (2) of her expected utility maximization where the wealth level $W(\sigma)$ is induced by the portfolio choice according to (5). Since the representative investor must hold the market portfolio, the first order conditions with $\theta = \xi$ determine the equilibrium pricing of the financial assets, i.e. the coefficients $a_{ij}, j \geq 2$ of the risk premium expansion (3); these coefficients solve for all levels
\( \sigma \) of risk:

\[
E [u'(W(\sigma))(R_i(\sigma) - R_f)] = 0 \text{ for } i = 1, \ldots, n, \tag{6}
\]

subject to \( W(\sigma) = q \left[ R_f + \sum_{i=1}^{n} \xi_i (R_i(\sigma) - R_f) \right] \),

where \( R_i(\sigma) - R_f = \pi_i(\sigma) + \sigma Y_i \) for \( i = 1, \ldots, n \).

We are then able to prove:

**Theorem 1** The equilibrium risk premium \( \pi_i(\sigma) = E(R_i) - R_f = \sum_{j=2}^{\infty} a_{ij} \sigma^j \) fulfills

\[
a_{i2} = \frac{1}{\tau} b_{i,M}, \quad a_{i3} = -\frac{\rho}{\tau^2} c_{i,M}, \quad a_{i4} = \frac{\kappa}{\tau^3} d_{i,M} + 1 - 3\rho V_M b_{i,M},
\]

and \( a_{i5} = -\frac{\chi}{\tau^5} f_{i,M} + \frac{3\kappa - \rho(1-\rho)}{\tau^4} V_M c_{i,M} + \frac{\kappa - \rho(1-2\rho)}{\tau^4} S_M b_{i,M} \),

where

\[
\tau = -\frac{u'(qR_f)}{qu''(qR_f)}, \quad \rho = \frac{u'(qR_f) u^{[3]}(qR_f)}{2[u''(qR_f)]^2},
\]
\[
\kappa = \frac{[u'(qR_f)]^2 u^{[4]}(qR_f)}{6[u''(qR_f)]^3}, \quad \chi = \frac{[u'(qR_f)]^3 u^{[5]}(qR_f)}{24[u''(qR_f)]^4}.
\]

Theorem 1 puts forward the preference parameters \( \tau, \rho, \kappa, \chi \) that characterize the representative investors’ concerns for higher order moments. Following Scott and Horvath (1980), we expect investors to have a negative preference for even moments and a positive preference for odd ones. Caballe and Pomansky (1996) have shown that this property is displayed by utility functions exhibiting “mixed risk aversion”, that is having all odd derivatives positive and all even derivatives negative. As noted by Pratt and Zeckhauser (1987), the latter property means that utility functions have completely monotone first derivatives, a property known to correspond to Laplace transforms of probability distributions on the positive real line. Eeckhoudt and Schlesinger (2006) have provided some theoretical underpinnings for this property in terms of “risk apportionment”. We will then rely upon this literature to maintain throughout the following assumptions:

\( u^{[k]}(qR_f) \leq 0 \text{ for } k \text{ even} \), and \( u^{[k]}(qR_f) \geq 0 \text{ for } k \text{ odd} \).

Under this assumption, all the preference parameters \( \tau, \rho, \kappa, \chi \) of Theorem 1 are non-negative; since they contain successively a higher order derivative of the representative investor’s utility.
function \( u \), we call them risk tolerance, skew tolerance, kurtosis tolerance and pentosis tolerance, respectively⁵. Theorem 1 shows that when preference parameters \( \rho, \kappa, \chi \) for higher order moments are rescaled by the right power of risk tolerance, they determine the risk premium contribution of the corresponding market co-moment.

It was the key insight of Samuelson (1970), that an approximation of the risk premium \( (E(R_i) - R_f) \) by the first term \( a_{i2}\sigma^2 \) would provide an asset pricing model that matches the traditional Sharpe-Lintner CAPM. Thus, to understand the price of skewness risk, it is necessary to study at least an expansion with one order higher. It has been well known since Kraus and Litzenberger (1976) that it is the co-skewness \( c_{i,M} \) of an asset and not its skewness that determines its risk premium. Theorem 1 matches that intuition.

It is helpful to compare our result in theorem 1 with the usual approach of Taylor expansions of the representative investor’s marginal utility (see e.g. Dittmar (2002)). To study higher terms let us now consider the risk premium up to the fourth term based on theorem 1:

\[
\begin{align*}
& a_{i2}\sigma^2 + a_{i3}\sigma^3 + a_{i4}\sigma^4 + a_{i5}\sigma^5 \\
= & \left( \frac{1}{\tau^3} + \frac{1 - 3\rho}{\tau^3} Var(R_M) + \frac{\kappa - \rho(1 - 2\rho)}{\tau^4} E[(R_M - E(R_M))^3] \right) Cov[R_i, (R_M - E(R_M))] \\
& + \left( -\frac{\rho}{\tau^2} + \frac{3\kappa - \rho(1 - \rho)}{\tau^4} Var(R_M) \right) Cov[R_i, (R_M - E(R_M))^2] \\
& + \frac{\kappa}{\tau^3} Cov[R_i, (R_M - E(R_M))^3] - \frac{\chi}{\tau^4} Cov[R_i, (R_M - E(R_M))^4].
\end{align*}
\]

Note that the pricing impact of market beta \( Cov[R_i, (R_M - E(R_M))] \) is driven not only by the inverse of risk tolerance \( \tau \) but also \( Var(R_M) \) together with the skew tolerance and the third moment of the market return together with kurtosis tolerance matter; a similar statement can be made for co-skewness. However, along the lines of a Taylor series expansion of the stochastic discount factor (see e.g. Dittmar (2002)) we would expect that the pricing impact of co-moments of an asset \( i \) with higher order market terms \( (R_M - E(R_M))^j (j = 1, 2, 3, 4) \) is driven only by the preference parameters \( \tau, \rho, \kappa, \chi \), respectively. (This is akin to expect that only the terms

⁵Skewness tolerance is related to the prudence concept of Kimball (1990); kurtosis tolerance is related to the temperance concept of Kimball (1992) and to risk vulnerability of the utility function in the presence of background risks, see Gollier (2001). Eeckhoudt and Schlesinger (2006) study lotteries with the property that prudence and temperance can be fully characterized by a preference relation over these lotteries. Pentosis tolerance is driven by the fifth derivative of the utility function. It is related to the concept of edginess that Lajeri-Chaherli (2004) introduced to study decision making in the presence of background risks. Eeckhoudt (2012) discusses the sign of various derivatives (third, fourth, fifth) of the utility function.
\( \frac{\alpha}{\gamma} d_{i,M} \) and \( -\frac{\alpha}{\gamma} f_{i,M} \) determine \( a_{44} \) and \( a_{55} \), respectively.) The proof of theorem 1 shows that this omission is akin to making only the series expansion of the marginal utility while forgetting that the optimality condition (whose expansion is called for) actually involves the product of marginal utility times net return: this is akin to overlooking that (higher order) moments of the product of marginal utility with risky assets’ returns are characterized through all products of suitable (lower order) moments of marginal utility and of suitable (lower order) moments of risky assets.

For illustration of the economic impact, let us consider for a moment the case of a utility function with constant relative risk aversion:

\[
   u'(w) = w^{-\alpha}, \alpha > 0.
\]

For this utility function we have:

\[
   \rho = \frac{1}{2} + \frac{1}{2\alpha} > \frac{1}{2}
\]

Theorem 1 then implies that for an asset \( i \) with positive market beta and positive market co-kurtosis, the risk premium for large co-kurtosis is smaller than expected in the Taylor series approach (see, e.g. Dittmar (2002)), since it overlooks the correction term proportional to \( 1 - 3\rho < 0 \). A similar comment could be made about the risk premium for negative co-pentosis.

### 3.2 Case of a representative investor over two periods

This subsection continues to consider a representative investor who maximizes expected utility \( E[u(W)] \) derived from the terminal wealth \( W \) that she obtains by investing some initial wealth \( q \) in a portfolio. The difference to the previous subsection is that we take into account that this terminal wealth may be the result of a dynamic investment strategy over several consecutive periods. For sake of expositional simplicity, we consider only two periods with constant risk-free rate \( R_f = R_{f,0} = R_{f,1} \): the first period runs from date 0 to date 1, the second period runs from date 1 to date 2. In other words, the dynamic portfolio of the representative investor is defined in terms of shares of wealth invested \( \theta = (\theta_0^i, \theta_1^i)_{1 \leq i \leq n} \) to give terminal wealth:

\[
   W_2 = W_1 \cdot \left[ R_f + \sum_{i=1}^{n} \theta_1^i (R_{i,2} - R_f) \right], \quad W_1 = q \cdot \left[ R_f + \sum_{i=1}^{n} \theta_0^i (R_{i,1} - R_f) \right]. \tag{7}
\]

\( ^6 \)Allowing the interest rate to vary would lead to additional terms that characterize the ratio. That would not affect our results qualitatively but would considerably complicate our notation.
The second period portfolio holdings may depend on the realization of returns over the first period, but for simplicity of exposition we do not write out this dependence explicitly throughout this subsection. Throughout, we consider a scale-location model for the vector of returns that is defined recursively over consecutive periods:

\[ R(1) = \left( R_i; 1 \leq i \leq n \right) = E(R(1)) + \sigma Y(1),\ \text{with}\ E(Y(1)) = 0, \]

and

\[ R(2) = \left( R_i; 1 \leq i \leq n \right) = E \left( R(2) | Y(1) \right) + \sigma Y(2),\ \text{with}\ E \left( Y(2) | Y(1) \right) = 0. \]

Note that we assume without loss of generality (up to a proper rescaling of the two random vectors \( Y(1) \) and \( Y(2) \)) that the scale parameter \( \sigma \) is the same in the two periods.

As in the previous subsection, we assume that the supply of risky assets is independent of \( \sigma \). At the beginning of the first period, the supply is given by a vector \( \xi \) of shares of total (date 0) wealth \( q \) invested in the \( n \) risky assets. Similarly, we denote \( \xi^1 = (\xi^1_i); 1 \leq i \leq n \) the vector of total date 1 wealth invested in the risky assets at the beginning of the second period. These define the market returns \( R_{M,1} \) and \( R_{M,2} \) over both periods by setting

\[ R_{M,1} = \xi^1 \perp E(R(1)) + \sigma \xi^1 \perp Y(1), \quad R_{M,2} = \xi^1 \perp E \left( R(2) | Y(1) \right) + \sigma \xi^1 \perp Y(2). \]

It is important to note that the date 1 vector of invested wealth \( \xi^1 \) depends on the relative prices of risky assets and that these (may) change over the first period. We assume that the total supply of asset \( i \) is given over the two consecutive periods through the quantities \( x_{i,0} \) and \( x_{i,1} \) of units of asset \( i \); then we can write

\[ \xi^1_i = \xi_i \frac{R_{i,1}}{R_{M,1}} \frac{x_{i,1}}{x_{i,0}}. \]

For sake of expositional simplicity, we assume:

\[ x_{i,1} = \frac{R_{M,1}}{R_{i,1}} x_{i,0}. \]

This simplifying assumption implies \( \xi^1 = \xi \); in other words, we impose some neutrality in the public offering of asset \( i \): it ensures the invariance (with respect to both state variables \( Y \) and scale of risk \( \sigma \)) of the market portfolio between the two dates in terms of shares of invested wealth. Our simplifying assumption could be relaxed at the cost of more complicated pricing formulas.
without changing the substance of the pricing of skewness that is the focus of our interest. We can then write
\[ R_{M,2} = \xi^1 E(R_{(2)|Y(1)}) + \sigma \xi^2 Y(2). \]

Based on Lemma 1, the scale parameter \( \sigma \) allows us to describe the pattern of risk premiums from their series expansions:

\[ \pi_{i,1}(\sigma) = E(R_{i,2}|Y(1)) - R_f = \sum_{j=2}^{\infty} a_{1, j}^1 \sigma^j \quad \text{and} \quad \pi_{i,0}(\sigma) = E(R_{i,1}) - R_f = \sum_{j=2}^{\infty} a_{0, j}^1 \sigma^j. \tag{10} \]

At date 0, the representative investor faces the following dynamic optimization program:

\[
\begin{align*}
\max_{q^0, q^1} E[u(W_2(\sigma))] &= \max_{q^0} E[J(W_1(\sigma))], \\
\text{where } J(W_1(\sigma)) &= \max_{\theta^1} E[u(W_2(\sigma)) | Y(1)], \\
W_2(\sigma) &= W_1(\sigma) \left[ R_f + \sum_{i=1}^{n} \theta^1_i (\pi_{i,1}(\sigma) + \sigma Y_{i,2}) \right], \\
W_1(\sigma) &= q \left[ R_f + \sum_{i=1}^{n} \theta^0_i (\pi_{i,0}(\sigma) + \sigma Y_{i,1}) \right].
\end{align*}
\]

Using the date 1 indirect utility function \( J \) defined above and the law of iterated expectations, we can then write the investor’s first order conditions for date 0 portfolio optimization for \( i = 1, \ldots, n \) as

\[
E \left[ u'(W_2(\sigma)) \frac{\partial W_2}{\partial W_1}(\pi_{i,0}(\sigma) + \sigma Y_{i,1}) \right] = E \left[ \frac{\partial J}{\partial W_1} \frac{\partial W_1}{\partial \theta^0_i} \right] = 0, \tag{11}
\]

where the terminal wealth level \( W_2 = W_2(\sigma) \) is induced by the portfolio choice according to (7). This describes the date 0 first order conditions.

We are solving this through backward induction. At date 1, given the observed realization of the state vector \( Y(1) \), the representative investor is faced with a given supply \( \xi \) of risky assets and return prospects whose randomness is defined by the scale parameter \( \sigma \) and the conditional probability distribution of \( Y(2) \) given \( Y(1) \). We can then apply the one-period market results derived in the former subsection to conclude that the (conditional) date 1 equilibrium risk premium for each asset \( i = 1, \ldots, n \) in the characterization of equation (10) is

\[
a_{1,2}^1 = \frac{1}{\tau_1(qR_{M,1})} b_{1, M}, \quad a_{1,3}^1 = -\frac{\rho_1(qR_{M,1})}{\tau_1^2(qR_{M,1})} c_{1, M}. \tag{12}
\]
Here, preference parameters are defined through the date 1 risk tolerance and skew tolerance functions
\[
\tau_1(w) = -\frac{u'(wR_f)}{wu''(wR_f)}, \quad \rho_1(w) = \frac{u'(wR_f) w^{[3]}(wR_f)}{2[u''(wR_f)]^2}.
\] (13)

Note that these are analogous to the risk and skew tolerances that we used in the single period case, see theorem 1. Seen from date 0, these functions will be evaluated at \( qR_M \) and so they are random variables. Our approach allows us to view them through a series expansion in the scale parameter and therefore we introduce the date 0 preference parameter
\[
\gamma_0 = q\frac{\partial \tau_1}{\partial w}(qR_f).
\] (14)

Note that \( \tau_1/R_f \) is equal to the common (relative) risk tolerance function, evaluated at \( wR_f \). Therefore, unless preferences are described through the so-called constant relative risk aversion, the function \( \tau_1 \) will not be constant and wealth changes over the first period will affect the investor’s risk tolerance at date 1. The term \( \gamma_0 \) describes the first order change in the risk tolerance function \( \tau_1 \) due to changes in investor’s wealth. Therefore, we call this preference parameter the wealth tolerance.

Date 1 (conditional) systematic co-moments are defined here analogous to definition 1 by\(^7\)
\[
b_{i,M}^1 = \frac{1}{\sigma^2} Cov \left[ R_{1,2}, (R_{M,2} - E(R_{M,2}|R_1)) \right] = Cov \left[ Y_{i,2}, (\xi^1 Y_{(2)}) | Y_{(1)} \right],
\] (15)
\[
c_{i,M}^1 = \frac{1}{\sigma^2} Cov \left[ R_{i,2}, (R_{M,2} - E(R_{M,2}|R_i))^2 \right] = Cov \left[ Y_{i,2}, (\xi^1 Y_{(2)})^2 | Y_{(1)} \right].
\] (16)

Moreover we need appropriate date 0 co-moments:

**Definition 2** For asset \( i = 1, \ldots, n \), systematic date 0 co-moments are defined by
\[
b_{i,M}^0 = \frac{1}{\sigma^2} Cov \left[ R_{i,1}, (R_{M,1} - E(R_{M,1})) \right] = Cov \left[ Y_{i,1}, (\xi^1 Y_{(1)}) \right],
\]
\[
c_{i,M}^0 = \frac{1}{\sigma^2} Cov \left[ R_{i,1}, (R_{M,1} - E(R_{M,1}))^2 \right] = Cov \left[ Y_{i,1}, (\xi^1 Y_{(1)})^2 \right],
\]
\[
c_{i,M}^{0,*} = \frac{1}{\sigma^2} Cov \left[ R_{i,1}, (R_{M,2} - E(R_{M,2}|R_1))^2 \right] = Cov \left[ Y_{i,1}, (\xi^1 Y_{(2)})^2 \right].
\]

Note that similar to definition 1 for the one-period case, we also standardize the date 0 co-moments by the scale of risk, see our discussion there. However, differently to definition 1 we

\(^7\)Here and throughout the paper all co-moments are expressed in terms of the random variables \((Y_i)_{1 \leq i \leq n}\) and also in terms of original returns \((R_i)_{1 \leq i \leq n}\) (appropriately rescaled by \( \sigma \)). Conditioning w.r.t. to the random variable \( Y_1 \) is the same as conditioning on \( R_1 \), since \( \sigma > 0 \).
only define the co-moments up to order 3, since we will study below only the first two terms $a_{i2}\sigma^2$ and $a_{i3}\sigma^3$ in our expansion.

Note that $c_{i,M}^{0,*}$ describes a form of intertemporal third order co-moment; it should not be confused with the time-zero expectation of the market co-skewness at time 1, since

$$E[c_{i,M}^1] = \frac{1}{\sigma^2}Cov\left[R_{i,2}, (R_{M,2} - E(R_{M,2}|R_1))^2\right] = Cov[Y_{i,2}, (\xi^1Y(2))^2].$$

This term does not vanish if and only if the market return over the second period is conditionally heteroskedastic given first period returns:

$$c_{i,M}^{0,*} = \frac{1}{\sigma^3}Cov\left[R_{i,1}, Var[R_{M,2}|R_1]\right] = Cov\left[Y_{i,1}, Var[\xi^1Y(2) | Y(1)]\right]. \quad (17)$$

Aggregating gives

$$\sum_{i=1}^n \xi_i c_{i,M}^{0,*} = \frac{1}{\sigma^3}Cov\left[R_{M,1}, Var[R_{M,2}|R_1]\right] = L_M. \quad (18)$$

Here, $L_M$ is the market leverage (or volatility feedback) effect. We expect it to be negative: bad news on the market leads to soaring expectations of future volatility. The term $c_{i,M}^{0,*}$ defined in equation (17) describes the contribution of each asset to the market leverage effect, i.e. a co-leverage (or volatility co-feedback) effect.

We solve for the date 0 risk premiums by plugging in the definition of $W_2(\sigma)$, the value of risk premium terms $\pi_{i,1}(\sigma)$ implied by formulas (12), using $\theta_{(1)} = \xi$ and solving first order conditions (11) with respect to unknown premiums $\pi_{i,0}(\sigma)$ when plugging in $\theta_{(0)} = \xi$; we are then able to prove:

**Theorem 2** The date 0 equilibrium risk premium $\pi_{i,0}(\sigma) = E(R_{i,1}) - R_f = \sum_{j=2}^{\infty} a_{i,j}^{0}\sigma^j$ is characterized by

$$a_{i2}^0 = \frac{1}{\tau_0} b_{i,M}^0, \quad a_{i3}^0 = -\frac{\rho_0}{\tau_0^2} c_{i,M}^0 + \frac{1 - \rho_0 + \gamma_0}{\tau_0^2} c_{i,M}^{0,*}$$

where

$$\tau_0 = -\frac{u'(qR_f^2)}{qR_f u''(qR_f^2)}, \quad \rho_0 = \frac{u'(qR_f^2) u''(qR_f^2)}{2[u''(qR_f^2)]^2}$$

and the preference term $\gamma_0$ is defined in equation (14).
This theorem has interesting implications for equilibrium risk premiums. In this theorem 2 the preference parameters \( \tau_0, \rho_0 \) are analogous the preference parameters \( \tau, \rho \) in theorem 1. The main difference is that the terms in theorem 1 are defined using the (single-period) riskfree return \( R_f \), whereas those in theorem 2 make use of the two-period riskfree return \( R_f^2 \), since utility is evaluated at wealth two periods from date 0.

Note that here we do not make explicit higher order terms of risk premium like \( a_{i4}^0 \) and \( a_{i5}^0 \); this is different from theorem 1 for the one-period case. Although we consider only the first two terms \( a_{i2}^0, a_{i3}^0 \), we can already see important differences in the two period situation compared to the one period case of the previous subsection. The main message here is that dynamic pricing introduces \( c_{i,M,0}^{0,*} \), another concept of market co-skewness at date 0. This additional term is the date 0 risk premium due to the sensitivity of changes in the date 1 risk premium through changes in market leverage (volatility feedback). The price of volatility risk is given here through \( \frac{1-\gamma_0}{\rho_0^{\delta}} \).

Note that the presence of \( c_{i,M,0}^{0,*} \) is typically due to intertemporal optimization behavior and is characterized through the wealth tolerance \( \rho_0 \). (The impact of \( c_{i,M,0}^{0,*} \) on risk premiums is not limited to wealth tolerance \( \gamma_0 \). For example, the pricing impact disappears in case of myopic behavior implied by the logarithmic utility function, since then \( \tau_0 = R_f, \rho_0 = 1, \gamma_0 = 0 \).) We note that a non-zero preference for positive skewness \( (\rho_0 > 0) \) gives rise to risk premiums for asset \( i \) not only for its market co-skewness but also for its market co-leverage. This matches the well-documented empirical fact that leverage effect creates skewness through temporal aggregation (see e.g. Meddahi and Renault (2004)).

### 3.3 Quadratic SDF

A convenient way to describe an asset pricing model is through the stochastic discount factor (SDF), see e.g. Cochrane (2001), i.e. through a random variable \( H \) with the property that risk premiums are given by

\[
E[R_i] - R_f = -R_f \text{Cov}(H, R_i), \text{ for } i = 1, \ldots, n.
\]

The literature usually considers time-series applications, where expectations are conditional on the information available at each date. In line with our earlier analysis and for simplicity we

\[\text{We only discuss the date 0 equilibrium risk premium and not the date 1 risk premium. The latter can be interpreted through theorem 1 with a conditional viewpoint.}\]
do not write out this dependence explicitly. Thus, essentially, our analysis appears to boil down to a single-period analysis but the reader should keep in mind the conditional viewpoint. We study first the SDF of the first subsection, i.e. the case of a single period case; then we study the two-period extension of the second subsection and will see that it leads us to consider an additional pricing factor in the SDF.

It is well known that the Sharpe-Lintner CAPM is tantamount to an SDF that is affine w.r.t. the market return. To accommodate some widely documented departures from the CAPM, empirical asset pricing may consider higher-order polynomials in the market return as a candidate SDF. For instance a quadratic SDF should accommodate pricing of skewness, see, among others, Harvey and Siddique (2000) and Dittmar (2002); up to a constant, a quadratic SDF, \( H(2) \), has the form

\[
R_f H(2) = \lambda_1 R_M + \lambda_2 R_M^2,
\]

for suitable parameters \( \lambda_1, \lambda_2 \). It is well-known that the quadratic SDF leads to a linear pricing relationship for excess returns:

\[
E \left[ r_i \right] = \lambda_1 \text{Cov}_t [r_i, r_M] + \lambda_2 \text{Cov}_t \left[ r_i, r_M^2 \right],
\]

where \( r_i = R_i - R_f, r_M = R_M - R_f \) stand for excess returns on asset \( i \) and the market, respectively.

It is tempting to apply the linear pricing relationship (20) to the squared market return; to do so, let us assume for a moment and for purely illustrative reasons that \( r_M^2 \) corresponds to a portfolio available in the market. (We will discuss in the next section that \( R_M^2 \) and thus \( r_M^2 \) may not be a portfolio in incomplete markets.) We denote by \( \eta \) the price and by \( \tilde{r}_M = r_M^2 / \eta - R_f \) the excess return on the squared market portfolio; applying the linear pricing relationship (20) to this excess return gives

\[
E \left[ \frac{r_M^2}{\eta} \right] - R_f = \lambda_1 \text{Cov}_t \left( r_M, \frac{r_M^2}{\eta} \right) + \lambda_2 \text{Cov}_t \left( r_M^2, \frac{r_M^2}{\eta} \right).
\]

The linear pricing relationship (20) can also be applied to the market excess return; this gives a system of two equations that can be resolved:

\[
\lambda_1 = \frac{\text{Var}(r_M) E[r_M] - \text{Cov}(r_M, r_M^2) (E[r_M^2] - \eta R_f)}{\text{Var}(r_M) \text{Var}(r_M^2) - (\text{Cov}(r_M, r_M^2))^2},
\]

\[
\lambda_2 = \frac{\text{Var}(r_M) (E[r_M^2] - \eta R_f) - \text{Cov}(r_M, r_M^2) E[r_M]}{\text{Var}(r_M) \text{Var}(r_M^2) - (\text{Cov}(r_M, r_M^2))^2}.
\]
One may be tempted to set $\eta = 0$; this characterization of $\lambda_1, \lambda_2$ then coincides with formulas (7b, 7c) put forward by Harvey and Siddique (2000). However, this appears to be at odds with a no-arbitrage condition, since $\eta$ is the price of a strictly positive payoff and should therefore be strictly positive. Also, when markets are incomplete we will stress in the next section that the squared market returns may not be in the asset span, such that its price cannot be observed easily.

Our analysis in subsection 3.1 provides a foundation for the common practice of quadratic SDF and allows us to relate the terms $\lambda_1, \lambda_2$ to preference parameters. Based on our analysis of $a_2 \sigma^2 + a_3 \sigma^3$ we can identify the parameters $\bar{\lambda}_1, \bar{\lambda}_2$ in the quadratic SDF of equation (19); we have, up to terms higher then $\sigma^3$:

$$
\frac{1}{\tau} b_{i,M} \sigma^2 - \frac{\rho}{\tau^2} c_{i,M} \sigma^3 = a_2 \sigma^2 + a_3 \sigma^3 = E[R_i] - R_f = -R_f \text{Cov}(H(2), R_i).
$$

Based on definition 1, this puts forward the quadratic SDF up to a constant

$$
R_f H(2) = -\frac{1}{\tau} R_M + \frac{\rho}{\tau^2} (R_M - E(R_M))^2. \tag{21}
$$

Our characterization of the SDF in equation (21) allows us to identify to the SDF in equation (19), up to a constant:

$$
\lambda_1 = -\frac{1}{\tau} - 2 \frac{\rho}{\tau^2} E[r_M], \quad \lambda_2 = \frac{\rho}{\tau^2}.
$$

The price of skewness is contained in the sensitivity $\lambda_2$: when it is zero we are back in a mean-variance pricing world, while $\lambda_2 \neq 0$ leads to mean-variance-skewness pricing. Note that here $\lambda_2$ is something like a structural invariant, only varying through the value of preference parameters computed from the derivatives of the utility function at $R_f$. Ultimately, estimating the size of $\lambda_2$ is at the core of empirical studies, see, e.g., Harvey and Siddique (2000).

Overall, we find that a single period analysis of a representative agent confirms the common practice of quadratic SDF. Ultimately, we will elaborate below from two angles that this paradigm breaks down: first we elaborate in the remainder of this (sub-)section that this may not hold in a two period analysis of a representative agent; second, section 5 will take up the issue in great detail with heterogeneous agents in a single period analysis and point out that there are priced factors in incomplete markets that do not show up in complete markets.

Let us now study the implications of our two-period extension in the previous subsection; for this we denote the SDF by $\hat{H}(2)$ to distinguish it from the prior SDF. We derive from the
analysis of the first two terms $a_i \sigma^2 + a_i^3 \sigma^3$ in theorem 2 that the date 0 one-period risk premium is (up to higher order terms)

$$-R_f \text{Cov} \left( \tilde{H}(2), R_{i,t} \right) = E (R_{i,t}) - R_f = \frac{1}{\tau_0} \theta_{i,M}^0 \sigma^2 - \frac{\rho_0}{\tau_0^2} \sigma_{i,M}^0 \sigma^3 + \frac{1}{\tau_0^2} \gamma_{0} \sigma_{i,M}^{0,*} \sigma^3.$$

Up to a constant, this puts forward the date 0 one-period SDF

$$\tilde{H}(2) = \tilde{H}(2, 1) + \tilde{H}(2, 2),$$

where

$$R_f \tilde{H}(2, 1) = R_f H(2) = -\frac{1}{\tau_0} R_{M,1} + \frac{\rho_0}{\tau_0} (R_{M,1} - E R_{M,1})^2,$$

$$R_f \tilde{H}(2, 2) = -\frac{1 - \rho_0 + \gamma_0}{\tau_0^2} \text{Var} [R_{M,2} | R_1].$$

Compared to the one-period SDF in the static model, there is an additional pricing factor which is the volatility risk factor $\text{Var} [R_{M,2} | R_1]$. The price of risk of this factor is determined not only by the risk preferences (risk and skew tolerance), but also through a wealth effect via the wealth tolerance $\gamma_0$.

One message of this section is that a two period analysis may be at odds with the common
quadratic SDF assumption, except if one maintains two rather restrictive assumptions about
the volatility risk factor $\text{Var}[R_{M,2} | R_1]$. These assumptions are statistical in nature and not underpinned by any economic theory: First, we need to assume that an aggregated volatility model is optimal:

$$\text{Var}[R_{M,2} | R_1] = \text{Var}[R_{M,2} | R_{M,1}] \quad (22)$$

Second, even under this assumption we only get the quadratic SDF specification by assuming an
ARCH(1) specification (Engle (1982)) for this aggregated volatility model:

$$\text{Var}[R_{M,2} | R_{M,1}] = \omega + \varsigma R_{M,1}^2$$

Even an asymmetric ARCH (Glosten et al. (1993)), where the sign of $R_{M,1}$ would matter (to accomodate an aggregate leverage effect) would invalid the naive quadratic SDF. Note in addition that the aggregation assumption (22) is all about the cross section of risky assets. We always assume that investors are homogeneous regarding their expectation of future returns. Aggregation of heterogeneous beliefs and/or asymmetric information is beyond the scope of this paper.
4 Portfolio separation and pricing of skewness risk with heterogeneous agents

This previous section studied the case of a single (representative) investor. Throughout the remainder of this paper we assume the economy is populated by $S$ (heterogeneous) investors $s = 1, ..., S$; each investor $s$ maximizes expected utility $E[u_s(W_s)]$ of her terminal wealth $W_s$ obtained by investing her initial wealth $q_s$ in a portfolio.

This section discusses demand of our heterogeneous investors and the pricing of skewness risk when we focus on the first two terms $a_{i2}\sigma^2 + a_{i3}\sigma^3$; along the way we also look into the impact of market incompleteness. We study this in two separate subsections first for the case of heterogeneous investors over one period and then for the case of heterogeneous investors over two periods.

4.1 Case of heterogeneous investors over one period

Throughout, this subsection we assume that there is only one period of investment. The portfolio of each investor $s = 1, ..., S$ is defined in terms of shares of wealth invested $\theta_s = (\theta_{is})_{1 \leq i \leq n}$; this gives terminal wealth

$$W_s = q_s \left[ R_f + \sum_{i=1}^{n} \theta_{is} (R_i - R_f) \right].$$

(23)

Note that this is the heterogeneous investor analogue (for each investor $s = 1, ..., S$) of the single period wealth dynamic that we considered in the representative investor case, equation (5). The demand $\theta_s$ of investor $s$ for risky assets solves the first order conditions of her expected utility maximization, where the first order condition for each investor $s = 1, ..., S$ is merely equation (2) written out for the utility function $u_s$ and terminal wealth $W_s$ based on demand $\theta_s$ of each investor.

Samuelson (1970) characterized the impact of higher order return moments on the demand for risky assets through a series expansion of the optimal portfolio $\theta_s = (\theta_{is})_{1 \leq i \leq n}$:

$$\theta_{is} = \sum_{j=0}^{\infty} \theta_{isj} \sigma^j.$$  

(24)

In section 2 we assumed that the total offer of risky assets was a fixed vector $\xi = (\xi_i)_{1 \leq i \leq n}$ of shares of the total amount $S\bar{q} = \sum_{s=1}^{S} q_s$ of invested wealth and we continue to adopt this...
assumption. Note that $\sum_{s=1}^{S} q_{s} \theta_{s} q_{s} = \sum_{s=1}^{S} \theta_{is} q_{s}$ is the aggregate demand of any asset $i = 1, ..., n$. Therefore, we have the following market clearing conditions for each risky asset $i = 1, ..., n$ and all orders $j > 0$:

$$\sum_{s=1}^{S} \theta_{is} q_{s} = \xi_{i} S \bar{q}, \quad \text{and} \quad \sum_{s=1}^{S} \theta_{is} q_{s} = 0. \quad (25)$$

This differs from our representative investor analysis (the previous section) in several ways. There, the total offer was $\xi$ which effectively limited us to $\theta = \xi$, i.e. all terms of order higher than 1 had to vanish. Here, this does not have to be the case at the level of individual investors; this means that investors can use higher order terms to hedge risks stemming from powers of market return not being in the asset span (in incomplete markets); this in turn will have an impact on risk premiums that only shows up when investors are heterogeneous.

The market clearing conditions (25) together with the first order conditions for each investor determine the equilibrium asset demand and pricing of financial assets; the coefficients $a_{ij}, j \geq 2$ of the risk premium expansion (3) solve for all levels $\sigma$ of risk:

$$E\left[U_{s}'(W_{s})(R_{i} - R_{f})\right] = 0, \quad \text{for} \quad i = 1, ..., n, \quad (26)$$

subject to $W_{s}(\sigma) = q_{s}\left[R_{f} + \sum_{i=1}^{n} \theta_{is}(\sigma) \cdot (R_{i}(\sigma) - R_{f})\right]$, where $R_{i}(\sigma) - R_{f} = \pi_{i}(\sigma) + \sigma Y_{i}$, for $i = 1, ..., n$.

Samuelson’s key insight that we extend in this paper is the following: for any investor $s$, the first $K$ coefficients $\theta_{isj}, j = 0, ..., K - 1$, only depend on the first $(K + 1)$ derivatives of the utility functions $u_{s}, s = 1, ..., S$, computed at the level of wealth $q_{s}R_{f}$. (This level can be seen as the terminal wealth resulting from investing the initial wealth $q_{s}$ only in the safe asset.) This permits successive determination of demand and market clearing, starting from terms of order zero and going up one order at a time. We are then able to prove:

**Theorem 3** The equilibrium risk premium $\pi_{i}(\sigma) = E(R_{i}) - R_{f} = \sum_{j=2}^{\infty} a_{ij} \sigma^{j}$ fulfills

$$a_{i2} = \frac{1}{\bar{\tau}} b_{i,M}, \quad a_{i3} = -\frac{\bar{\rho}}{\bar{\tau}^{2}} c_{i,M}, \quad (27)$$

where individual ($\tau_{s}$) and average ($\bar{\tau}$) risk tolerance coefficients as well as individual ($\rho_{s}$) and average ($\bar{\rho}$) skew tolerance coefficients are defined by

$$\tau_{s} = -\frac{u_{s}'(q_{s}R_{f})}{q_{s}u_{s}''(q_{s}R_{f})} \bar{\tau} = \frac{1}{S \bar{q}} \sum_{s=1}^{S} \tau_{s} q_{s}, \quad \rho_{s} = \frac{u_{s}'(q_{s}R_{f})u_{s}''(q_{s}R_{f})}{2 (u_{s}''(q_{s}R_{f}))^{2}}, \quad \bar{\rho} = \frac{\sum_{s=1}^{S} q_{s} \tau_{s} \rho_{s}}{\sum_{s=1}^{S} q_{s} \tau_{s}}. \quad (28)$$
The equilibrium demand of investor $s$ is given through

$$\theta_{s0} = \frac{\tau_s}{\bar{\tau}} \xi, \quad \theta_{s1} = \frac{\tau_s (\rho_s - \bar{\rho})}{\bar{\tau}^2} \Sigma^{-1} c,$$

(29)

where $\theta_{s0} = (\theta_{is0})_{1 \leq i \leq n}, \theta_{s1} = (\theta_{is1})_{1 \leq i \leq n}$ describe the zero, first order demand vector and $c = (c_i, M)_{1 \leq i \leq n}$ describes the vector of market co-skewness.

Theorem 3 has interesting implications for equilibrium risk premiums. Approximating the risk premium $(E(R_i) - R_f)$ by its first two terms $\sigma^2 a_{i2} + \sigma^3 a_{i3}$ is similar to extending the CAPM to an asset pricing model that takes into account skewness risk. Skewness risk is priced in equilibrium as soon as the properly defined average skew tolerance is non-zero and some assets display a non-zero co-skewness (while the market return may well be symmetric). Our result displays the aggregate impact on the skewness premium in case of investors heterogeneity.

Heterogeneity across investors’ risk tolerances and skew tolerances $\tau_s, \rho_s$ ($s = 1, \ldots, S$) may come either from heterogeneity of their utility functions $u_s$ or from heterogeneity of their invested wealth $q_s$. In both cases we find here the well-known result that heterogeneity does not really matter for mean-variance-skewness pricing; only the average risk tolerance coefficient $\bar{\tau}, \bar{\rho}$ matters. Note in particular that for homogeneous investors (same utility function $u = u_s$, same wealth $q = \bar{q}$ invested), the risk premium on variance and on skewness risk would simply be that of Theorem 1.

Let us now take a look the implications of theorem 3 for equilibrium asset demand. We have seen that the Sharpe-Lintner CAPM gives a correct picture of actual risk premiums insofar as we focus on the first term $a_{i2} \sigma^2$ of their series expansion. Theorem 3 shows that the same applies to the equilibrium demand for risky assets, i.e. a standard mean-variance approach is sufficient insofar as we focus on the first term $\theta_{s0}$ of the series expansion for the equilibrium demand of investor $s$. This is the celebrated two-fund separation theorem, where all investors hold the same portfolio of risky assets defined by portfolio weights $\xi$. By analogy with the CAPM relationship, let us study an additional fund defined by portfolio weights $\xi^{sk}$ that lead to a return $R^{sk} = R_f + \sum_{i=1}^{n} \xi^{sk}_i (R_i - R_f)$ with the property

$$\sigma^3 a_{j3} = -\frac{\bar{\rho}}{\bar{\tau}^2} \text{Cov}(R_j, R^{sk}), \text{ for all } j = 1, \ldots, n.$$
Note that this equation defines $\xi^{sk}$ as a unique solution of a linear system of $n$ equations with non-singular matrix $Var(R)$. Theorem 3 states that

$$\xi^{sk} = \sigma\Sigma^{-1}c.$$ \hspace{1cm} (30)

This new portfolio plays the same role for pricing skewness risk that the market portfolio played to price (co-)variance risk. Therefore, throughout this paper we refer to it as the skewness portfolio. Not surprisingly, it also appears in investors’ equilibrium demand for risky assets, i.e. we find:

$$\sigma\theta_{s1} = \frac{\tau_s(\rho_s - \bar{\rho})}{\tau^2} \xi^{sk}.$$ \hspace{1cm} (31)

Equations (29) establish a three-fund theorem: All investors hold the same two portfolios of risky assets (so-called mutual funds) defined by portfolio weights $\xi$ and $\xi^{sk}$, respectively. The only difference in actual holdings comes from wealth heterogeneity (ceteris paribus, the amount invested is proportional to wealth), heterogeneity in risk tolerances (the amount invested is proportional to risk tolerance at this wealth level) and, regarding the second mutual fund, heterogeneity in skew tolerances (the amount invested is proportional to relative spread between the skew tolerance of investor $s$ and the average one). Thus, an investor with stronger or weaker preferences for skewness than the average must underdiversify with respect to mean-variance efficiency.

Recall that the wealth weighted sum of $\tau_s\rho_s$ in relation to the wealth weighted sum of $\tau_s$ gives the average skew tolerance coefficient $\bar{\rho}$. Therefore, although investors’ holdings in the skewness portfolio may be different from zero, aggregate demand of it vanishes$^9$.

It is worth elaborating in more detail why investors will have non-zero positions in the skewness portfolio when their preferences for skewness depart from the average skewness preference. Comparing (27) and (30) shows that for all assets $i = 1, \ldots, n$:

$$Cov[R_i, R^{sk}] = Cov [R_i, (R_M - E(R_M))^2].$$ \hspace{1cm} (32)

In other words, we have the following decomposition for the squared market return:

$$\left(R_M - E(R_M)\right)^2 = R^{sk} - E(R^{sk}) + T^{sk},$$ \hspace{1cm} (33)

$^9$Essentially this is due to our simplifying assumption that the offer of risky assets does not depend on the scale parameter $\sigma$. Changing this assumption may introduce higher order dependence.
where the so-called tracking error \( (T^{sk} - E(T^{sk})) \) is uncorrelated with all traded asset returns \( R_i, i = 1, \ldots, n \). Up to an additive constant, the skewness portfolio return \( R^{sk} \) can therefore be interpreted as an affine regression of the squared (demeaned) market return on the traded asset returns. In other words, the skewness portfolio is the mutual fund that is best able to track the squared market return; thus, demand for the skewness portfolio is induced by the demand for tracking the squared market return\(^{10}\).

In complete markets, the squared market return can be replicated through traded assets and the tracking error disappears. In incomplete markets, however, the squared market return may not be in the asset span. The variance \( \text{Var}(T^{sk}) \) of the tracking error can then be interpreted as a measure of market incompleteness. The bigger this variance, the less accurate is the feasible hedge of the squared market return that a representative agent would like to hold for the sake of skewness preferences (see e.g. Dittmar (2002)). This interpretation in terms of market incompleteness will be confirmed in the next section.

4.2 Case of heterogeneous investors over two periods

This subsection extends the discussion of equilibrium with heterogeneous investors from one to two periods. Our extension here is analogous to the extension in the representative agent case from one to two periods in subsection 3.2.

This subsection continues to consider \( s = 1, \ldots, S \) investors who maximize their expected utility \( E[u_s(W_s)] \) derived from the terminal wealth \( W_s \) that they obtain by investing their initial wealth \( q_s \) in a portfolio. The difference to the previous subsection is that we take into account that this terminal wealth may be the result of a dynamic investment strategy over several consecutive periods. For sake of expositional simplicity, we consider only two periods with constant risk-free rate \( R_f = R_{f,0} = R_{f,1} \): the first period runs from date 0 to date 1, the second period with runs from date 1 to date 2. In other words, the dynamic portfolio of the representative investor is defined in terms of shares of wealth invested \( \theta_s = (\theta^0_{is}, \theta^1_{is})_{1 \leq i \leq n} \) to give terminal wealth:

\[
W_{s,2} = W_{s,1} \cdot \left[ R_f + \sum_{i=1}^{n} \theta^1_{is} (R_{i,2} - R_f) \right], \quad W_{s,1} = q_s \cdot \left[ R_f + \sum_{i=1}^{n} \theta^0_{is} (R_{i,1} - R_f) \right].
\]

\(^{10}\)In line with this reasoning, a careful examination of option markets would show that preferences for skewness introduce non-zero demand in asymmetric assets like options, see Judd and Leisen (2010).
The second period portfolio holdings may depend on the realization of returns over the first period, but for simplicity of exposition we do not write out this dependence explicitly throughout this subsection.

We continue to use the same scale-location model for the vector of returns that we defined recursively over consecutive periods in equations (8, 9) of subsection 3.2, i.e.

\[ R_{(1)} = (R_{i,1})_{1 \leq i \leq n} = E(R_{(1)}) + \sigma Y_{(1)}, \quad \text{with } E(Y_{(1)}) = 0, \]  

and

\[ R_{(2)} = (R_{i,2})_{1 \leq i \leq n} = E(R_{(2)}|Y_{(1)}) + \sigma Y_{(2)}, \quad \text{with } E(Y_{(2)}|Y_{(1)}) = 0. \]

As in subsection 3.2 we adopt the simplifying assumption that \( \xi^1 = \xi \), see our discussion there. The associated market returns \( R_{M,1} \) and \( R_{M,2} \) over both periods are then given as

\[ R_{M,1} = \xi^1 E(R_{(1)}) + \sigma \xi^1 Y_{(1)}, \quad R_{M,2} = \xi^1 E(R_{(2)}|Y_{(1)}) + \sigma \xi^1 Y_{(2)}. \]

Similar to the series expansion of demand in the equation (24) of the previous subsection we characterize the impact of these higher order moments on the demand for risky assets at both dates 0 and 1 through a series expansion of the optimal portfolio \( \theta_s^0 = (\theta_{is}^0)_{1 \leq i \leq n} \) and \( \theta_s^1 = (\theta_{is}^1)_{1 \leq i \leq n} \), respectively:

\[ \theta_{is}^0 = \sum_{j=0}^{\infty} \theta_{isj}^0 \sigma^j \quad \text{and} \quad \theta_{is}^1 = \sum_{j=0}^{\infty} \theta_{isj}^1 \sigma^j. \]  

Aggregate demand of any asset \( i = 1, ..., n \) is \( \sum_{s=1}^{S} \theta_{is0}^0 q_s \) and \( \sum_{n=1}^{S} \theta_{is0}^1 W_{s,1} \) at dates 0 and 1. Therefore, we have the following market clearing conditions for each risky asset \( i = 1, ..., n \) and all orders \( j > 0 \):

\[ \sum_{s=1}^{S} \theta_{is0}^0 q_s = \xi_i S \bar{q}, \quad \text{and} \quad \sum_{s=1}^{S} \theta_{is0}^0 q_s = 0 \quad \text{at date 0}; \]  

\[ \sum_{s=1}^{S} \theta_{is0}^1 W_{s,1} = \xi_i \sum_{s=1}^{S} W_{s,1}, \quad \text{and} \quad \sum_{s=1}^{S} \theta_{isj}^1 W_{s,1} = 0 \quad \text{at date 1}. \]

Based on Lemma 1, the scale parameter \( \sigma \) allows us to describe the pattern of risk premiums from their series expansions:

\[ \pi_{i,1}(\sigma) = E(R_{i,2}|Y_{(1)}) - R_f = \sum_{j=2}^{\infty} a_{ij}^1 \sigma^j \quad \text{and} \quad \pi_{i,0}(\sigma) = E(R_{i,1}) - R_f = \sum_{j=2}^{\infty} a_{ij}^0 \sigma^j. \]
At date 0, every investor \( s = 1, \ldots, S \) faces the following dynamic optimization program:

\[
\max_{\theta_1^s, \theta_2^s} \mathbb{E}[u_s(W_{s,2}(\sigma))] = \max_{\theta_1^s} \mathbb{E}[J_s(W_{s,1}(\sigma))],
\]

where \( J_s(W_{s,1}(\sigma)) = \max_{\theta_1^s} \mathbb{E}[u(W_{s,2}(\sigma)) | Y(1)] \),

subject to the associated wealth dynamics in equation (34). Using the date 1 indirect utility function \( J_s \) defined above and the law of iterated expectations, we can then write the investor’s first order conditions for date 0 portfolio optimization for \( i = 1, \ldots, n \) as

\[
\mathbb{E}\left[u'(W_{s,2}(\sigma)) \frac{\partial W_{s,2}}{\partial W_{s,1}} (\pi_{i,0}(\sigma) + \sigma Y_{i,1}) \right] = \mathbb{E}\left[ \frac{\partial J_s}{\partial W_{s,1}} \frac{\partial W_{s,1}}{\partial \theta_0^{s}} \right] = 0, \quad (41)
\]

where the terminal wealth level \( W_{s,2} = W_{s,2}(\sigma) \) is induced by the portfolio choice according to (34). This describes the date 0 first order conditions for each investor \( s = 1, \ldots, S \). These date 0 first order conditions, together with the date 0 market clearing conditions (38) determine jointly the date 0 equilibrium allocation (demand, risk premiums)\(^{11}\).

We are solving this through backward induction. Our two date return setup here mirrors the two date return setup with representative agent in subsection 3.2. Therefore, we continue to use the definitions of date 1 market beta and market co-skewness that we introduced in equations (15, 16) analogously definition 1. At date 1, given the observed realization of the state vector \( Y(1) \), each investor determines her demand based on the return prospects whose randomness is defined by the scale parameter \( \sigma \) and the conditional probability distribution of \( Y(2) \) given \( Y(1) \); the market clearing condition (38) then determines the equilibrium risk premiums, which also determines the optimal demand of each investor \( s = 1, \ldots, S \).

To apply the one-period market results derived in the former subsection we define first individual and aggregate preference parameters through the date 1 (aggregate) risk tolerance and (aggregate) skew tolerance functions

\[
\tau_{s,1}(w) = - \frac{u'(wR_f)}{wu''(wR_f)}, \quad \rho_{s,1}(w) = \frac{u'(wR_f)u^{[3]}(wR_f)}{2[w''(wR_f)]^2}, \quad (42)
\]

\[
\bar{\tau}_1(w_1, \ldots, w_S) = \frac{\sum_s w_s \tau_{s,1}(w_s)}{\sum_s w_s}, \bar{\rho}_1(w_1, \ldots, w_S) = \frac{\sum_s w_s \rho_{s,1}(w_s) \tau_{s,1}(w_s)}{\sum_s w_s \tau_{s,1}(w_s)} \quad (43)
\]

Note that these are analogous to the risk and skew tolerances that we used in the single period case, see equation (28) in theorem 3. The individual risk and skew tolerance functions

\(^{11}\)The date 1 equilibrium is defined analogous the equilibrium in the previous subsection, but with a conditional viewpoint. Our focus is on the date 0 equilibrium and therefore we do not write this out explicitly here.
will be evaluated at \( W_{s,1} \); the aggregate risk and skew tolerance functions will be evaluated at \((W_{s,1})_{s=1,\ldots,S}\).

We conclude from the one-period market results derived in the former subsection that the (conditional) date 1 equilibrium risk premium for each asset \( i = 1, \ldots, n \) in the characterization of equation (40) is

\[
a_{s,2}^{i} = \frac{1}{\tilde{\tau}_1(W_{1,1}, \ldots, W_{S,1})} b_{i,M}^{1} \quad a_{s,3}^{i} = -\frac{\tilde{\rho}_1(W_{1,1}, \ldots, W_{S,1})}{\tilde{\tau}_1^2(W_{1,1}, \ldots, W_{S,1})} c_{i,M}^{1}.
\]

We stress that \( \tilde{\tau}_1, \tilde{\rho}_1 \) are random variables conditional on the realization of wealth \((W_{s,1})_{s=1,\ldots,S}\) for all investors. In the two-period case of the representative agent (subsection 3.2) a wealth effect appeared that we characterized through a preference parameter called (date 0) wealth tolerance. Here we define analogously for each investor \( s = 1, \ldots, S \) the date 0 wealth tolerance \( \gamma_s \) by setting

\[
\gamma_{s,0} = q_s \frac{\tau_{s,0}}{\tilde{\tau}_0} \frac{\partial \tilde{\tau}_1}{\partial w_s}(q_s R_f), \quad \tilde{\gamma}_0 = \frac{\sum_{s=1}^{S} q_s \gamma_{s,0} \tau_{s,0}}{\sum_{s=1}^{S} q_s \tau_{s,0}}.
\]

Here we used the date 0 risk tolerances

\[
\tau_{s,0} = -\frac{u'_s(q_s R_f^2)}{R_f q_s u''_s(q_s R_f^2)}, \quad \tilde{\tau}_0 = \frac{\sum_{s=1}^{S} q_s \gamma_{s,0} \tau_{s,0}}{\sum_{s=1}^{S} q_s}.
\]

In addition, we introduce the date 0 skew tolerances

\[
\rho_{s,0} = \frac{u''_s(q_s R_f^2)}{2 [u'_s(q_s R_f^2)]^2} \left[ u''_s(q_s R_f^2) \right] \quad \tilde{\rho}_0 = \frac{\sum_{s=1}^{S} q_s \rho_{s,0} \gamma_{s,0} \tau_{s,0}}{\sum_{s=1}^{S} q_s \gamma_{s,0} \tau_{s,0}}.
\]

Our two date return setup here mirrors the two date return setup with representative agent in subsection 3.2. Therefore, we continue to use the definitions of date 0 systematic co-moments introduced in definition 2; for completeness we repeat these here:

\[
b_{i,M}^0 = \frac{1}{\sigma^2} Cov[R_{i,1}, (R_{M,1} - E(R_{M,1}))] = Cov[Y_{i,1}, (\xi Y_{i,1})],
\]

\[
c_{i,M}^0 = \frac{1}{\sigma^2} Cov[R_{i,1}, (R_{M,1} - E(R_{M,1}))^2] = Cov[Y_{i,1}, (\xi^2 Y_{i,1})^2],
\]

\[
c_{i,M}^{0,*} = \frac{1}{\sigma^2} Cov[R_{i,1}, (R_{M,2} - E(R_{M,2}|R_{i,1}))^2] = Cov[Y_{i,1}, (\xi^2 Y_{i,2})^2].
\]

We are then able to prove:
Theorem 4 The date 0 equilibrium risk premium $\pi_{i,0}(\sigma) = E(R_{i,1}) - R_f = \sum_{j=2}^{\infty} a_{ij}^0 \sigma^j$ is characterized by
\[
a_{i2}^0 = \frac{1}{\bar{\tau}^0} \theta_{i,M}^0, \quad a_{i3}^0 = \frac{1 - \bar{\rho}^0 + \bar{\gamma}^0}{\bar{\tau}^0} \bar{c}_M^0 - \frac{\bar{\rho}^0 \bar{c}_M^0}{\bar{\tau}^0}.
\]
where $\tau_{s,0}, \bar{\tau}_0, \rho_{s,0}, \bar{\rho}_0, \gamma_{s,0}, \bar{\gamma}_0$ are given through equations (45-47). The date 0 equilibrium demand of investor $s = 1, \ldots, S$ fulfills
\[
\theta_{s0}^0 = \frac{\tau_{s,0}}{\bar{\tau}_0} \xi,
\]
\[
\theta_{s1}^0 = \frac{\tau_{s,0}}{\bar{\tau}_0} \left( \rho_{s,0} - \bar{\rho}_0 \right) \Sigma_0^{-1} \bar{c}_M^0 + \left\{ \frac{\tau_{s,0} \left( \rho_{s,0} - \bar{\rho}_0 \right)}{\bar{\tau}_0} - \frac{\tau_{s,0} \left( \gamma_{s,0} - \bar{\gamma}_0 \right)}{\bar{\tau}_0} \right\} \Sigma_0^{-1} \bar{c}_M^0,
\]
where $\theta_{s0}^0 = (\theta_{is0})_{1 \leq i \leq n}$ and $\theta_{s1}^0 = (\theta_{is1})_{1 \leq i \leq n}$ describe the (date 0) zero and first order demand vector and $\Sigma_0$ the (date 0) variance matrix $\text{Var}(R_{(1)})$.

This theorem has interesting implications for equilibrium risk premiums\(^{12}\). Although investors may be heterogeneous in their preference parameters $\tau_s, \rho_s, \gamma_s$ ($s = 1, \ldots, S$), we find that heterogeneity does not really matter for pricing; only aggregate preference parameters $\bar{\tau}_0, \bar{\rho}_0, \bar{\gamma}_0$ are relevant. In addition, we note that with these aggregate preference parameters, theorem 4 (two period case, heterogeneous investors) matches the pricing result of theorem 2 (two-period case, representative investor). In line with this, but looking at it from a different angle we compare theorem 4 with theorem 3 and find that dynamic pricing introduces $c_{i,M}^0$. In subsection 3.2 we argued that this concept of market co-skewness is related to market leverage (volatility feedback effect) and referred to $c_{i,M}^0$ as the co-leverage (or volatility co-feedback) effect.

Let us now take a look at the implications of theorem 4 for equilibrium asset demand. We discussed in the previous subsection after theorem 3 that skewness leads to a three-fund theorem. Analogous to there, we define a fund with portfolio weights $\xi_{sk}^0$ leading to a return $R_{1k}^0 = R_f + \sum_{i=1}^{n} \xi_{sk}^0 (R_{i,1} - R_f)$ with the property that the contribution of the following part to the risk premium $\sigma^3 a_{i3}^0$ is given through:
\[
-\frac{\bar{\rho}_0}{\bar{\tau}_0} c_{i,M}^0 \sigma^3 = -\frac{\bar{\rho}_0}{\bar{\tau}_0} \text{Cov}(R_{i,1}, R_{1k}^0) \quad \text{for} \quad i = 1, \ldots, n, \quad \text{i.e.} \quad \xi_{sk}^0 = \sigma \Sigma_0^{-1} c_{i,M}^0;
\]

\(^{12}\)We only discuss the date 0 equilibrium demand and risk premiums and not the date 1 allocation. The latter can be interpreted through theorem 3 with a conditional viewpoint.
we refer to $\xi_0^{sk}$ as the date 0 skewness portfolio. We note that $\text{Cov}(R_{i,1}, R^{sk}) = \text{Cov}(R_{i,1}, (R_{M,1} - E(R_{M,1}))^2)$, such that there is a decomposition of the squared market return into the skewness portfolio with return $R_1^{sk}$ and a tracking error $T_1^{sk} = E(T_1^{sk})$ that is uncorrelated with all traded asset returns $R_i, i = 1, \ldots$. The skewness portfolio can therefore be seen as the mutual fund best able to track the squared market return. We find, again, that investors hold at date 0 the same portfolio of risky assets (so-called mutual funds) defined by portfolio weights $\xi$ and $\xi_0^{sk}$, respectively. As before, the difference in holdings comes from heterogeneity in risk and heterogeneity in skew tolerances.

However, in our dynamic extension, an additional term due to the co-leverage effect shows up in demand. Similarly to the skewness portfolio we define a fund with portfolio weights $\xi_0^{L}$ leading to a return $R_1^{L} = R_f + \sum_{i=1}^{n} \xi_i^{L}(R_{i,1} - R_f)$ with the property that the contribution of the following part to the risk premium $\sigma^3 a_i^{0,*}$ is given through:

$$\frac{1 - \bar{\rho}_0 + \bar{\gamma}_0}{\bar{\tau}_0^2} c_{i,M}^{0,*} = \frac{1 - \bar{\rho}_0 + \bar{\gamma}_0}{\bar{\tau}_0^2} \text{Cov}(R_{i,1}, R_1^{L})$$

for $i = 1, \ldots, n$, i.e. $\xi_0^{L} = \sigma \Sigma^{-1} c_{M}^{0,*};$ (52)

we refer to $\xi_0^{L}$ as the date 0 leverage portfolio. We note that

$$\text{Cov}(R_{i,1}, R^{L}) = \text{Cov}(R_{i,1}, (R_{M,2} - E(R_{M,2}|R_1))^2) = \text{Cov}(R_{i,1}, \text{Var}(R_{M,2}|R_1)) = \sigma^3 c_{i,M}^{0,*},$$

such that aggregating across the market gives

$$\sum_{i=1}^{n} \xi_i \text{Cov}(R_{i,1}, R^{L}) = \text{Cov} [R_{M,1}, \text{Var}(R_{M,2}|R_1)] = L_M,$$

i.e. the market leverage (volatility feedback) effect, see also equation (18). Therefore, we view this as a decomposition of the market leverage (volatility feedback) $\text{Var}(R_{M,2}|R_1)$ into the leverage portfolio with return $R_1^{L}$ and a tracking error $T_1^{L} = E(T_1^{L})$ that is uncorrelated with all traded asset returns $R_i, i = 1, \ldots, n$. The leverage portfolio can therefore be seen as the mutual fund best able to track the market leverage (volatility feedback) $\text{Var}(R_{M,2}|R_1)$.

Ultimately, theorem 4 provides a four-fund theorem: all investors hold at date 0 the same portfolio of risk assets (so-called mutual funds) defined by portfolio weights $\xi$, $\xi_0^{sk}$ and $\xi_0^{L}$, respectively; the difference in holdings comes from heterogeneity in risk, heterogeneity in skew tolerances and heterogeneity in wealth tolerances. Our dynamic extension of the theorem 4 leads us consider one more (mutual) fund: the leverage portfolio. The leverage portfolio can be
seen as providing some hedge against the market leverage (volatility feedback), i.e. it protects against future changes in market volatility. This is a form of intertemporal hedge demand that is common in dynamic portfolio selection problems, see, e.g. Liu (2007), and Chacko and Viceira (2005).

5 Unspanned SDF in polynomials of market return

Our analysis showed that a dynamic extension introduces an additional priced factor that is overlooked in the standard approach of a Taylor series expansion of the representative agents utility function in single period models. This section focuses on the single-period setup and studies whether tracking errors in incomplete markets lead to additional priced factors that have been overlooked in the current literature.

So far, we approximated the risk premium in the single period case through the first two terms \( a_2 \sigma^2 + a_3 \sigma^3 \) of its expansion (3); in this approximation, a polynomial of degree two in the market return is a sufficient statistic to define the SDF, see equation (21); this is in accordance with the representative agent model of Dittmar (2002). However, this ignores the impact of higher order terms in the expansion (3). This section carries out a careful investigation of higher order terms \( a_{ij} \sigma^j \) \((j = 4, 5)\) in the expansion (3) and shows that polynomials in the market return do not capture completely the pricing impact of investors’ heterogeneity in incomplete markets.

5.1 The spanning issue with polynomials of market return

Following the logic of Harvey and Siddique (2000) as well as based on the representative agent models in Dittmar (2002) and Chung et al. (2006), we may expect that additional terms in the expansion (3) lead to a characterization of the risk premium on asset \( i, i = 1, \ldots, n \) as a linear function that consists not only of the standard market beta \( \sigma^2 b_{i,M} \) and of the standard co-skewness \( \sigma^3 c_{i,M} \), but also of the co-kurtosis \( \sigma^4 d_{i,M} \), the co-pentosis \( \sigma^5 f_{i,M} \), and so on, as we go further in the expansion. This would lead to SDFs represented as linear functions of \( R_M \) and \((R_M - E(R_M))^j, j = 2, 3, 4, \ldots, \) i.e. as polynomials in the market return \( R_M \).

The problem with this reasoning is that it overlooks a form of market incompleteness, namely the impossibility to get a perfect hedge for powers of the market return from the payoff of linear portfolios. Let us characterize this issue for the power \( j = 2 \). As explained in section 4 above
(see equation (33)) there is in general a tracking error:

\[(R_M - E(R_M))^2 = R_{sk} - E(R_{sk}) + T_{sk}.\]

By construction, the tracking error \((T_{sk} - E(T_{sk}))\) is uncorrelated with any primitive asset return, so that the squared market return captures perfectly the price of skewness risk:

\[\sigma^3 c_{i,M} = Cov[R_t, R_{sk} - E(R_{sk})] = Cov[R_t, R_{sk}], \text{ for all } i = 1, \ldots, n.\]

Unfortunately, this may not be true anymore when it comes to the price of kurtosis risk and that of pentosis risk. In the former case we have

\[\sigma^4 d_{i,M} = Cov[R_t, (R_M - E(R_M))(R_{sk} - E(R_{sk})) + Cov[R_t, (R_M - E(R_M))T_{sk}],\]

while we find in the latter case

\[\sigma^5 f_{i,M} = Cov[R_t, (R_{sk} - E(R_{sk}))^2] + Cov[R_t, (T_{sk})^2] + 2 Cov[R_t, (R_{sk} - E(R_{sk}))T_{sk}]
= -Cov[R_t, (R_{sk} - E(R_{sk}))^2]
+ 2 Cov[R_t, (R_{sk} - E(R_{sk}))^2(R_{sk} - E(R_{sk}))] + Cov[R_t, (T_{sk})^2].\]

This is the reason why we may expect that proper pricing of higher order risks will involve pricing factors like \((R_M - E(R_M))(R_{sk} - E(R_{sk}))(R_{sk} - E(R_{sk}))^2\) and \((R_M - E(R_M))^2(R_{sk} - E(R_{sk}))^2\) in order to span the otherwise unhedgeable risk brought by \((R_M - E(R_M))T_{sk}\) and \((T_{sk})^2\) respectively. In other words, in case of market incompleteness, spanning the SDF may require not just polynomials in the single variable \((R_M - E(R_M))\), but polynomials in two variables: \((R_M - E(R_M))\) and, in addition, \((R_{sk} - E(R_{sk}))\). Note that only these two variables show up because we have limited our analysis so far to the tracking error on the skewness portfolio.

Higher order terms in the expansions take polynomials with additional variables corresponding to returns on additional mimicking portfolios. These additional mimicking portfolios consist of two types: first, there are mimicking portfolios of higher powers of returns. We will illustrate this remark in subsection 5.3 below by introducing, similarly to the skewness portfolio, a kurtosis portfolio with return \(R_{kurt}\) and a tracking error \((T_{kurt} - E(T_{kurt}))\) on it:

\[(R_M - E(R_M))^3 = R_{kurt} - E(R_{kurt}) + T_{kurt}, \quad (53)\]
where $T^{kurt}$ is uncorrelated with all primitive asset returns $R_i, i = 1, ..., n$. Second, there are mimicking portfolios for products of market returns with tracking errors. In the next subsection we will introduce, similarly to the skewness portfolio, a co-cross skewness portfolio portfolio with return $R^{*sk}$ and a tracking error $(T^{*sk} - E(T^{*sk}))$ on it:

$$ (R_M - E(R_M))(R^{sk} - E(R^{sk})) = R^{*sk} - E(R^{*sk}) + T^{*sk}, $$

where $T^{*sk}$ is uncorrelated with all primitive asset returns $R_i, i = 1, ..., n$. We will then illustrate in subsection 5.3 the pricing impact of products of the market return with this co-cross skewness portfolio. Overall, we expect pricing kernels that capture pentosis risk to be polynomials of four variables: $(R_M - E(R_M))$, $(R^{sk} - E(R^{sk}))$, $R^{*sk} - E(R^{*sk})$ and $(R^{kurt} - E(R^{kurt}))$.

Of course, market incompleteness, as manifest in the non-zero tracking errors $(T^k - E(T^k))$, $(T^{*sk} - E(T^{*sk}))$ and $(T^{kurt} - E(T^{kurt}))$ will have a pricing impact only if it matters for individual investors in equilibrium, i.e. these investors are heterogeneous in terms of preferences for higher order moments.

### 5.2 Co-kurtosis pricing and unspanned co-cross skewness

In this subsection, we go one step further than subsection 3.3. We want to characterize the pricing factors which matter to describe the approximation of the risk premium $E(R_i) - R_f$ through the first three terms $a_{i2}\sigma^2 + a_{i3}\sigma^3 + a_{i4}\sigma^4$ of its expansion (3). We show that the addition of the third term $\sigma^{\alpha}a_{i4}$ may introduce two additional pricing factors: First, we find an additional power of the market return, namely the cubic one, to go beyond quadratic SDF of section 4, as in the representative agent model of Dittmar (2002). However, second, we also find the co-cross skewness factor $(R_M - E(R_M))(R^{sk} - E(R^{sk}))$. This factor will be priced in equilibrium if and only if investors display some heterogeneity in their skew tolerances.

Here we extend definitions of aggregate characteristics of preferences introduced earlier as follows:

$$ \tilde{\rho} = \frac{\sum_{s=1}^{S} q_s \tau_s \rho_s}{\sum_{s=1}^{S} q_s \tau_s}, \quad \tilde{\kappa} = \frac{\sum_{s=1}^{S} q_s \tau_s \kappa_s}{\sum_{s=1}^{S} q_s \tau_s}, \quad Var(\rho) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \tilde{\rho})^2}{\sum_{s=1}^{S} q_s \tau_s}. $$

(55)

We remind the definition of individual skew-tolerance $\rho_s$ and define kurtosis-tolerance $\kappa_s$ by

$$ \rho_s = \frac{u'_s(q_s R_f) u''_{ss}(q_s R_f)}{2[u''_{ss}(q_s R_f)]^2}, \quad \kappa_s = \frac{|u'_s(q_s R_f)|^2 u''_{sss}(q_s R_f)}{6[u''_{ss}(q_s R_f)]^3}. $$

(56)

The approximation $E(R_i) - R_f \approx a_{i2}\sigma^2 + a_{i3}\sigma^3 + a_{i4}\sigma^4$ leads to complete theorem 3 by
\textbf{Theorem 5}  With heterogeneous investors \( s = 1, \ldots, S \) we complete the characterization of the equilibrium risk premium \( \pi_i(\sigma) = E(R_i) - R_f = \sum_{j=2}^{\infty} a_{i}^{j} \sigma^{j} \) in theorem 3 by

\[
\sigma^{4} a_{i4} = \frac{\bar{\kappa}}{\bar{\tau}^{3}} d_{i,M} + \frac{1 - 3\bar{\rho}}{\bar{\tau}^{3}} Var(\rho) \left( R_M - E(R_M) \right)(R_{sk} - E(R_{sk})), \tag{57}
\]

where \( \bar{\tau} \) is defined in equation (28) and \( \bar{\rho}, \bar{\kappa}, Var(\rho) \) are defined in equation (55).

Comparing the description of \( a_{i4} \) in theorem 5 to that in theorem 1, i.e. with the fourth order risk premium term in the case of a representative agent, we see that the covariance of individual asset returns \( i \) with the co-cross skewness factor \( (R_M - E(R_M))(R_{sk} - E(R_{sk})) \) enters into pricing. We will discuss below that this covariance may not vanish. Before doing so, we take a look at risk premiums from the perspective of SDF and priced factors: the approximation (57) puts forward an SDF \( H(3) \) such that up to constants:

\[
(-R_f)H(3,1) = \left\{ \frac{1}{\bar{\tau}} + \frac{1 - 3\bar{\rho}}{\bar{\tau}^{3}} Var(\rho) \right\} R_M - \frac{\bar{\rho}}{\bar{\tau}^{2}} (R_M - E(R_M))^{2} + \frac{\bar{\kappa}}{\bar{\tau}^{3}} (R_M - E(R_M))^{3},
\]

\[
(-R_f)H(3,2) = -2 \frac{Var(\rho)}{\bar{\tau}^{3}} (R_M - E(R_M))(R_{sk} - E(R_{sk})). \tag{59}
\]

Here, \( H(3,1) \) is the classical cubic SDF provided by representative agent models, see, e.g. Dittmar (2002). As usual, the three pricing factors \( R_M, (R_M - E(R_M))^{2} \) and \( (R_M - E(R_M))^{3} \) have prices proportional to risk aversion, skew tolerance and kurtosis tolerance, respectively. Note that adding the expansion term \( \sigma^{4} a_{4} \) has led us to adding the correction term \( (1 - 3\bar{\rho})Var(\rho)/\bar{\tau}^{3} \) to the standard weight \((1/\bar{\tau})\) of the market return; note also, however, that this correction term is of lower order in small noise expansions since \( Var(R_M) \) is proportional to \( \sigma^{2} \).

More important, by comparison with the classical SDF that is a polynomial in the market return, our setting may add one pricing factor \( (R_M - E(R_M))(R_{sk} - E(R_{sk})) \). It is worth analyzing what may make it necessary to account explicitly for this additional factor within the SDF, beyond the common cubic market return \( (R_M - E(R_M))^{3} \). First we note that by definition of the tracking error

\[
(R_M - E(R_M))(R_{sk} - E(R_{sk})) \neq (R_M - E(R_M))^{3} \iff T^{sk} \neq 0.
\]
But, even more important we have:

\[
\text{Cov} \left[ (R_M - E(R_M))(R_{sk} - E(R_{sk})), R_i \right] \neq \text{Cov} \left[ (R_M - E(R_M)^3, R_i \right]
\]

\[\iff \text{Cov} \left[ (R_M - E(R_M))T_{sk}, R_i \right] \neq 0 \iff E \left[ (R_M - E(R_M))T_{sk}R_i \right] \neq 0.\]

In other words, the tracking error, albeit uncorrelated with all asset returns, may become correlated when multiplied by the market return. The resulting additional pricing factor has a price proportional to the cross-sectional variance \(\text{Var}(\rho)\) of skew tolerances across investors. In particular, this additional factor is priced in equilibrium if and only if investors are heterogeneous in terms of skew tolerances. We call co-cross skewness of asset \(i\) the risk quantity \(\text{Cov}[(R_M - E(R_M))(R_{sk} - E(R_{sk})), R_i]\) that shows up in the risk premium formula (57). The importance of the cross-sectional variance \(\text{Var}(\rho)\) in \(H(3, 2)\) should not come as a surprise: it matches a similar result of Constantinides and Duffie (1996), where market incompleteness (here: tracking error \((T_{sk} - E(T_{sk}))\)) gives rise to an additional pricing factor with a price proportional to the variance of the distribution of heterogeneity.

As far as market incompleteness is concerned, it is worth noting that non-zero individual kurtosis tolerances will not lead the agents to invest directly in the cubic market return but only in its best tracking portfolio with return \(R^{kurt}\) corresponding to the affine regression defined in (53):

\[
R^{kurt} = R_f + \sum_{i=1}^{n} \xi^{kurt}_i (R_i - R_f), \text{ where } \xi^{kurt} = (\xi^{kurt}_i)_{1 \leq i \leq n} = \sigma^2 \Sigma^{-1} d.
\]

Here, \(d_M = (d_{i,M})_i\) is the vector of market co-kurtosis of asset \(i\) introduced in definition 1, \(\xi^{kurt}\) defines the so-called kurtosis portfolio that is hold in equilibrium by agent \(s\) insofar as her kurtosis tolerance differs from the average. Similarly, due to market incompleteness the agent cannot invest into \((R_M - E(R_M))(R_{sk} - E(R_{sk}))\) but only in the return \(R^{sk}\) of the best tracking portfolio \(\xi^{sk}\), corresponding to its affine regression defined in (54):

\[
R^{sk} = R_f + \sum_{i=1}^{n} \xi^{sk}_i (R_i - R_f), \text{ where } \xi^{sk} = \sigma^2 \Sigma^{-1} c(\xi, \xi^{sk}).
\]

where we used the vector of co-cross skewness \(c(\xi, \xi^{sk}) = (c_i(\xi, \xi^{sk}))_{1 \leq i \leq n}\) with

**Definition 3** The co-cross skewness of asset \(i\) is defined as

\[
c_i(\xi, \xi^{sk}) = \frac{1}{\sigma^4} \text{Cov}[(R_M - E(R_M))(R_{sk} - E(R_{sk})), R_i] = \text{Cov}[(\xi^{sk})^\top Y)(\xi^{sk})^\top Y, Y_i].
\]
Similarly to (31), we show in the appendix that the equilibrium portfolio of investor \( s \) involves a third order term \( \sigma^2 \theta_{s2} \) such that:

\[
\sigma^2 \theta_{s2} = -\frac{\tau_s}{\bar{\tau}^4} \left\{ \left( \kappa_s - \bar{\kappa} \right) \xi^{kurt} - 2[\rho_s(\rho_s - \bar{\rho}) - Var(\rho)] \xi^{*sk} - 3(\rho_s - \bar{\rho})Var(R_M) \xi \right\}.
\] (60)

Note that, by contrast with (31), this expansion term involves two departures from the mean variance portfolio: Most important, the kurtosis portfolio leads to a portfolio adjustment when the kurtosis tolerance of investor \( s \) is different from average. In addition to that, the co-cross skewness portfolio \( \xi^{*sk} \) is reflected in the portfolio holdings of investor \( s \) when the term \( \rho_s(\rho_s - \bar{\rho}) \) differs from its cross-sectional average \( Var(\rho) \).

5.3 Co-pentosis pricing and unspanned squared skewness portfolio return

In this subsection, we go one step further than in the former one. We want to characterize the pricing factors which matter to describe the approximation of the risk premium \( (E(R_i) - R_f) \) through the first four terms \( \sigma^2(a_{i2} + \sigma a_{i3} + \sigma^2 a_{i4} + \sigma^3 a_{i5}) \) of its expansion (3). We show that the addition of the fourth term \( \sigma^5 a_{i5} \) may lead to introduce five additional pricing factors: Most important, we find an additional power of the market return, namely the quartic one, to go beyond cubic pricing kernels of the previous subsection as in the representative agent model of Dittmar (2002). However, we also find the squared skewness portfolio factor \( (R_{sk} - E(R_{sk}))^2 \) (second), the co-cross skewness portfolio \( (R_M - E(R_M))(R_{sk} - E(R_{sk})) \) (third), the co-cross-kurtosis portfolio \( (R_M - E(R_M))(R_{kurt} - E(R_{kurt})) \) (fourth) and the co-cross quadratic market-skewness portfolio \( (R_M - E(R_M))^2(R_{sk} - E(R_{sk})) \) (fifth additional factor). These factors will be priced in equilibrium when investors display heterogeneous skew tolerances (for a non-zero price of the squared skewness portfolio factor), and when these skew tolerances display some cross sectional correlation with kurtosis tolerances (for a non-zero price of the co-cross-kurtosis and the co-cross quadratic market-skewness portfolios).

Here we extend the definitions of aggregate characteristics of preferences introduced earlier by introducing \( A_1(\rho, \kappa) \), \( A_2(\rho, \kappa) \) as follows:

\[
A_1(\rho, \kappa) = -\bar{\rho} + 2\bar{\rho}^2 - 4Var(\rho) + \bar{\kappa}, \quad A_2(\rho, \kappa) = -\bar{\rho} + \bar{\rho}^2 - 8Var(\rho) + 3\bar{\kappa}.
\] (61)
The approximation $E(R_i) - R_f \approx a_{i2}\sigma^2 + a_{i3}\sigma^3 + a_{i4}\sigma^4 + a_{i5}\sigma^5$ leads to complete theorem 3 by

**Theorem 6** With heterogeneous investors $s = 1, \ldots, S$ we complete the characterization of the equilibrium risk premium $\pi_i(\sigma) = E(R_i) - R_f = \sum_{j=2}^{\infty} a_{ij}\sigma^j$ in theorems 3, 5 by

$$\bar{\pi}^5\sigma_{45} = A_1(\rho, \kappa)S_Mb_{i,M} + A_2(\rho, \kappa)V_Mc_{i,M} - Var(\rho)Cov[(R^{sk} - E(R^{sk}))^2, R_i] - \bar{\chi}f_{i,M}$$

$$-4\{Skew(\rho) + \bar{\rho}Var(\rho)\} Cov[(R_M - E(R_M))(R^{sk} - E(R^{sk})), R_i]$$

$$+ Cov(\rho, \kappa)\left\{2Cov[(R_M - E(R_M))(R^{kurt} - E(R^{kurt})), R_i] + 3Cov[(R_M - E(R_M))^2(R^{sk} - E(R^{sk})), R_i] \right\}.$$

(62)

where the quantities $A_1(\rho, \kappa), A_2(\rho, \kappa)$ are defined in equation (61).

Comparing this to theorem 1, i.e. with the fifth order risk premium term in the representative agent case, we see additional pricing factors due to the cross-sectional distribution of the preference parameters and to the cross-sectional correlation of preference parameters. The cross-sectional distributions as well as the cross-sectional correlation of investors’ preferences are known to be empirically relevant to understand individual investors’ portfolio choice, see, e.g. Noussair et al. (2014).

Regarding the impact of heterogeneity on the price of different risk factors, several remarks are in order. First, not surprisingly, the quartic market return has a non-zero price insofar as the investors display in average a non-zero pentosis tolerance:

$$\chi_s = \left[\frac{u_s'(q_sR_f)^3u_s^{[5]}(q_sR_f)}{24[u_s''(q_sR_f)]^4}\right], \quad \bar{\chi} = \frac{\sum_{s=1}^{S} q_s\tau_s\chi_s}{\sum_{s=1}^{S} q_s\tau_s}.$$

Second, the cross-sectional variance $Var(\rho)$ of skew-tolerances scales not only the price of the co-cross-skewness portfolio (as seen in the previous subsection) but also the price of the squared skewness portfolio. Third, the cross-sectional covariance $Cov(\rho, \kappa)$ between skew tolerances and kurtosis tolerances scales the price of both the co-cross-kurtosis and the co-cross quadratic market-skewness portfolios. Here, analogous to the previous definitions of cross sectional averages and variances, this covariance is defined by

$$Cov(\rho, \kappa) = \frac{\sum_{s=1}^{S} q_s\tau_s(\rho_s - \bar{\rho})(\kappa_s - \bar{\kappa})}{\sum_{s=1}^{S} q_s\tau_s}.$$
Fourth, the cross-sectional skewness of skew tolerances

\[ Skew(\rho) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \bar{\rho})^3}{\sum_{s=1}^{S} q_s \tau_s} \]

scales the product of the co-cross skewness portfolio \( R^{sk} - E[R^{sk}] \) with the market return.

Finally, the quantities \( A_1(\rho, \kappa) \) and \( A_2(\rho, \kappa) \) come to modify at higher orders the prices of the market return and the skewness portfolio respectively.

Adding \( \sigma a_5 \) of equation (62) to the approximation (57) leads us to the following SDF, up to a constant:

**Theorem 7** Up to risk premium terms of order higher than five in \( \sigma \), the SDF is, up to a constant:

\[
H(4) = H(4, 1) + H(3, 2) + H(4, 2) + H(4, 3) + H(4, 4) + H(4, 5),
\]

where

\[
(-R_f)H(4, 1) = \left\{ \frac{1}{\tau} + \frac{1 - 3\bar{\rho}}{\tau^3} Var(R_M) + \frac{A_1(\rho, \kappa)}{\tau^4} Skew(R_M) \right\} R_M
\]

\[-\left\{ \bar{\rho} - \frac{A_2(\rho, \kappa)}{\tau^4} Var(R_M) \right\} (R_M - E(R_M))^2
\]

\[+ \frac{\bar{\kappa}}{\tau^3} (R_M - E(R_M))^3 - \frac{\bar{\chi}}{\tau^3} (R_M - E(R_M))^4.\]

The additional pricing factor \( H(3, 2) \) is defined in (59). The additional pricing factors \( H(4, j) \) \((j=2,3,4,5)\) are defined as

\[
(-R_f)H(4, 2) = 2 \frac{Cov(\rho, \kappa)}{\tau^4} (R_M - E(R_M)) (R^{kurt} - E(R^{kurt})),
\]

\[
(-R_f)H(4, 3) = -4 \frac{Skew(\rho) + \bar{\rho} Var(\rho)}{\tau^4} (R_M - E(R_M)) (R^{sk} - E(R^{sk})),
\]

\[
(-R_f)H(4, 4) = 3 \frac{Cov(\rho, \kappa)}{\tau^4} (R_M - E(R_M))^2 (R^{sk} - E(R^{sk})),
\]

\[
(-R_f)H(4, 5) = - \frac{Var(\rho)}{\tau^4} (R^{sk} - E(R^{sk}))^2.
\]

Note that \( H(4, 1) \) is a fourth order polynomial in the market return that corresponds to the classical quartic SDF provided by representative agent models, see, e.g. Dittmar (2002). As usual, the four pricing factors \( R_M, (R_M - E(R_M))^2, (R_M - E(R_M))^3 \) and \( (R_M - E(R_M))^4 \) have

39
prices proportional to risk aversion, skew tolerance, kurtosis tolerance and pentosis tolerance, respectively.

In comparison to the cubic SDF (58), note that adding the expansion term $\sigma^5a_3$ has led us to adding two correction terms. First, a further correction by $A_1(\rho, \kappa)/\bar{\tau}^4S_M$ to the standard weight $(1/\bar{\tau})$ of the market return; note this correction is in addition to the earlier term $(1 - 3\bar{\rho})V_M/\bar{\tau}^3$ that showed up in (58) as a correction to the quadratic SDF of (21). Second, a correction $-A_2(\rho, \kappa)/\bar{\tau}^4V_M$ to the standard weight $(\bar{\rho}/\bar{\tau}^2)$ of the squared market return. Note that both correction terms are of lower order in small noise expansions since $V_M$ is proportional to $\sigma^2$ and $S_M$ is proportional to $\sigma^3$.

Similar to $H(3, 2)$ these pricing factors embody a form of market incompleteness, here the impossibility to hedge the squared market return, the cubic market return as well as the cross product between market return and the return on the skewness portfolio. Ultimately, the SDF will be a polynomial of all these factors at various powers; note that only the powers that appear through $H(3, 2)$ contribute to fourth moments (kurtosis pricing), while only the terms in $H(4, j)$ $(j=2, 3, 4, 5)$ lead to fifth moments (pentosis pricing).

The additional factors $H(3, 2)$ and $H(4, j)$ $(j=2, 3, 4, 5)$, defined in equations (63-63) above, will contribute to pricing whenever there is some heterogeneity in preferences for higher order moments ($\text{Var}(\rho), \text{Cov}(\rho, \kappa)$ and $\text{Skew}(\rho)$). In particular, note that even in a world with heterogeneous risk averse investors that have only non-vanishing skew tolerances, i.e. investors with $\kappa_s = \chi_s = 0$ for all $s = 1, \ldots, S$, the additional factors $H(3, 2), H(4, 3)$ and $H(4, 5)$ will be priced as long as there is heterogeneity of preferences for higher order moments with $\text{Var}(\rho) \neq 0$ or $\text{Skew}(\rho) \neq 0$.

Overall, this section shed more light on how heterogeneity of investors’ preferences do have an impact on the spanning of the pricing kernel. The main message is threefold:

(i) In a world without investor heterogeneity, the only (possibly) relevant pricing factors are the powers of the market return $(R_M - E(R_M))^j$, $j = 1, 2, 3, 4, \ldots$ as in the representative agent model of Dittmar (2002).

(ii) In a world with heterogeneity, the pricing factors defined by powers of the market return still show up with a non-zero price if and only if there is a non-zero aggregated preference
for the corresponding moment.

(iii) However, in a world with heterogeneity of preferences, additional pricing factors must be considered based on tracking errors as explained above. The equilibrium prices of associated risks can be interpreted as measures of heterogeneity of preferences for associated higher order moments.

6 Conclusion

This paper showed a three-fund separation theorem for investors with mean-variance-skewness preferences: agents hold the risk-free asset, the market portfolio and a skewness portfolio. In complete markets, i.e. when the skewness portfolio is the portfolio that replicates the squared market return, the skewness risk premium is driven by co-variation with the squared market such that pricing is characterized by a stochastic discount factor that is a quadratic polynomial in the market return. In incomplete markets, however, the skewness portfolio is the projection of the squared market return on the available assets leading to a tracking error. In world with heterogeneity of preferences, cross moments of the tracking error and the market return at various orders contribute to pricing in addition to the skewness portfolio; in stochastic discount factors this shows up as priced factors in addition to factors that are powers of market returns. In a two-period extension of our baseline case, we introduce non-myopic investors: we find that such investors hold an additional fund, called leverage portfolio, to replicate (intertemporal) changes in the market return variance, a market volatility (leverage feedback) effect; the market volatility then shows up as another priced factor, that is in addition to powers of market returns and in addition to the products of tracking error portfolios with market returns at various orders. Empirical studies of skewness risk based on stochastic discount factors that are polynomials in the market return miss all these additional factors. Even more pricing factors should have been introduced if, besides accommodating heterogeneity of preferences, we had allowed some investors’ heterogeneity regarding their expectations of future market volatility.
Appendix

A Single-period Proofs

Proof of Lemma 1. The first order conditions (2) can be rewritten as

\[ \pi_i(\sigma) E[u'(W(\sigma))] + \sigma E[u'(W(\sigma))Y_i] = 0. \]

Differentiating w.r.t. \( \sigma \) gives

\[ \pi_i'(\sigma) E[u'(W(\sigma))] + \pi_i(\sigma) \frac{\partial E[u'(W(\sigma))]}{\partial \sigma} + E[u'(W(\sigma))Y_i] + \sigma \frac{\partial E[u'(W(\sigma))Y_i]}{\partial \sigma} = 0. \]

Knowing that \( \pi_i(0) = 0 \) we find for \( \sigma = 0 \) that

\[ \pi_i'(0) u'(W(0)) + u'(W(0)) E(Y_i) = 0. \]

We conclude \( \pi_i'(0) = 0 \) since \( u'(W(0)) \neq 0 \) and \( E(Y_i) = 0 \).

For our derivations in the appendix, we introduce the concepts of co-skewness, co-kurtosis, and co-pentosis of asset \( i \) with a portfolio \( \theta \) by

\[ c_i(\theta) = \text{Cov} \left( Y_i, (\theta^\top Y)^2 \right), \quad d_i(\theta) = \text{Cov} \left( Y_i, (\theta^\top Y)^3 \right), \quad f_i(\theta) = \text{Cov} \left( Y_i, (\theta^\top Y)^4 \right), \]

and we denote the co-cross-skewness and the co-cross-kurtosis with portfolios \( \theta, \tilde{\theta} \) by

\[ c_i(\theta, \tilde{\theta}) = \text{Cov} \left( Y_i, (\theta^\top Y) (\tilde{\theta}^\top Y) \right), \quad d_i(\theta, \tilde{\theta}) = \text{Cov} \left( Y_i, (\theta^\top Y)^2 (\tilde{\theta}^\top Y) \right). \]

In addition we denote the third moment of a a portfolio \( \theta \) by

\[ Sk(\theta) = E \left( (\theta^\top Y)^3 \right). \]

We will also write these as vectors as follows: \( c(\theta) = (c_i(\theta))_{1 \leq i \leq n}, \ d(\theta) = (d_i(\theta))_{1 \leq i \leq n}, \ f(\theta) = (f_i(\theta))_{1 \leq i \leq n}, \ c(\theta, \tilde{\theta}) = (c_i(\theta, \tilde{\theta}))_{1 \leq i \leq n}, \ d(\theta, \tilde{\theta}) = (d_i(\theta, \tilde{\theta}))_{1 \leq i \leq n} \). Finally, for \( \theta = \xi \), we consider the following market quantities:

\[ c = c(\xi), \quad d = d(\xi), \quad f = f(\xi), \quad Sk = Sk(\xi), \]

\[ \xi_{sk}^k = \Sigma^{-1} c, \quad \xi_{sk}^{ssk} = \Sigma^{-1} c(\xi, \xi_{sk}^k), \quad \xi_{sk}^{kurt} = \Sigma^{-1} d. \]
A.1 Characterizing the expansion for the product of marginal utility and net returns

This subsection derives an expansion of the product of marginal utility with the net return for an asset. We are interested in this expansion since the first order conditions of optimal demand tell us that the expectation of this product vanishes for all assets at the optimal demand. We will use this representation in the next subsection A.2 to derive risk premiums in the representative agent analysis; we will also use it in subsections A.3, A.4 to characterize the equilibrium with heterogeneous investors.

Assuming for sake of expositional simplicity that the utility function $u_s$ is analytical,

$$h_{is}(\sigma) = u_s'(W_s(\theta_s(\sigma))) \cdot (R_i(\sigma) - R_f)$$

is an analytical function of the scale parameter $\sigma$ with a zero constant term since $R_i(0) = R_f$.

Thus, we write:

$$h_{is}(\sigma) = \sum_{j=1}^{\infty} h_{isj} \sigma^j.$$ 

We denote $r(\sigma) = (R_i(\sigma) - R_f)_{1 \leq i \leq n}$ the vector of net risky returns. We have by definition:

$$r_i(\sigma) = \sigma Y_i + \sum_{j=2}^{\infty} a_{ij} \sigma^j, i = 1, \ldots, n \quad (67)$$

We compute the series coefficients $h_{isj}$ by multiplying the expansion (67) with the series expansion of $u_s'(W_s(\theta(\sigma)))$. We first write a Taylor series around $q_s R_f$:

$$u_s'(W_s(\theta)) = u_s'(q_s R_f) + u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{ks} r_k \right) + \frac{1}{2} u_s'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{ks}^2 r_k \right)^2 \quad (68)$$

$$+ \frac{1}{6} u_s''''(q_s R_f) q_s^3 \left( \sum_{k=1}^{n} \theta_{ks}^3 r_k \right) + \frac{1}{24} u_s^{(5)}(q_s R_f) q_s^4 \left( \sum_{k=1}^{n} \theta_{ks}^4 r_k \right)^4 + \ldots$$

Now, we deduce from (67) the series expansion of portfolio returns:

$$\sum_{k=1}^{n} \theta_{ks}(\sigma) r_k = \sigma \sum_{k=1}^{n} \theta_{ks0} Y_k + \sigma^2 \sum_{k=1}^{n} (\theta_{ks1} Y_k + \theta_{ks0} a_{k2}) \quad (69)$$

$$+ \sigma^3 \sum_{k=1}^{n} (\theta_{ks2} Y_k + \theta_{ks1} a_{k2} + \theta_{ks0} a_{k3})$$

$$+ \sigma^4 \sum_{k=1}^{n} (\theta_{ks3} Y_k + \theta_{ks2} a_{k2} + \theta_{ks1} a_{k3} + \theta_{ks0} a_{k4}) + \ldots$$
Jointly, (68) and (69) imply:

\[
u_s'(W_s(\theta)) = u_s'(q_s R_f) + \sigma u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k \right) + \sigma^2 \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k) \right) + \frac{1}{2} u_s''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k \right)^2 \right\} + \sigma^3 \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k \right) \right\} + \sigma^4 \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k \right) \right\} + \ldots \]

Multiplying this with the expansion (67) of \( r_i \) and collecting terms, this now implies that:

\[
h_{i1} = u_s'(q_s R_f) Y_i,
\]

\[
h_{i2} = u_s'(q_s R_f) a_{i2} + u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k Y_i \right),
\]

\[
h_{i3} = u_s'(q_s R_f) a_{i3} + u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k a_{i2} \right) + \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k) \right) + \frac{1}{2} u_s''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k \right)^2 \right\} Y_i,
\]

that

\[
h_{i4} = u_s'(q_s R_f) a_{i4} + u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k \alpha} Y_k) a_{i3} + \sum_{k=1}^{n} (\theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k) a_{i2} \right) + \frac{1}{2} u_s''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{k \alpha} Y_k \right)^2 a_{i2} + \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k + \theta_{k \alpha} Y_k) \right) \right\} Y_i,
\]

44
and that
\[ h_{i5} = u_s'(q_s R_f) a_{i5} + u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{k0} Y_k \right) a_{i4} \]
\[ + \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k1} Y_k + \theta_{k0} q_k) (\theta_{k0} q_k) \right) + \frac{1}{2} u_s'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{k0} Y_k \right)^2 \right\} a_{i3} \]
\[ + \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k2} Y_k + \theta_{k1} a_k + \theta_{k0} q_k) (\theta_{k0} q_k) \right) + \frac{1}{6} u_s'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{k0} Y_k \right)^2 \right\} a_{i2} \]
\[ + \left\{ u_s''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{k3} Y_k + \theta_{k2} a_k + \theta_{k1} a_k + \theta_{k0} q_k) (\theta_{k0} q_k) \right) + \frac{1}{6} u_s'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{k0} Y_k \right)^2 \right\} a_{i1} \]
\[ + \left\{ \frac{1}{2} u_s'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} (\theta_{k1} Y_k + \theta_{k0} a_k) \right)^2 \right\} Y_i. \]

A.2 Proof of theorem 1

This subsection uses the results of the previous subsection to prove theorem 1. The representative investor economy can be interpreted as an economy with a single investor \( S = 1 \), utility function \( u_s = u \) and demand \( \theta = \xi \), i.e. \( \theta_{i0} = \xi_i, \theta_{ij} = 0, j = 1, 2, \ldots \).

We denote \( g_i(\sigma) = u'(W(\sigma))(R_i(\sigma) - R_f) \) the product of net return of asset \( i \) and marginal utility, evaluated with demand equal supply \( (\theta = \xi) \). Assuming for sake of expositional simplicity that the utility function \( u \) is analytical, \( g_i(\sigma) \) is an analytical function of the scale parameter \( \sigma \) with a zero constant term since \( R_i(0) = R_f \). Thus we write:

\[ g_i(\sigma) = \sum_{j=1}^{\infty} g_{ij} \sigma^j \]

The first-order conditions then read for \( i = 1, \ldots, n \):

\[ E(g_{ij}) = 0, \forall j = 1, 2, 3, \ldots \quad (70) \]

Our goal is to characterize the coefficients \( a_{ij}, j = 2, 3, \ldots \) implied by equations (70). Our analysis in A.1 described the expansion terms \( h_{ijs} \); here these become the \( g_{ij} \) terms and the analysis there can be used to describe the first five terms in the expansion of \( g_i(\sigma) \) for each \( i = 1, \ldots, n \). Recall that \( E(Y_i) = 0 \) for \( i = 1, \ldots, n \). Up to order five, the informative optimality conditions are as follows:
1. $E(g_{i2}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'(q_{Rf})a_{i2} + u''(q_{Rf})q\sum_{k=1}^{n}\xi_kCov(Y_k,Y_i),$$

i.e. $a_{i2} = -\frac{u''(q_{Rf})q}{u'(q_{Rf})}Cov(Y_i,\xi^\perp Y) = \frac{1}{\tau}b_{i,M}$.

2. $E(g_{i3}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'(q_{Rf})a_{i3} + \frac{1}{2}u_{s}^{[3]}(q_{Rf})q^2Cov\left[Y_i, (\xi^\perp Y)^2\right],$$

i.e. $a_{i3} = -\frac{1}{2}\frac{u_{s}^{[3]}(q_{Rf})q^2}{u'(q_{Rf})}c_{i,M} = -\frac{\rho}{\tau^2}c_{i,M}$.

3. $E(g_{i4}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'(q_{Rf})a_{i4} + u''(q_{Rf})q\left(\sum_{k=1}^{n}\xi_ka_{k2}a_{i2}\right) + \frac{1}{2}u_{s}^{[3]}(q_{Rf})q^2Var(\xi^\perp Y)a_{i2}$$

$$+u_{s}^{[3]}(q_{Rf})q^2\left(\sum_{k=1}^{n}\xi_ka_{k2}\right)Cov\left[Y_i, \xi^\perp Y\right] + \frac{1}{6}u_{s}^{[4]}(q_{Rf})q^3Cov\left[Y_i, (\xi^\perp Y)^3\right],$$

i.e.

$$a_{i4} = \frac{\kappa}{\tau^3}d_{i,M} + \frac{1}{\tau}(\xi^\perp a_2)a_{i2} - \frac{\rho}{\tau^2}Var(\xi^\perp Y)a_{i2} - \frac{2\rho}{\tau^2}(\xi^\perp a_2)b_{i,M}.$$

However, we have seen that:

$$\xi^\perp a_2 = \frac{\xi^\perp b_M}{\tau} = \frac{V_M}{\tau}.$$  

Using this and the formula $a_{i2} = \frac{b_{i,M}}{\tau}$, we conclude:

$$a_{i4} = \frac{\kappa}{\tau^3}d_{i,M} + \frac{1}{\tau^3}(1 - 3\rho)V_Mb_{i,M}.$$  

4. $E(g_{i5}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'(q_{Rf})a_{i5} + u''(q_{Rf})q\left(\xi^\perp a_2\right)a_{i3} + \frac{1}{2}u_{s}^{[3]}(q_{Rf})q^2Var(\xi^\perp Y)a_{i3}$$

$$+u''(q_{Rf})q\left(\xi^\perp a_3\right)a_{i2} + \frac{1}{6}u_{s}^{[4]}(q_{Rf})q^3E(\xi^\perp Y)^3a_{i2} + u_{s}^{[3]}(q_{Rf})q^2Cov\left(\xi^\perp Y, Y_i\right)(\xi^\perp a_3)$$

$$+\frac{1}{2}u_{s}^{[4]}(q_{Rf})q^3Cov\left(\left(\xi^\perp Y\right)^2, Y_i\right)(\xi^\perp a_2) + \frac{1}{24}u_{s}^{[5]}(q_{Rf})q^4Cov\left(\left(\xi^\perp Y\right)^4, Y_i\right),$$

that is

$$a_{i5} = -\frac{X}{\tau^2}f_{i,M} + \frac{V_M}{\tau^2}a_{i3} - \frac{\rho}{\tau^2}V_Ma_{i3}$$

$$+\frac{1}{\tau}(\xi^\perp a_3)\frac{b_{i,M}}{\tau} + \frac{\kappa}{\tau^3}S_M\frac{b_{i,M}}{\tau} - \frac{2\rho}{\tau^2}b_{i,M}(\xi^\perp a_3) + \frac{3\kappa}{\tau^3}c_{i,M}\frac{V_M}{\tau}.$$  

46
As above we have used here the known identities:

\[
\xi^\perp a_2 = \frac{\xi^\perp b_M}{\tau} = \frac{V_M}{\tau}, a_{i2} = \frac{b_{i,M}}{\tau}.
\]

If we now remember that similarly:

\[
\xi^\perp a_3 = -\frac{\rho}{\tau^2} S_M, a_{i3} = -\frac{\rho}{\tau^2} c_{i,M},
\]

we end up with

\[
a_{i5} = -\frac{\chi}{\tau^4} f_{i,M} + \frac{3\kappa - \rho(1 - \rho)}{\tau^4} V_M c_{i,M} + \frac{\kappa - \rho(1 - 2\rho)}{\tau^4} S_M b_{i,M}.
\]

A.3 Characterizing individual demand for given risk premiums

The first order conditions (26) then read for \(i = 1, \ldots, n\):

\[
E(h_{i,j}) = 0, \forall j = 1, 2, 3, \ldots \tag{71}
\]

Our earlier analysis in subsection A.1 characterized the first five terms in the expansion of \(h_{i,j}\) for each \(i = 1, \ldots, n\), we study in our final step the expansion of the first order conditions \(E(h_{i,j}) = 0\) for \(j = 1, \ldots, 5\), and all \(i = 1, \ldots, n\). Recall that \(E(Y_i) = 0\) for \(i = 1, \ldots, n\) by definition, such that \(E(h_{i,1}) = 0\) for \(i = 1, \ldots, n\). Up to order five, the informative first order conditions are as follows:

1. \(E(h_{i,2}) = 0\) gives for \(i = 1, \ldots, n\):

\[
0 = u_s'(q_s R_f) a_{i2} + u_s''(q_s R_f) q_s \sum_{k=1}^n \theta_{ks0} Cov(Y_k, Y_i).
\]

In vector-matrix notation this reads

\[
0 = u_s'(q_s R_f) a_2 + u_s''(q_s R_f) q_s \Sigma \theta_{s0} \quad \text{i.e.} \quad \theta_{s0} = \tau_s \Sigma^{-1} a_2, \tag{72}
\]

where

\[
\tau_s = -\frac{u_s'(q_s R_f)}{q_s u_s''(q_s R_f)}
\]

stands for the relative risk tolerance.

2. The condition \(E(h_{i,3}) = 0\) gives for \(i = 1, \ldots, n\):

\[
0 = u_s'(q_s R_f) a_{i3} + u_s''(q_s R_f) q_s \sum_{k=1}^n \theta_{ks1} Cov(Y_k, Y_i) + \frac{1}{2} u_s'''(q_s R_f) q_s^2 Cov \left( \sum_{k=1}^n \theta_{ks0} Y_k \right)^2, Y_i \right).
\]

47
Using the definition of co-skewness with a portfolio that we introduced at the beginning of this appendix, this reads in vector-matrix notation:

\[ 0 = u'_s(q_s R_f)a_3 + u''_s(q_s R_f)q_s \Sigma \theta_{s1} + \frac{1}{2} u'''_s(q_s R_f)q_s^2 c(\theta_{s0}). \]

This gives

\[ \theta_{s1} = -\frac{u'''_s(q_s R_f)q_s}{2u''_s(q_s R_f)q_s} \Sigma^{-1} c(\theta_{s0}) - \frac{u'_s(q_s R_f)}{u''_s(q_s R_f)q_s} \Sigma^{-1} a_3 = \tau_s \Sigma^{-1} \left[ c(\theta_{s0}) \frac{\rho_s}{\tau_s^2} + a_3 \right], \tag{73} \]

where

\[ \rho_s = \frac{u'_s(q_s R_f)u'''_s(q_s R_f)}{2[u''_s(q_s R_f)]^2} \]

is the skew tolerance.

3. The condition \( E(h_{is4}) = 0 \) gives for \( i = 1, \ldots, n \):

\[ 0 = u'_s(q_s R_f)a_4 + u''_s(q_s R_f)q_s(\theta_{s0}a_2)a_2 + \frac{1}{2} u'''_s(q_s R_f)q_s^2 \text{Var}(\theta_{s0}^2 Y) a_2 \]
\[ + u''_s(q_s R_f)q_s \text{Cov}(\theta_{s2}^2 Y, Y_i) + u'''_s(q_s R_f)q_s^2 \text{Cov}[(\theta_{s0}^2 Y) (\theta_{s1}^2 Y + \theta_{s0}^2 a_2), Y_i] \]
\[ + \frac{1}{6} u'''_s(q_s R_f)q_s^3 \text{Cov}[(\theta_{s0}^2 Y)^3, Y_i]. \]

Using the definitions of co-cross-skewness and co-kurtosis with a portfolio that we introduced at the beginning of this appendix, this reads in vector-matrix notation:

\[ 0 = u'_s(q_s R_f)a_4 + u''_s(q_s R_f)q_s(\theta_{s0}a_2)a_2 + \frac{1}{2} u'''_s(q_s R_f)q_s^2 \text{Var}(\theta_{s0}^2 Y) a_2 \]
\[ + u''_s(q_s R_f)q_s \Sigma \theta_{s2} + u'''_s(q_s R_f)q_s^2 c(\theta_{s0}, \theta_{s1}) + u'''_s(q_s R_f)q_s^2 (\theta_{s0}a_2) \Sigma \theta_{s0} + \frac{1}{6} u'''_s(q_s R_f)q_s^3 d(\theta_{s0}). \]

This gives

\[ q_s \Sigma \theta_{s2} = q_s \tau_s a_4 - q_s(\theta_{s0}^2 a_2) a_2 + \frac{q_s \rho_s}{\tau_s} \text{Var}(\theta_{s0}^2 Y) a_2 \]
\[ + 2 q_s \rho \frac{q_s}{\tau_s} c(\theta_{s0}, \theta_{s1}) + 2 q_s \rho_s \theta_{s0} a_2 \Sigma \theta_{s0} - \frac{q_s \kappa_s}{\tau_s^2} d(\theta_{s0}), \tag{74} \]

where

\[ \kappa_s = \frac{[u'_s(q_s R_f)]^2 u'''(q_s R_f)}{6[u''_s(q_s R_f)]^3} \]

is the kurtosis tolerance.
4. Finally, the condition \( E(h_{is}) = 0 \) gives for \( i = 1, \ldots, n \):

\[
0 = u'_s(q_sR_f)a_{i5} + u''_s(q_sR_f)q_s(\theta_{s0}\alpha_2)a_{i3} + \frac{1}{2}u'''_s(q_sR_f)q_s^2\text{Var}(\theta_{s0}^+Y)a_{i3} \\
+ u'_s(q_sR_f)q_s(\theta_{s1}^+a_2 + \theta_{s0}^+a_3)a_{i2} \\
+ u''_s(q_sR_f)q_s^2\text{Cov}[\theta_{s0}^+Y, \theta_{s1}^+Y]a_{i2} + \frac{1}{6}u'''_s(q_sR_f)q_s^3E[(\theta_{s0}^+Y)^3]a_{i2} \\
+ u'_s(q_sR_f)q_s\text{Cov}(\theta_{s2}^+Y, Y_i) + u''_s(q_sR_f)q_s^2\text{Cov}[\theta_{s0}^+Y] (\theta_{s2}^+Y + \theta_{s1}^+a_2 + \theta_{s0}^+a_3), Y_i] \\
+ \frac{1}{2}u'''_s(q_sR_f)q_s^2\text{Cov}[(\theta_{s1}^+Y + \theta_{s0}^+a_2)^2, Y_i] \\
+ \frac{1}{2}u'''_s(q_sR_f)q_s^3\text{Cov}[(\theta_{s0}^+Y)^2 (\theta_{s1}^+Y + \theta_{s0}^+a_2), Y_i] + \frac{1}{24}u^{(5)}_s(q_sR_f)q_s^4\text{Cov}[(\theta_{s0}^+Y)^4, Y_i].
\]

Using the definition of co-pentosis with a portfolio that we introduced at the beginning of this appendix, this reads in vector-matrix notation:

\[
0 = u'_s(q_sR_f)a_{i5} + u''_s(q_sR_f)q_s(\theta_{s0}\alpha_2)a_{i3} + \frac{1}{2}u'''_s(q_sR_f)q_s^2\text{Var}(\theta_{s0}^+Y)a_{i3} \\
+ u'_s(q_sR_f)q_s(\theta_{s1}^+a_2 + \theta_{s0}^+a_3)a_{i2} + \frac{1}{6}u'''_s(q_sR_f)q_s^3\text{Sk}(\theta_{s0})a_{i2} \\
+ u''_s(q_sR_f)q_s^2(\theta_{s0}\Sigma\theta_{s1})a_{i2} + u''_s(q_sR_f)q_s\Sigma\theta_{s3} \\
+ u''_s(q_sR_f)q_s^2(c(\theta_{s0}, \theta_{s2}) + (\theta_{s1}^+a_2 + \theta_{s0}^+a_3)(\Sigma\theta_{s0})) \\
+ \frac{1}{2}u'''_s(q_sR_f)q_s^2(c(\theta_{s1}) + 2(\theta_{s0}^+a_2)(\Sigma\theta_{s1})) \\
+ \frac{1}{2}u'''_s(q_sR_f)q_s^3(d(\theta_{s0}, \theta_{s1}) + (\theta_{s0}^+a_2)c(\theta_{s0})) + \frac{1}{24}u^{(5)}_s(q_sR_f)q_s^4f(\theta_{s0}),
\]

which gives

\[
q_s\Sigma\theta_{s3} = q_s\tau_{s5}a_{i5} - q_s(\theta_{s0}\alpha_2)a_{i3} + \frac{q_s\rho_s}{\tau_{s5}}\text{Var}(\theta_{s0}^+Y)a_{i3} - q_s(\theta_{s1}^+a_2 + \theta_{s0}^+a_3)a_{i2} \\
- \frac{q_s\kappa_s}{\tau_{s5}^2}\text{Sk}(\theta_{s0})a_{i2} + \frac{2q_s\rho_s}{\tau_{s5}}(\theta_{s0}\Sigma\theta_{s1})a_{i2} \\
+ 2\frac{q_s\rho_s}{\tau_{s5}}(c(\theta_{s0}, \theta_{s2}) + (\theta_{s1}^+a_2 + \theta_{s0}^+a_3)(\Sigma\theta_{s0})) + \frac{q_s\rho_s}{\tau_{s5}}(c(\theta_{s1}) + 2(\theta_{s0}^+a_2)(\Sigma\theta_{s1})) \\
- 3\frac{q_s\kappa_s}{\tau_{s5}^2}(d(\theta_{s0}, \theta_{s1}) + (\theta_{s0}^+a_2)c(\theta_{s0})) + \frac{q_s\kappa_s}{\tau_{s5}^2}f(\theta_{s0}),
\]

where

\[
\chi_s = \frac{[u'_s(q_sR_f)]^3u^{(5)}_s(q_sR_f)}{24[u''_s(q_sR_f)]^4}
\]

is the pentosis tolerance.
A.4 Proof of Theorem 3: Characterizing the equilibrium allocation with heterogeneous investors

This section uses the representation of the previous subsection to determine the market clearing equilibrium allocation. This provides the proof of Theorem 3.

Market clearing conditions are:

\[ \sum_{s=1}^{S} q_s \theta_{s0} = S \bar{q} \xi \quad \text{and} \quad \sum_{s=1}^{S} q_s \theta_{s j} = 0, \quad j = 1, 2, 3. \]

1. Plugging in the value of \( \theta_{s0} \) given by (72), the first market clearing condition gives

\[ \sum_{s=1}^{S} q_s \tau_s \Sigma^{-1} a_2 = S \bar{q} \xi, \quad \text{i.e.} \quad a_2 = \frac{1}{\bar{\tau}} b, \theta_{s0} = \frac{\tau_s}{\bar{\tau}} \xi, \quad (76) \]

where \( \bar{\tau} = \frac{\sum_{s=1}^{S} q_s \tau_s}{S \bar{q}}. \)

2. Plugging in the value of \( \theta_{s1} \) given by (73), the second market clearing condition (\( j = 1 \)) gives

\[ \sum_{s=1}^{S} q_s \tau_s \Sigma^{-1} \left[ c(\theta_{s0}) \frac{\rho_s}{\tau_s} + a_3 \right] = 0. \]

Equation (76) implies \( c(\theta_{s0}) = \frac{\tau_s^2}{\bar{\tau}^2} c \). We deduce

\[ a_3 = -\frac{\bar{\rho}}{\bar{\tau}^2} c, \theta_{s1} = \frac{\tau_s}{\bar{\tau}} (\rho_s - \bar{\rho}) \Sigma^{-1} c = \frac{\tau_s}{\bar{\tau}} (\rho_s - \bar{\rho}) \xi_0^{sk}, \]

where \( \bar{\rho} = \frac{\sum_{s=1}^{S} q_s \tau_s \rho_s}{\sum_{s=1}^{S} q_s \tau_s}. \)

3. Using the representation of \( \theta_{s2} \) given in (74), the third market clearing condition (\( j = 2 \)) gives:

\[ 0 = a_4 \sum_{s=1}^{S} q_s \tau_s - \sum_{s=1}^{S} \left( q_s (\theta_{s0}^\perp a_2) - \frac{q_s \rho_s}{\tau_s} Var \left( \theta_{s0}^\perp Y \right) a_2 \right) \]
\[ - \sum_{s=1}^{S} \left( -2 \frac{q_s \rho_s}{\tau_s} c(\theta_{s0}, \theta_{s1}) - 2 \frac{q_s \rho_s}{\tau_s} (\theta_{s0}^\perp a_2) \Sigma \theta_{s0} + \frac{q_s \kappa_s}{\tau_s^2} d(\theta_{s0}) \right). \]

Noting that

\[ \theta_{s0} = \frac{\tau_s}{\bar{\tau}} \xi, a_2 = \frac{1}{\bar{\tau}} b = \frac{1}{\bar{\tau}} \Sigma \xi, d(\theta_{s0}) = \frac{\tau_s^3}{\bar{\tau}^3} d, \]

50
we get

\[
0 = a_4 \sum_{s=1}^{S} q_s \tau_s - \sum_{s=1}^{S} \left( \frac{q_s \tau_s}{\bar{\tau}} (\xi^+ \Sigma \xi) a_2 - \frac{3 q_s \rho_s \tau_s}{\tau^2} (\xi^+ \Sigma \xi) a_2 - \frac{2 q_s \rho_s \tau_s}{\tau^3} (\rho_s - \bar{\rho}) c(\xi, \xi^s_k) + \frac{q_s \kappa \tau_s}{\tau^3} d \right),
\]

that is

\[
a_4 = \frac{1 - 3 \bar{\rho}}{\tau^3} (\xi^+ \Sigma \xi) b - 2 \frac{\text{Var}(\rho)}{\tau^3} c(\xi, \xi^s_k) + \frac{\bar{\kappa}}{\tau^3} d,
\]

where

\[
\bar{\kappa} = \frac{\sum_{s=1}^{S} q_s \tau_s \kappa_s}{\sum_{s=1}^{S} q_s \tau_s}. \text{Var}(\rho) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \bar{\rho})^2}{\sum_{s=1}^{S} q_s \tau_s}.
\]

Based on (74) we then find that

\[
\theta_{s2} = \frac{3 \tau_s}{\tau^3} (\rho_s - \bar{\rho}) (\xi^+ \Sigma \xi) \xi + 2 \frac{\tau_s}{\tau} (\rho_s (\rho_s - \bar{\rho}) - \text{Var}(\rho)) \xi^s_k - \frac{\tau_s}{\tau^3} (\kappa_s - \bar{\kappa}) \xi^k_{\text{Furt}}.
\]

4. Using the representation of \( \theta_{s3} \) given in (75), the fourth market clearing condition \((j = 3)\) gives:

\[
0 = a_5 \sum_{s=1}^{S} q_s \tau_s + \sum_{s=1}^{S} -q_s (\theta_{s0}^+ a_2) a_3 + \frac{q_s \rho_s}{\tau_s} \text{Var}(\theta_{s0}^+) a_3 - q_s (\theta_{s1}^+ a_2 + \theta_{s0}^+ a_3) a_2
\]

\[
+ 2 \sum_{s=1}^{S} \frac{q_s \rho_s}{\tau_s} (\theta_{s0}^+ \Sigma \theta_{s1}) a_2
\]

\[
+ 2 \sum_{s=1}^{S} \frac{q_s \rho_s}{\tau_s} (c(\theta_{s0}, \theta_{s2}) + (\theta_{s1}^+ a_2 + \theta_{s0}^+ a_3)(\Sigma \theta_{s0})) + \frac{q_s \rho_s}{\tau_s} (c(\theta_{s1}) + 2 (\theta_{s0}^+ a_2)(\Sigma \theta_{s1}))
\]

\[
- 3 \frac{q_s \kappa_s}{\tau^2_s} (d(\theta_{s0}, \theta_{s1}) + (\theta_{s0}^+ a_2) c(\theta_{s0})) + \frac{q_s \chi_s}{\tau^3_s} f(\theta_{s0}).
\]

Recalling that:

\[
\theta_{s0} = \frac{\tau_s}{\tau} \xi, a_2 = \frac{1}{\tau} \Sigma \xi, a_3 = -\frac{\bar{\rho}}{\tau^2} c, \theta_{s1} = \frac{\tau_s}{\tau^2} (\rho_s - \bar{\rho}) \Sigma^{-1} c, f(\theta_{s0}) = \frac{\tau^4_s}{\tau^4} f, \xi^+ c = Sk,
\]
we get

\[ 0 = a_5 \sum_{s=1}^{S} q_s \tau_s + \sum_{s=1}^{S} \frac{q_s \tau_s}{\bar{\tau}^4} (\xi \Sigma \xi) \bar{\rho} c - \frac{q_s \tau_s \rho_s}{\bar{\tau}^4} (\xi \Sigma \xi) \bar{\rho} c - \frac{q_s \tau_s (\rho_s - 2 \bar{\rho})}{\bar{\tau}^4} (Sk \cdot b) \]

\[ + \sum_{s=1}^{S} -\frac{q_s \tau_s \kappa_s}{\bar{\tau}^4} (Sk \cdot b) + 2 \frac{q_s \tau_s \rho_s (\rho_s - \bar{\rho})}{\bar{\tau}^4} (\xi \Sigma \xi) b + 6 \frac{q_s \tau_s \rho_s}{\bar{\tau}^4} (\rho_s - \bar{\rho}) (\xi \Sigma \xi) c \]

\[ + \sum_{s=1}^{S} 4 \frac{q_s \tau_s \rho_s}{\bar{\tau}^4} (\rho_s (\rho_s - \bar{\rho}) - \text{Var}(\rho)) c(\xi, \xi^s \bar{\kappa}) - 2 \frac{q_s \tau_s \rho_s}{\bar{\tau}^4} (\kappa_s - \bar{\kappa}) c(\xi, \xi^{sk}) \]

\[ + \sum_{s=1}^{S} 2 \frac{q_s \tau_s \rho_s}{\bar{\tau}^4} (\rho_s - 2 \bar{\rho}) (Sk \cdot b) + \frac{q_s \tau_s}{\bar{\tau}^4} (\rho_s - \bar{\rho})^2 c(\xi^s \bar{\kappa}) + 2 \frac{q_s \tau_s \rho_s}{\bar{\tau}^4} (\rho_s - \bar{\rho}) (\xi \Sigma \xi) c \]

\[ + \sum_{s=1}^{S} -3 \frac{q_s \tau_s \kappa_s}{\bar{\tau}^4} (\rho_s - \bar{\rho}) d(\xi, \xi^s \bar{\kappa}) - 3 \frac{q_s \tau_s \kappa_s}{\bar{\tau}^4} (\xi \Sigma \xi) c + \frac{q_s \tau_s}{\bar{\tau}^4} f, \]

that is

\[ a_5 = \frac{-\bar{\rho} + \bar{\rho}^2 - 8 \text{Var}(\rho) + 3 \bar{\kappa} (\xi \Sigma \xi) c + \bar{\rho} + 2 \bar{\rho}^2 - 4 \text{Var}(\rho) + \bar{\kappa} (Sk \cdot b)}{\bar{\tau}^4} \]

\[ -4 \frac{\text{Skew}(\rho) + \bar{\rho} \text{Var}(\rho)}{\bar{\tau}^4} c(\xi, \xi^{sk}) + \frac{\text{Cov}(\rho, \kappa)}{\bar{\tau}^4} \left\{ 2 c(\xi, \xi^{sk}) + 3 d(\xi, \xi^{sk}) \right\} \]

\[ - \frac{\text{Var}(\rho)}{\bar{\tau}^4} c(\xi^{sk}) / \bar{\tau}^4 f, \]

where

\[ \bar{\chi} = \frac{\sum_{s=1}^{S} q_s \tau_s \chi_s}{\sum_{s=1}^{S} q_s \tau_s}, \text{Cov}(\rho, \kappa) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \bar{\rho}) (\kappa_s - \bar{\kappa}) \sum_{s=1}^{S} q_s \tau_s}{\sum_{s=1}^{S} q_s \tau_s}, \text{Skew}(\rho) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \bar{\rho})^3}{\sum_{s=1}^{S} q_s \tau_s}. \]

In terms of higher moments of returns, one may want to reread formula (77) for risk premium as:

\[ \sigma^5 a_5 = A_1(\rho, \kappa) E \left[ (\xi Y)^3 \right] \text{Cov} \left[ Y, (\xi Y) \right] - 4 \frac{\text{Skew}(\rho) + \bar{\rho} \text{Var}(\rho)}{\bar{\tau}^4} \text{Cov}[((\xi Y)(\xi^{sk} Y), Y] \]

\[ + A_2(\rho, \kappa) \text{Var} \left[ (\xi Y) \right] \text{Cov} \left[ ((\xi Y)^2, Y \right] - \frac{\text{Var}(\rho)}{\bar{\tau}^4} \text{Cov} \left[ (\xi^{sk} Y)^2, Y \right] - \frac{\bar{\chi}}{\bar{\tau}^4} \text{Cov} \left[ (\xi Y)^4, Y \right] \]

\[ + \frac{\text{Cov}(\rho, \kappa)}{\bar{\tau}^4} \left\{ 2 \text{Cov}[((\xi Y)(\xi^{sk} Y), Y] + 3 \text{Cov}[((\xi Y)^2 (\xi^{sk} Y), Y] \right\}, \]

where \( A_1(\rho, \kappa), A_2(\rho, \kappa) \) have been defined in equation (61).

\section*{B Online Appendix: Two-period Extension}

This appendix studies the extension from a single time-period to two time-periods; it contains the proofs of the theorems 2 (representative investor case) and 4 (heterogeneous investor case).
proof of these two theorems runs largely in parallel. The first subsection derives a description of
the date 0 first order conditions of demand for given risk premiums that apply to any investor,
including the representative investor. The second subsection then uses this to describe the
equilibrium risk premium for the representative investor, while the third subsection describes
the equilibrium for heterogeneous investors.

With a conditional viewpoint, the individual demand schedule of any investor may depend on
her current wealth. To simplify our later derivation, we define for each investor \( s \) three dummy
functions \( f_s, g_s, \tilde{f}_s \) by setting

\[
f_s(w_s) = w_s \theta_{s0}, \tag{78}
g_s(w_s) = w_s \theta_{s1}, \tag{79}
\tilde{f}_s(w_s) = w_s \theta_{s0} a_1, \tag{80}
\]

for a given functional form of the expansion terms \( \theta_{s0}, \theta_{s1}, \) and \( a_1 \) (as functions of date 1) wealth
of investor \( s \). Note that these expansion terms may depend on the wealth of investors other than
\( s \); note also that they depend on the realization of the vector of random variables \( Y_1 \). However, to
simplify the exposition throughout this proof, we will not write out these dependences explicitly.

B.1 Deriving the expansion of the date 0 first order condition

As discussed in the main part of the paper, the date 0 first-order conditions are for \( i = 1, \ldots, n \)
and agent \( s = 1, \ldots, S \):

\[
0 = E_0 \left[ u'_s(W_{s,2}) \frac{\partial W_{s,2}}{\partial W_{s,1}} (\pi_0(\sigma) + \sigma Y_{(1)}) \right]. \tag{81}
\]

As in the single-period analysis, we interpret the date 0 first-order conditions as a \( \sigma \) series of
first order conditions. To simplify our exposition, we assume that

\[
h_s^0(\sigma) = u'_s(W_{s,2}) \frac{\partial W_{s,2}}{\partial W_{s,1}} (\pi_0(\sigma) + \sigma Y_{(1)})
\]

is an analytical function of the scale parameter \( \sigma \); it has a zero constant term since \( R_{(1)}(0) = R_f \).

Thus we write

\[
h_s^0(\sigma) = \sum_{j=1}^{\infty} h_{s,j}^0 \sigma^j. \tag{82}
\]
Note that equations (81, 82) imply that for all \( j = 1, 2, 3, \ldots \):

\[
E_0 \left[ h_{sj}^0 \right] = 0. \tag{83}
\]

Our goal in this subsection is to characterize these expansion terms. Throughout our analysis we use repeatedly

\[
\pi_0(\sigma) + \sigma Y(1) = \sigma Y(1) + \sum_{j=2}^{\infty} a_j^0 \sigma^j, \quad \text{and} \quad \pi_1(\sigma) + \sigma Y(2) = \sigma Y(2) + \sum_{j=2}^{\infty} a_j^1 \sigma^j. \tag{84}
\]

We note based on the series expansion of the individual demand schedule and (84) that terminal wealth can be expressed as

\[
W_{s,2} = W_{s,1} \left( R_f + \theta^{1,1}_s (\pi_1(\sigma) + \sigma Y(2)) \right)
\]

\[
= W_{s,1} R_f + \sigma W_{s,1} \left( \theta^{1,1}_s Y(2) \right) + \sigma^2 W_{s,1} \left\{ \theta^{1,1}_s Y(2) + \theta^{1,1}_s a_2^1 \right\} + O \left( \sigma^3 \right), \tag{85}
\]

and that date 1 wealth can be written

\[
W_{s,1} = q_s \left( R_f + \theta^{0,\perp}_s (\pi_0(\sigma) + \sigma Y(1)) \right)
\]

\[
= q_s R_f + q_s \sigma \left( \theta^{0,\perp}_s Y(1) \right) + q_s \sigma^2 \left\{ \theta^{0,\perp}_s Y(1) + \theta^{0,\perp}_s a_2^0 \right\} + O \left( \sigma^3 \right). \tag{86}
\]

To determine the \( \sigma \) series expansion of the first order conditions, we derive next the \( \sigma \) series expansion of the first two product terms on the left-hand side of equation (82). First, let us derive the \( \sigma \) series expansion of \( \frac{\partial W_{s,2}}{\partial W_{s,1}} \): Using the definitions in equations (78-80) we can write the investor’s terminal wealth \( W_{s,2} \) as

\[
W_{s,2} = R_{f,1} W_{s,1} + \sigma Y_{(2)}^\perp f_s (W_{s,1}) + \sigma^2 \left\{ Y_{(2)}^\perp g_s (W_{s,1}) + \tilde{f}_s (W_{s,1}) \right\} + O \left( \sigma^3 \right). \tag{87}
\]

This gives

\[
\frac{\partial W_{s,2}}{\partial W_{s,1}} = R_{f,1} + \sigma Y_{(2)}^\perp f_s' (W_{s,1}) + \sigma^2 \left\{ Y_{(2)}^\perp g_s' (W_{s,1}) + \tilde{f}_s' (W_{s,1}) \right\} + O \left( \sigma^3 \right). \tag{88}
\]

For further calculations below we will need the \( \sigma \) series expansion of \( \frac{\partial W_{s,2}}{\partial W_{s,1}} \). For this, we start with a series expansion of the three dummy functions around \( q_s R_f \):

\[
\begin{align*}
    f_s'(W_{s,1}) &= f_s'(q_s R_f) + f_s''(q_s R_f) (W_{s,1} - q_s R_f) + \ldots, \\
    g_s'(W_{s,1}) &= g_s'(q_s R_f) + g_s''(q_s R_f) (W_{s,1} - q_s R_f) + \ldots, \\
    \tilde{f}_s'(W_{s,1}) &= \tilde{f}_s'(q_s R_f) + \tilde{f}_s''(q_s R_f) (W_{s,1} - q_s R_f) + \ldots.
\end{align*}
\]
Based on this and equation (87) we can now write out the $\sigma$ series expansion of (89) as:

$$
\frac{\partial W_{s,2}}{\partial W_{s,1}} = R_f + \sigma Y_{(2)} f'_s(q_s R_f) + \sigma^2 \left\{ Y_{(2)} f''_s(q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g'_s(q_s R_f) + \tilde{f}_s(q_s R_f) \right\} + O(\sigma^3) \tag{90}
$$

Next, let us derive the $\sigma$ series expansion of marginal utility derived from terminal wealth $u'_s(W_{s,2})$. Analogous to the derivation of (90), it can be shown that terminal wealth (88) can be written as

$$
W_{s,2} = R_f W_{s,1} + \sigma Y_{(2)} f_s(q_s R_f) + \sigma^2 \left\{ Y_{(2)} f'_s(q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g_s(q_s R_f) + \tilde{f}_s(q_s R_f) \right\} + O(\sigma^3). \tag{91}
$$

Equation (91) jointly with (87) gives:

$$
W_{s,2} = q_s R_f^2 + \sigma \left\{ Y_{(2)} f_s(q_s R_f) + q_s R_f \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right\} + \sigma^2 \left\{ Y_{(2)} f'_s(q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g_s(q_s R_f) + \tilde{f}_s(q_s R_f) \right\} + O(\sigma^3). \tag{92}
$$

Finally, we exploit (92) and perform a Taylor expansion of $u'_s(W_{s,2})$ around $q_s R_f^2$:

$$
u'_s(W_{s,2}) = u'_s(q_s R_f^2) + \sigma u''_s(q_s R_f^2) \left( Y_{(2)} f_s(q_s R_f) + q_s R_f \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right) + \frac{1}{2} \sigma^2 \left\{ u''_s(q_s R_f^2) \left[ Y_{(2)} f'_s(q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g'_s(q_s R_f) + \tilde{f}'_s(q_s R_f) \right] \right\} + O(\sigma^4). \tag{93}
$$

Now that we have a characterization of the $\sigma$ series expansion of the first two product terms on the left-hand side of equation (82), we put these together with (84). First, we note that equation (90) jointly with (84) gives

$$
\frac{\partial W_{s,2}}{\partial W_{s,1}} \left( \pi_0(\sigma) + \sigma Y_{(1)} \right) = \sigma (R_f Y_{(1)}) + \sigma^2 \left\{ (Y_{(2)} f'_s(q_s R_f)) Y_{(1)} + R_f a_0^0 \right\} + \sigma^3 \left\{ \left( Y_{(2)} f''_s(q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g'_s(q_s R_f) + \tilde{f}'_s(q_s R_f) \right) Y_{(1)} \right\} + O(\sigma^4) \tag{94}
$$
By definition, multiplying (95) and (93), we can characterize the terms in equation (82):

\[ h_{s1}^0 = (R_f Y_{(1)}) u'_s (q_s R_f^2) \]

\[ h_{s2}^0 = u''_s (q_s R_f^2) \left( Y_{(2)} f'_s (q_s R_f) + q_s R_f \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right) (R_f Y_{(1)}) \]

\[ + u'_s (q_s R_f^2) \left\{ \left( Y_{(2)} f'_s (q_s R_f) \right) Y_{(1)} + R_f a_2^0 \right\} \]

\[ h_{s3}^0 = (R_f Y_{(1)}) \frac{1}{2} \left\{ u'''_s (q_s R_f^2) \left[ Y_{(2)} f'_s (q_s R_f) + q_s R_f \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right] \right. \]

\[ + \left. 2u''_s (q_s R_f^2) \left( Y_{(2)} f'_s (q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g_s (q_s R_f) + \theta_{s1}^{0,1} Y_{(1)} + \theta_{s0}^{0,1,0} \right) \right\} \]

\[ + \left\{ \left( Y_{(2)} f'_s (q_s R_f) \right) Y_{(1)} + R_f a_2^0 \right\} u''_s (q_s R_f^2) \left( Y_{(2)} f'_s (q_s R_f) + q_s R_f \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right) \]

\[ + u'_s (q_s R_f^2) \left\{ \left( Y_{(2)} f'_s (q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) + Y_{(2)} g'_s (q_s R_f) + \theta_{s1}^{0,1} Y_{(1)} \right) \right\} \]

\[ + u'_s (q_s R_f^2) \left\{ \left( Y_{(2)} f'_s (q_s R_f) \right) a_2^2 + R_f a_3^0 \right\} \]

For further analysis we note that \( f_s, f'_s, f''_s, g_s, \tilde{f}_s \) depend on \( Y_{(1)} \) but do not depend on \( Y_{(2)} \).

By definition, \( E_1[Y_{(2)} | Y_{(1)}] = 0 \) and \( E_0[Y_{(1)}] = 0 \), which implies \( E_0 \left[ h_{s1}^0 \right] = 0 \) and

\[ E \left[ Y_{(1)} \left( Y_{(2)} f'_s (q_s R_f) \right) \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right] = E[Y_{(2)} g_s (q_s R_f) Y_{(1)}] = E \left[ Y_{(1)} \left( \theta_{s0}^{0,1} a_2^0 \right) \right] = 0, \]

\[ E \left[ \left( Y_{(2)} f'_s (q_s R_f) \right) Y_{(1)} \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right] = E[Y_{(2)} f'_s (q_s R_f) a_2^0] = E \left[ Y_{(2)} \left( \theta_{s0}^{0,1} Y_{(1)} \right) \right] = 0, \]

\[ E \left[ Y_{(2)} f''_s (q_s R_f) q_s \left( \theta_{s0}^{0,1} Y_{(1)} \right) Y_{(1)} \right] = E \left[ Y_{(2)} g'_s (q_s R_f) Y_{(1)} \right] = E \left[ Y_{(2)} f'_s (q_s R_f) a_2^0 \right] = 0. \]

Up to order three, the informative first-order conditions (83) are then as follows:

1. \( E_0 \left[ h_{s2}^0 \right] = 0 \) is equivalent to

\[ 0 = u''_s (q_s R_f^2) q_s R_f^2 \sum_0 \theta_{s0}^0 + u'_s (q_s R_f^2) R_f a_2^0. \]  \( \quad \text{(96)} \)

We recall that \( \tau_{s,0} \), defined in equation (46), stands for the (date 0) risk-tolerance of investor \( s \), then divide equation (96) by \( -u'_s (q_s R_f^2) R_f \) and then find that

\[ 0 = \frac{1}{\tau_{s,0}} \sum_0 \theta_{s0}^0 - a_2^0. \]  \( \quad \text{(97)} \)

2. Next we analyze \( E_0 \left[ h_{s3}^0 \right] = 0 \); it is equivalent to

\[ \frac{1}{2} u'''_s (q_s R_f^2) R_f \left( E_0 \left[ \left( Y_{(2)} f'_s (q_s R_f) \right)^2 Y_{(1)} \right] + q_s^2 R_f^2 Cov_0 \left( \left( \theta_{s0}^{0,1} Y_{(1)} \right)^2, Y_{(1)} \right) \right) \]

\[ + u''_s (q_s R_f^2) R_f Cov_0 \left( f'_s (q_s R_f), Y_{(1)} \right) + u''_s (q_s R_f^2) q_s R_f^2 Cov_0 \left( \left( \theta_{s1}^{0,1} Y_{(1)} \right), Y_{(1)} \right) \]

\[ + u'_s (q_s R_f^2) E_0 \left[ \left( Y_{(2)} f'_s (q_s R_f) \right) \left( Y_{(2)} f'_s (q_s R_f) \right) Y_{(1)} \right] \]

\[ + u'_s (q_s R_f^2) Cov_0 \left( f'_s (q_s R_f), Y_{(1)} \right) + u'_s (q_s R_f^2) R_f a_3^0 = 0. \]  \( \quad \text{(98)} \)
We recall that $\rho_{s,0}$, defined in equation (47), stands for the (date 0) skew-tolerance of agent $s$, then divide equation (98) by $-u'_s(q_sR_f^2)R_f$ and use the definitions of $\tau_{s,0}, \rho_{s,0}$ to get:

$$0 = -\frac{\rho_{s,0}}{\tau_{s,0} q_s R_f^2} E_0 \left[ \left( Y_{(2)}(q_sR_f) \right)^2 Y_{(1)} \right] - \frac{\rho_{s,0}}{\tau_{s,0}^2} Cov_0 \left( \left( \theta_{s,0}^{(1)} Y_{(1)} \right)^2, Y_{(1)} \right)$$

$$+ \frac{1}{\tau_{s,0} q_s R_f} Cov_0 \left( \tilde{f}_s(q_sR_f), Y_{(1)} \right) + \frac{1}{\tau_{s,0}} Cov_0 \left( \left( \theta_{s,1}^{(1)} Y_{(1)} \right), Y_{(1)} \right)$$

$$+ \frac{1}{R_f} Cov_0 \left( \tilde{f}_s(q_sR_f), Y_{(1)} \right) - a_3^0.$$  \hspace{1cm} (99)

### B.2 Representative Agent Risk Premium

Based on our single-period analysis, but with a date 1 conditional viewpoint we know the risk premium up to order 3:

$$a_2^1 = \frac{1}{\tau_1} b_M^1, \text{ and } a_3^1 = -\frac{\rho_1}{\tau_1^2} c_1$$

where we recall that

$$\tau_1(w) = -\frac{u'(wR_f)}{wu''(wR_f)}, \rho_1(w) = \frac{u'(wR_f) u''(wR_f)}{2 [u''(wR_f)]^2}.$$  \hspace{1cm}

We also recall that the representative agent’s date 1 demand is given through the aggregate supply at date 1, i.e.

$$\theta^1_0 = \xi \text{ and } \theta^1_1 = 0.$$  \hspace{1cm}

This implies for the functions in equation (78-80):

$$f(w) = w\xi, f'(w) = \xi, g(w) = 0, \tilde{f}(w) = w\xi^1 a_2^1 = \frac{w}{\tau_1} \xi \xi^1 b_M^1.$$  \hspace{1cm}

We recall the definitions of $\tau_0$ and of $\rho_0$,

$$\tau_0 = -\frac{u'(qR_f^2)}{qR_f u''(qR_f^2)} \text{ and } \rho_0 = \frac{u'(qR_f^2) u''(qR_f^2)}{2 [u''(qR_f^2)]^2},$$  \hspace{1cm}

and that the representative agent’s date 1 demand is given through the aggregate supply at date 0, i.e.

$$\theta^0_0 = \xi \text{ and } \theta^0_1 = 0.$$  \hspace{1cm} (100)

57
Up to order three, the informative first-order conditions (83) are characterized through equations (97, 99). Here we apply them with a single representative agent \( u_s = u, \tau_{s,0} = \tau_0, \rho_{s,0} = \rho_0 \) and the supply (100):

1. Based on (97), \( E_0 \left[ h_{s2}^0 \right] = 0 \) gives

\[
a_2^0 = \frac{1}{\tau_0} \Sigma_0 \xi = \frac{1}{\tau_0} b_M^0.
\]

2. Based on (99), \( E_0 \left[ h_{s3}^0 \right] = 0 \) and noting that \( \xi \perp \Sigma_1 \xi = \xi \perp b_M^1 \), we get:

\[
a_3^0 = -\frac{\rho_0}{\tau_0^2} Cov_0 \left[ \xi \perp \Sigma_1 \xi, Y_{(1)} \right] - \frac{\rho_0}{\tau_0^2} Cov_0 \left( \left( \xi \perp Y_{(1)} \right)^2, Y_{(1)} \right) + \frac{1}{\tau_0} \left( \frac{1}{\tau_1(qR_f)} + \frac{1}{R_f} \right) Cov_0 \left[ \xi \perp \Sigma_1 \xi, Y_{(1)} \right] - \frac{1}{R_f} Cov_0 \left( \tilde{f}'(qR_f), Y_{(1)} \right).
\]

Using the product rule we calculate that

\[
\tilde{f}'(qR_f) = \xi \perp b_M^1 \left( \frac{1}{\tau_1(qR_f)} - qR_f \frac{\partial \tau_1}{\partial w}(qR_f) \frac{1}{\tau_1^2(qR_f)} \right).
\]

Finally, we then conclude from this and noting \( \tau_1(qR_f) = \tau_0 \), that

\[
a_3^0 = -\frac{\rho_0}{\tau_0^2} Cov_0 \left[ \xi \perp \Sigma_1 \xi, Y_{(1)} \right] - \frac{\rho_0}{\tau_0^2} Cov_0 \left( \left( \xi \perp Y_{(1)} \right)^2, Y_{(1)} \right) + \frac{1}{\tau_0} Cov_0 \left[ \xi \perp \Sigma_1 \xi, Y_{(1)} \right] - \frac{\rho_0}{\tau_0^2} Cov_0 \left( \left( \xi \perp Y_{(1)} \right)^2, Y_{(1)} \right),
\]

where we recall that

\[
\gamma_0 = q \frac{\partial \tau_1}{\partial w}(qR_f).
\]

### B.3 Heterogeneous Agent Equilibrium

#### B.3.1 The date 1 (conditional) demand schedule

Based on our single-period analysis, but with a date 1 conditional viewpoint we know the equilibrium up to order 3:

\[
a_2^1 = \frac{1}{\tau_1} b_{M1}, \theta_{s0}^1 = \frac{\tau_{s,1}}{\tau_1} \xi,
\]

\[
a_3^1 = -\frac{\rho_1}{\tau_1^2} c_1, \theta_{s1}^1 = \frac{\tau_{s,1}}{\tau_1} \left( \rho_{s,1} - \bar{\rho}_1 \right) \xi_{0,1}^k.
\]
where

$$\tau_{s,1}(w_s) = - \frac{u_s'(w_s R_f)}{w_s u_s''(w_s R_f)}, \rho_{s,1}(w_s) = \frac{u_s'(w_s R_f) u_s'''(w_s R_f)}{2 |u_s''(w_s R_f)|^2},$$

$$\bar{\tau}_1 = \frac{\sum_s w_s \tau_{s,1}(w_s)}{\sum_s w_s}, \bar{\rho}_1 = \frac{\sum_s w_s \rho_{s,1}(w_s) \tau_{s,1}(w_s)}{\sum_s w_s \tau_{s,1}(w_s)},$$

$$\xi^{sk}_{0,1} = \Sigma_1^{-1} c_1.$$

Note that

$$\tau_{s,1}(q_s R_f) = \tau_{s,0}; \tag{106}$$

where we recall that

$$\tau_{s,0} = - \frac{u_s'(q_s R_f^2)}{R_f q_s u_s''(q_s R_f^2)} \tag{107}$$

stands for the (date 0) relative risk-tolerance. The above representation of the date 1 conditional equilibrium implies

$$f_s(w_s) = w_s \frac{\tau_{s,1}}{\bar{\tau}_1} \xi, g_s(w_s) = w_s \frac{\tau_{s,1}}{\bar{\rho}_1} (\rho_{s,1} - \bar{\rho}_1) \xi^{sk}_{0,1}, \text{ and } \tilde{f}_s(w_s) = w_s \frac{\tau_{s,1}}{\bar{\tau}_1} (\xi \perp b_M^1). \tag{108}$$

### B.3.2 The date 0 demand schedule knowing the functional form of date 1 equilibrium

Up to order three, the informative first-order conditions (83) are as follows:

1. Based on equation (97), $E_0 [h_{s2}^0] = 0$ is equivalent to

$$\theta_{s0}^0 = \tau_{s,0} \Sigma_2^{-1} a_2^0, \tag{109}$$

where we recall $\tau_{s,0}$ as in equation (107).

2. Next we analyze $E_0 [h_{s3}^0] = 0$. Based on equation (99), using the representation in (108) and noting that $\xi \perp \Sigma \xi = \xi \perp b_M^1$ we find

$$0 = - \frac{\rho_{s,0}}{(\bar{\tau}_1(q_s R_f))^2} Cov_0(\xi \perp \Sigma^1 \xi, Y_1) - \frac{\rho_{s,0}}{\tau_{s,0}} Cov_0 \left( \left( \theta_{s0}^0 Y_1 \right)^2, Y_1 \right)$$

$$+ \frac{1}{(\bar{\tau}_1(q_s R_f))^2} \left( \frac{\partial}{\partial w_s} \left( w_s \frac{\tau_{s,1}}{\bar{\tau}_1} (q_s R_f) \right) \right) Cov_0(\xi \perp \Sigma^1 \xi, Y_1)$$

$$+ \frac{1}{R_f \bar{\tau}_1(q_s R_f)} \left( \frac{\partial}{\partial w_s} \left( w_s \frac{\tau_{s,1}}{\bar{\tau}_1} (q_s R_f) \right) \right) Cov_0(\xi \perp \Sigma^1 \xi, Y_1)$$

$$- \frac{1}{R_f} \left( \frac{\partial}{\partial w_s} \left( w_s \frac{\tau_{s,1}}{\bar{\tau}_1} (q_s R_f) \right) \right) Cov_0(\xi \perp \Sigma^1 \xi, Y_1) - a_3^0. \tag{110}$$
B.3.3 Date 0 market clearing conditions knowing the functional form of date 1 equilibrium

We first determine the equilibrium at order 0. We multiply (109) by \( q_s \), take the sum for \( s = 1, \ldots, S \) and use the market clearing condition for \((\theta^0_{s0})_s\), i.e. \( \sum_{s=1}^{S} q_s \theta^0_{s0} = \xi \sum_{s=1}^{S} q_s \), to find

\[
\Sigma_0 \left( \xi \sum_{s=1}^{S} q_s \right) = \sum_{s=1}^{S} q_s \theta^0_{s0} - a^0_2 \sum_{s=1}^{S} q_s \tau_{s,0} = 0
\]

Now, we have

\[
a^0_2 = \frac{1}{\tau_0} \sum_0 \xi = \frac{1}{\tau_0} \beta^0_M,
\]

where we recall

\[
\bar{\tau}_0 = \frac{\sum_{s=1}^{S} q_s \tau_{s,0}}{\sum_{s=1}^{S} q_s}.
\]

We replace \( a^0_2 \) in (109) to get

\[
\theta^0_{s0} = \frac{\tau_{s,0}}{\bar{\tau}_0} \xi.
\]

(111)

Next, we determine the equilibrium at order 1. Using the product rule, we calculate

\[
\frac{\partial \left( w_s \frac{\tau_{s,1}}{\bar{\tau}_1} \right)}{\partial w_s} = \frac{1}{\bar{\tau}_1} \frac{\partial \left( w_s \frac{\tau_{s,1}}{\bar{\tau}_1} \right)}{\partial w_s} - \frac{\partial}{\partial w_s} \frac{\tau_{s,1}}{\bar{\tau}_1}.
\]

(112)

Evaluating this at \( q_s R_f \) and using equation (106) gives

\[
\frac{1}{R_f \bar{\tau}_1(q_s R_f)} \left[ \frac{\partial \left( w_s \frac{\tau_{s,1}}{\bar{\tau}_1} \right)}{\partial w_s} (q_s R_f) \right] - \frac{1}{R_f} \left[ \frac{\partial}{\partial w_s} \frac{\tau_{s,1}}{\bar{\tau}_1} (q_s R_f) \right] = q_s \frac{\tau_{s,1}(q_s R_f)}{(\bar{\tau}_1(q_s R_f))^2} \frac{\partial \bar{\tau}_1}{\partial w_s} (q_s R_f) = \frac{\gamma_{s,0}}{(\bar{\tau}_1(q_s R_f))^2},
\]

(113)

where we recall that

\[
\gamma_{s,0} = q_s \frac{\tau_{s,0}}{\bar{\tau}_0} \frac{\partial \bar{\tau}_1}{\partial w_s} (q_s R_f).
\]

Using equation (113) and equation (111) as well as noting \( \bar{\tau}_1(q_R) = \bar{\tau}_0 \), we get:

\[
\frac{\rho_{s,0} - 1 - \gamma_{s,0}}{\bar{\tau}_0^2} \text{Cov}_0 \left( \xi \Sigma_t \xi, Y_{(1)} \right) + \frac{\rho_{s,0} \gamma_{s,0}}{\bar{\tau}_0^2} \text{Cov}_0 \left( \left( \xi^1 Y_{(1)} \right)^2, Y_{(1)} \right) - \frac{1}{\tau_{s,0}} \sum_0 \theta^0_{s1} + a^0_3 = 0.
\]

(114)

This equation allows us to determine individual date 0 demand \( \theta^0_{s1} \) based on the risk premium \( a^0_3 \); in a second step we would then infer the risk premium that clears the market and in a final step determine the equilibrium demand. However, the structure of the representation (114)
permits a shortcut to infer the equilibrium risk premium. For this, we multiply (114) by \( q_s \tau_{s,0} \) and take the sum for \( s = 1, \ldots, S \)

\[
\sum_{s=1}^S q_s \tau_{s,0} \rho_{s,0} - \sum_{s=1}^S q_s \tau_{s,0} \gamma_{s,0} + \sum_{s=1}^S q_s \tau_{s,0} \gamma_{s,0} Cov_0 \left( \xi^\top \Sigma_1 \xi, Y_{(1)} \right)
\]

(115)

\[
+ \sum_{s=1}^S \frac{q_s \tau_{s,0} \rho_{s,0}}{\tau_0^2} Cov_0 \left( \left( \xi^\top Y_{(1)} \right)^2, Y_{(1)} \right) - \Sigma_0 \left( \sum_{s=1}^S q_s \theta_{s,1}^0 \right) + \sum_{s=1}^S q_s \tau_{s,0} \alpha_3^0 = 0.
\]

(116)

We recall the market clearing conditions for \( \{ \theta_{s,1}^0 \} \), i.e. \( \sum_{s=1}^S q_s \theta_{s,1}^0 = 0 \), divide equation (116) by \( \sum_{s=1}^S q_s \tau_{s,0} \) and get

\[
\alpha_3^0 = -\frac{\bar{\rho}_0 - 1 - \bar{\gamma}_0}{\tau_0^2} Cov_0 \left( \xi^\top \Sigma_1 \xi, Y_{(1)} \right) - \frac{\bar{\rho}_0}{\tau_0^2} Cov_0 \left( \left( \xi^\top Y_{(1)} \right)^2, Y_{(1)} \right),
\]

(117)

where we recall that \( \bar{\gamma}_0 \) and \( \bar{\rho}_0 \) are given as

\[
\bar{\gamma}_0 = \frac{\sum_{s=1}^S \gamma_{s,0} q_s \tau_{s,0}}{\sum_{s=1}^S q_s \tau_{s,0}}, \quad \bar{\rho}_0 = \frac{\sum_{s=1}^S \rho_{s,0} q_s \tau_{s,0}}{\sum_{s=1}^S q_s \tau_{s,0}}.
\]

Now, we multiply (114) by \( \tau_{s,0} \), and replace (117) to find

\[
\Sigma_0 \theta_{s,1}^0 = \frac{\tau_{s,0} \rho_{s,0} - \tau_{s,0} - \tau_{s,0} \gamma_{s,0}}{\tau_0^2} Cov_0 \left( \xi^\top \Sigma_1 \xi, Y_{(1)} \right) + \frac{\tau_{s,0} \rho_{s,0} \gamma_{s,0}}{\tau_0^2} Cov_0 \left( \left( \xi^\top Y_{(1)} \right)^2, Y_{(1)} \right) + \tau_{s,0} \alpha_3^0
\]

\[
= \left\{ \frac{\tau_{s,0} \left( \rho_{s,0} - \bar{\rho}_0 \right)}{\tau_0^2} - \frac{\tau_{s,0} \left( \gamma_{s,0} - \bar{\gamma}_0 \right)}{\tau_0^2} \right\} Cov_0 \left( \xi^\top \Sigma_1 \xi, Y_{(1)} \right)
\]

(118)

\[
+ \frac{\tau_{s,0} \left( \rho_{s,0} - \bar{\rho}_0 \right)}{\tau_0^2} Cov_0 \left( \left( \xi^\top Y_{(1)} \right)^2, Y_{(1)} \right)
\]

Since \( Cov_0 \left( \xi^\top \Sigma_1 \xi, Y_{(1)} \right) = Var_1 \left[ \xi^\top Y_{(2)} \right] \), we find the stated representation of \( \theta_{s,1}^0 \).

References


