Identifying the Class of Term Structure Models Possessing Closed-Form Solutions for Bond and Bond-Option Prices: An Expectation Approach

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Abstract

What properties must a term structure model possess in order to have closed-form bond price solutions? Are there undiscovered models which offer simple bond price formula? Why do models that have closed-form bond price solutions also tend to have closed-form bond-option solutions? This paper attempts to answer these questions from an expectation approach.

It is well known that the price of a derivative security can be expressed either as a solution to an expectation, or as a solution to a second order partial differential equation. Except for normally distributed random processes, the expectation approach has been difficult to implement. In this paper we demonstrate that when the expectation is expressed as a path integral, and the law of iterated expectations is used, implementation becomes straightforward. This technique takes advantage of the fact that, in the continuous time limit, all innovations are normally distributed (excluding jump processes). Hence, for those models possessing simple closed-form solutions, implementation is quite easy, since only normally distributed processes need to be dealt with.

Using this technique, the class of models which have closed form bond and bond-option solutions is identified. The approach also explains why closed-form solutions typically exist for both bond and bond-options, or neither. In addition, the approach allows one to evaluate bond-option prices when the model is infinite factor, where no PDE representation even exists.

1 Introduction

Consider the class of term structure models that possess closed-form bond price solutions. What features do they have in common? Why do these same models typically also possess closed-form solutions for option prices? Where should we look for other models with simple closed-form solutions? This paper attempts to answer these questions by solving for asset prices via an expectation approach, rather than taking the more traditional approach of solving a partial differential equation (PDE).

Black and Scholes (1973) and Merton (1973) create an instantaneous riskfree portfolio to obtain a second order PDE that any derivative security must satisfy. Their arguments are one of arbitrage: if the derivative did not satisfy the relevant PDE (and appropriate boundary conditions), then costless portfolios would exist which would guarantee the owner possible gain, with no chance of loss. Harrison and Kreps (1979) demonstrate that the absence of arbitrage implies the existence of equivalent martingale measures, under which an option price can be expressed as an expectation. These two approaches are known to be equivalent through the theorem of Feynman and Kac. It is interesting to note that in many cases the martingale approach proves to be more elegant and powerful. Yet, except when the underlying random processes are normally distributed, explicit calculation of the expectation can be quite difficult. In this paper, we demonstrate how such an expectation can be calculated explicitly.

The technique proposed expresses the appropriate expectation as a path integral, and then uses the law of iterated expectations so that each expectation we consider involves only normally distributed random variables.[1] This technique offers much insight into derivative pricing. For example, Girsanov's Theorem can be shown to be just a trivial change of variables. Best of all, mathematical implementation of the technique is straightforward: one needs only the knowledge of the projection theorem (for multi-factor models), and the fact that, if $\tilde{x}$ is normally distributed, $x \sim \Phi\left[\mu\,,\,\sigma^2\right]$, then

$$\mathrm{E}\left[e^{a\tilde{x}}\right] = e^{a\mu + \frac{a^2}{2}\sigma^2}\ .$$

Whereas solving the PDE often requires the investigator to 'guess' a particular form that a solution will take, the expectation approach *predicts* that form. The approach therefore offers guidance into where to look for other models which have closed-form solutions, and offers an explanation of why both bond and bond-options tend to have closed-form expressions for the

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[1] Merton (1971) uses a continuous time framework to demonstrate that it is unnecessary to assume either quadratic utility functions or discrete-time normally distributed returns in order for the CAPM to hold. The CAPM obtains because, in the continuous time limit, infinitesimal returns *are* normally distributed. Basically, a similar argument is being used here.

same models. In addition, the technique used here is the only one available when the model possesses an infinite number of factors. For these random field models (Kennedy (1994, 1997), Goldstein (1997)), no PDE representation exists.

The rest of the paper is as follows. In Section 2, we discuss the theorem of Feynman and Kac, and demonstrate how the expectation can be calculated by use of path integrals and iterated expectations. In Section 3, we look at the one-factor term structure models which have closed-form bond and bond-option expressions. In Section 4 we look at multi-factor models. In Section 5, we discuss infinite-factor models. We conclude in Section 6.

2 Pricing Securities Using an Expectation Approach

In this paper, we limit our discussion to pricing bond and bond-options. It is well known that the value of these securities at date v, with maturity date s, can be written as the expected value of time-discounted final payout under the risk-neutral measure:

$$W(v) = \mathrm{E}_v^Q \left[e^{-\int_v^s du\, r_u} \, W(s) \right] .$$

For the discount bond, $W(s) = P^s(s) = 1$. For the call option on a bond, where the underlying matures at date T, $W(s) = C(s,s,T) = \max[P^T(s) - K, 0]$, where K is the strike price.

Assume that the spot rate can be written as a stationary Markov process

$$dr = \mu(r)\, dt + \sigma(r)\, dZ^Q(t) \tag{1}$$

under the risk neutral measure. Then the Theorem of Feynman and Kac states that bond and bond-options must satisfy the PDE

$$W_t + \mu(r)\, W_r + \frac{1}{2}\, \sigma^2(r)\, W_{rr} = r\, W \, .$$

For specified boundary conditions and terminal value, this PDE can be solved to obtain bond and bond-option formula. Although the expectation approach and the PDE approach are equivalent, typically explicit calculation of the expectation is quite difficult. Here, we discuss a technique which permits the expectation to be solved explicitly.

First, let us look at bond prices:

$$P^s(v) = \mathrm{E}_v^Q \left[e^{-\int_v^s du\, r_u} \right] .$$

The first step is to discretize time. For example, define $v \equiv n\, \Delta t$ and $s \equiv N \Delta t$. At the end of the calculation, we take $\Delta t \to 0$ to obtain the continuous time limit. Thus we have

$$
\begin{aligned}
P^N(n) &= \mathrm{E}_n^Q\left[e^{-\sum_{j=n}^{N-1}\Delta t\, r_j}\right] \\
&= \mathrm{E}_n^Q\left[e^{-\Delta t\, r_n}\, e^{-\Delta t\, r_{(n+1)}}\, e^{-\Delta t\, r_{(n+2)}} \dots e^{-\Delta t\, r_{(N-2)}} e^{-\Delta t\, r_{(N-1)}}\right] \\
&= e^{-\Delta t\, r_n}\left[\mathrm{E}_n^Q\, e^{-\Delta t\, r_{(n+1)}}\ \mathrm{E}_{(n+1)}^Q\ e^{-\Delta t\, r_{(n+2)}} \dots \mathrm{E}_{(N-3)}^Q\, e^{-\Delta t\, r_{(N-2)}}\ \mathrm{E}_{(N-2)}^Q\, e^{-\Delta t\, r_{(N-1)}}\right],
\end{aligned}
$$

where we have used the law of iterated expectations in the last step. This can be written compactly as

$$
P^N(n) = e^{-\Delta t r_n} \prod_{j=n}^{N-2} \left\{ \mathrm{E}_j^Q \left[e^{-\Delta t\, r_{(j+1)}} \right] \right\} ,
$$

It is important to interpret this equation correctly, since certain operations do not commute: one must write out the expectations in ascending time order, and then perform the calculations from right to left. Indeed, let us look at the expectation furthest to the right:

$$
\mathrm{E}_{(N-2)}^Q \left[e^{-\Delta t\, r_{(N-1)}} \right] .
$$

The Ito process in Equation 1 implies that, for sufficiently small Δt, $dr_{(N-2)} \equiv \left(r_{(N-1)} - r_{(N-2)} \right)$, conditional upon $r_{(N-2)}$, is normally distributed:

$$
\left(r_{(N-1)} - r_{(N-2)} \right) | \mathcal{F}(N-2) \ \sim\ \Phi \left[\mu(r_{(N-2)})\, \Delta t \,,\, \sigma^2(r_{(N-2)})\, \Delta t \right] ,
$$

implying,

$$
r_{(N-1)} | \mathcal{F}(N-2) \ \sim\ \Phi \left[r_{(N-2)} + \mu(r_{(N-2)})\, \Delta t \,,\, \sigma^2(r_{(N-2)})\, \Delta t \right] .
$$

The form of this expectation is well known:

$$
\mathrm{E}_{(N-2)}^Q \left[e^{-\Delta t\, r_{(N-1)}} \right] = e^{-\Delta t \left(r_{(N-2)} + \mu(r_{(N-2)})\, \Delta t \right) + \frac{\Delta t^2}{2}\, \sigma^2 (r_{(N-2)})\, \Delta t} . \tag{2}
$$

Of course, as $\Delta t \rightarrow 0$, we have an infinite number of such expectations to calculate. But, if the analytic form of each term remains invariant, then each successive calculation can be performed. Indeed, the class of models with simple closed-form bond and bond-option formula has been identified: it is the class of models which remain invariant in structure after successive expectations have been performed.

3 One Factor Models

A quick look at Equation 2 predicts most of the familiar term structure models that possess closed-form bond and bond-option formula. In particular, if the drift and diffusion are of the form

$$
\mu(r) = a + br \,, \qquad \sigma^2(r) = c + dr \,,
$$

then the expectation in Equation 2 produces a structure linear in $r_{(N-2)}$. Indeed, each successive iteration will remain linear in the spot rate. This analysis includes the models of Merton (1973), Vasicek (1977), Cox, Ingersoll and Ross (CIR, 1985b), and its extended version (Pearson and Sun (1990)).

Here, we demonstrate the proposed technique more thoroughly with some examples. We shall see why it is that when we get closed-form bond price solutions, we also typically get closed-form option price solutions. We also discuss why the proposed method implies that there may not exist any other one-factor models with (simple) closed-form bond price formulas.

3.1 CIR Square Root Process

Consider the spot rate process

$$dr = \kappa\,(\theta - r)\;dt + \sigma\,\sqrt{r}\,dZ^Q\;.$$

The price of a discount bond at time v, maturing at time s can be written formally as

$$\begin{aligned} P^s(v) &= \mathrm{E}^Q_v\left[e^{-\int_v^s du\, r_u}\right] \\ &= \lim_{\Delta t\to 0} e^{-\Delta t\, r_n} \prod_{j=n}^{N-2} \left\{\mathrm{E}^Q_j\left[e^{-\Delta t\, r_{(j+1)}}\right]\right\}\;, \end{aligned}$$

where $v = n\,\Delta t$, $s = N\Delta t$. Once again we emphasize that this last expression must be written in ascending time order, and then calculated from right to left. Let us look at the last expectation:[2]

$$\mathrm{E}^Q_{(N-2)}\left[e^{-\Delta t\, r_{(N-1)}}\right] = e^{-\Delta t\left[r_{(N-2)}+\kappa(\theta-r_{(N-2)})\Delta t\right]+\frac{1}{2}\,(\Delta t)^2\,\sigma^2 r_{(N-2)}\,\Delta t}\;.$$

Note that the price of a bond at time $(N-2)$ which matures at time N is equal to

$$\begin{aligned} P^N(N-2) &= e^{\Delta t\, r_{(N-2)}}\,\mathrm{E}^Q_{(N-2)}\left[e^{-\Delta t\, r_{(N-1)}}\right] \\ &\equiv e^{A(\Delta t) - r_{(N-2)}\,B(\Delta t)}\;, \end{aligned}$$

that is, linear in the spot rate. Indeed, it is easy to convince oneself that, after each successive expectation has been calculated, the bond price will maintain this form. Also note that if we had considered the extended case, where the volatility takes the form

$$\sigma(r) = \sigma\,\sqrt{r+\gamma}\,,$$

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[2] It is well known that the square-root process does not permit negative interest rates. Thus, it might be surprising that we can consider the spot rate to be normally distributed. It is important to emphasize that this is true only in the limit $\Delta t \to 0$. Indeed, the density is known to have a non-central χ^2 distribution. But when one looks at the moments of this distribution, one can see that, as $\Delta t \to 0$, all moments higher than the second vanish faster than $\Delta t^{(1)}$.

that the bond price would still have maintained this affine structure.

All that remains is to determine the form of the coefficients $A(\tau)$, $B(\tau)$. The simplest procedure at this point is to use the PDE to find what differential equations $A(\tau)$ and $B(\tau)$ satisfy. Unfortunately, such a technique will not be available when we consider infinite factor models. Therefore, we demonstrate an alternative technique here.

Above, we demonstrated that the bond price is of the form

$$P^s(v) \equiv e^{A(s-v) - r_v\, B(s-v)}\,.$$

Hence,

$$\begin{aligned} P^s(v-\Delta t) &= e^{-\Delta t\, r_{(v-\Delta t)}}\, \mathrm{E}^Q_{(v-\Delta t)}\left[P^s(v)\right] \\ &= e^{-\Delta t\, r_{(v-\Delta t)}}\, e^{A(s-v)}\, e^{-B(s-v)\left[r_{(v-\Delta t)} + \kappa(\theta - r_{(v-\Delta t)})\Delta t\right] + \frac{1}{2}\, B^2(s-v)\,\sigma^2\, r_{(v-\Delta t)}} \end{aligned}$$

By definition of $A(\cdot)$, $B(\cdot)$, this is equivalent to

$$P^s(v-\Delta t) \equiv e^{A(s-v+\Delta t) - r_{(v-\Delta t)}\, B(s-v+\Delta t)}\,.$$

Collecting terms of order $r^1_{(v-\Delta t)}$, $r^0_{(v-\Delta t)}$, we find

$$\begin{aligned} B(s-v+\Delta t) &= B(s-v)\,(1-\kappa\,\Delta t) + \Delta t - \frac{\sigma^2}{2}\, B^2(s-v)\,\Delta t \\ A(s-v+\Delta t) &= A(s-v) + \kappa\,\theta\, B(s-v)\,\Delta t\,. \end{aligned}$$

These can be rewritten as ($\tau \equiv s - v$)

$$\begin{aligned} \frac{dB}{d\tau} &= -\kappa\, B + 1 - \frac{\sigma^2}{2}\, B^2 \\ \frac{dA}{d\tau} &= -\kappa\,\theta\, B\,. \end{aligned}$$

These can be solved to obtain the solution of Cox, Ingersoll, and Ross (1985b).

To obtain the bond-option formula, similar arguments are used. Actually, it is easier to solve the problem using forward risk adjusted measures. Under these measures, the bond-option can be written

$$C(v,s,T) = P^T(v)\, \mathrm{E}^{R(T)}_v\left[1_{(P^T(s)>K)}\right] - K\, P^s(v)\, \mathrm{E}^{R(s)}_v\left[1_{(P^T(s)>K)}\right]\,. \tag{3}$$

Here, we solve for the first term; the second is obtained in an identical fashion.

The transformation from the risk-neutral measure to the T-forward risk adjusted measure is obtained through

$$dZ^{R(T)}(t) = dZ^Q(t) + \sigma\,\sqrt{r}\, B(T-t)\, dt\,,$$

where $B(\cdot)$ is obtained from the bond price formula above. Hence, the spot rate process under FRAM(T) is

$$\begin{aligned} dr &= \kappa\,(\theta - r)\,dt + \sigma\,\sqrt{r}\,\left[dZ^{R(T)}(t) - \sigma\,\sqrt{r}\,B(T-t)\,dt\right] \\ &= \left[\kappa\theta - r\left(\kappa + \sigma^2\,B(T-t)\right)\right]\,dt + \sigma\,\sqrt{r_t}\,dZ^{R(T)}(t) \end{aligned}$$

The first term in the bond-option formula is

$$\begin{aligned} C^1(v,s,T) &= P^T(v)\,\mathrm{E}_v^{R(T)}\left[1_{(P^T(s)>K)}\right] \\ &= P^T(v)\,\mathrm{E}_v^{R(T)}\left[1_{(\log(P^T(s))>\log(K))}\right] \\ &= P^T(v)\,\mathrm{E}_v^{R(T)}\left[1_{(A(T-s)-r_s\,B(T-s)-\log(K)>0)}\right]\,. \end{aligned}$$

That is, we need the probability, under FRAM(T), that r_s will be less than $\left(\frac{A(T-s)-\log(K)}{B(T-s)}\right)$. To obtain the probability density of r_s, we first calculate the characteristic function,

$$G(\lambda,(s-v)) \equiv \mathrm{E}_v^{R(T)}\left[e^{i\,\lambda\,r_s}\right]\,. \tag{4}$$

We then obtain the probability density by applying a Fourier transform to the characteristic function.[3] The final step is to integrate over the probability density to obtain a cumulative distribution.

Note that Equation 4 has the same affine structure in the spot rate as did the expectation for the bond price formula. This implies that whenever a model possesses a closed-form solution for the bond price, it will also have a closed form solution for the characteristic function.[4] This is the reason why bond-options tend to have closed-form solutions whenever the bond prices do.

To price the option, we proceed in the same fashion as above. We discretize time, and use iterated expectations. As demonstrated for the bond price, each expectation will produce a term linear in the spot rate. Hence,

$$G(\lambda,(s-v)) = e^{H(s-v)-r_v\,F(s-v)}\,.$$

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[3] The probability function $\Pi(x)$ is obtained from the characteristic function $G(\lambda)$ via

$$\Pi(x) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} d\lambda\,e^{-i\lambda x}\,G(\lambda)\,.$$

[4] There is a jump in logic here, since we transformed from the risk-neutral to the forward risk adjusted measure. However, the option price can be obtained using the risk-neutral measure– the algebra is just slightly more complicated.

To obtain the functional form of $H(\cdot)$ and $F(\cdot)$, we calculate

$$\begin{aligned} \mathrm{E}^{R(T)}_{(v-\Delta t)}\, G(\lambda, (s-v)) &= \mathrm{E}^{R(T)}_{(v-\Delta t)} \left[e^{H(s-v) - r_v\, F(s-v)} \right] \\ &\equiv \left[e^{H(s-v+\Delta t) - r_{(v-\Delta t)}\, F(s-v+\Delta t)} \right] . \end{aligned}$$

It is convenient to write $u \equiv (s-v)$ and $\tau \equiv (T-s)$. In the limit $\Delta t \to 0$, we then obtain:

$$\begin{aligned} F_u &= -F \left[\kappa + \sigma^2\, B(\tau+u) \right] - \frac{\sigma^2}{2}\, F^2 , & F(u=0) &= -i\lambda \\ H_u &= -\kappa\, \theta\, F , & H(u=0) &= 0. \end{aligned}$$

The solution to $F(\cdot)$ is

$$F(u) = \frac{i\, \lambda\, P(u)}{1 - i\, \lambda\, Q(u)} ,$$

where

$$\begin{aligned} P(u) &= -e^{-\kappa u}\, e^{-\sigma^2 \int_0^u dv\, B(\tau+v)} \\ \frac{d}{du} Q(u) &= -\frac{\sigma^2}{2} P(u) . \end{aligned}$$

The solution to $H(\cdot)$ is

$$H(u) = -\frac{2\, \kappa\, \theta}{\sigma^2}\, \log \left[1 - i\, \lambda\, Q(u) \right] .$$

Thus, the characteristic function is

$$G(\lambda, (s-v)) = (1 - i\, \lambda\, Q(s-v))^{\left(\frac{-2\, \kappa\, \theta}{\sigma^2} \right)}\; e^{-r_v \left(\frac{i\, \lambda\, P(u)}{1 - i\, \lambda\, Q(u)} \right)} .$$

This characteristic function leads to a non-central χ^2 distribution, with the form obtained as in CIR (1985b).

It is interesting to note that this method permits one to extend the model to time-dependent parameters $\kappa(t)$, $\theta(t)$, and $\sigma(t)$, and to confirm the results of Jamshidian (1992). Indeed, the same technique used here can be used on models within a Heath, Jarrow, and Morton (1990, 1992) framework to test whether closed form option prices exist. For example, Jamshidian (1992) notes that the function

$$\frac{\kappa(t)\, \theta(t)}{\sigma^2(t)}$$

must be time independent if the transition density is to remain χ^2. However, even if this is not the case, the characteristic function can be solved in closed-form, possessing an affine structure in the current spot rate. Hence, only a Fourier transform (integral) need be solved in order to obtain the transition density, an ultimately, the bond-option price.

3.2 Quadratic Model

Above, we assumed that the spot rate was a sufficient statistic to price all term structure securities. More generally, we can consider the spot rate to be a function of some state variable, and write $r = r(x)$. Assume that the evolution of the state variable x follows

$$dx = \mu(x)\, dt + \sigma(x)\, dZ^Q \,.$$

Then the discount bond satisfies

$$P^s(v) = \mathrm{E}_v^Q \left[e^{-\int_v^s du\, r(x_u)} \right] .$$

Again we discretize time, and use iterated expectations, to write this as

$$P^N(n) = e^{-\Delta t\, r(x_n)} \prod_{j=n}^{N-2} \left\{ \mathrm{E}_j^Q \left[e^{-\Delta t\, r(x_{j+1})} \right] \right\} .$$

The x-process implies

$$x_{j+1} | \mathcal{F}_j \;\sim\; \Phi \left(x_j + \mu(x_j)\, \Delta t \,,\, \sigma^2(x_j)\, \Delta t \right) .$$

Hence, the expectation furthest to the right is

$$\mathrm{E}_{N-2}^Q \left[e^{-\Delta t\, r(x_{(N-1)})} \right] = \int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt{2\pi\sigma^2(x_{(N-2)})\Delta t}} \times$$
$$e^{\left(\frac{-1}{2\sigma^2(x_{(N-2)})\Delta t}\right)\left[x_{(N-1)} - x_{(N-2)} - \mu(x_{(N-2)})\Delta t\right]^2} e^{-\Delta t\, r(x_{(N-1)})} \,. \tag{5}$$

Consider the mapping $r(x_j) = x_j^2$, and the process

$$dx = \kappa\, (\theta - x)\; dt + \sigma\, dZ^Q \,.$$

The quadratic mapping permits successive expectations to have a form

$$P^N(j) = e^{A(N-j) + B(N-j)\, x_j + C(N-j)\, x_j^2} \,.$$

Attempting to generalize the x-process beyond an Ornstein-Uhlenbeck form will lead to a structure where the expectations cannot be performed.

3.3 Other 1 Factor Models

The search for other models which offer simple closed-form expressions for bond prices and bond-option prices looks formidable. A look at integral tables implies that solutions to integrals of the form

$$\int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi\sigma^2(x_{(N-2)})\Delta t}} \, e^{(\frac{-1}{2\sigma^2(x_{(N-2)})\Delta t})\left[x_{(N-1)}-x_{(N-2)}-\mu(x_{(N-2)})\Delta t\right]^2} \, e^{-\Delta t \, r(x_{(N-1)})} \, .$$

only have simple solutions for the affine term structure models and the quadratic model mentioned above. Note that other closed-form bond price solutions for one-factor models do exist. For example, see Dothan (1977) and Goldstein and Keirstead (1997). There, bond prices are written as an infinite sum, or integral, of eigenvectors of the infinitesimal generator. But the search for simple closed-form bond price formula for one factor models may be exhausted.

4 Multi-Factor Models

Assume the spot rate can be written as a function of two state variables, $r_t = r(x_t, y_t)$. Assume the Markov dynamics

$$\begin{aligned} dx &= \mu_x(x,y)\,dt + \sigma_x(x,y)\,dZ_x \\ dy &= \mu_y(x,y)\,dt + \sigma_y(x,y)\,dZ_y \, , \end{aligned}$$

where $dZ_x \, dZ_y = \rho(x,y)\,dt$. Again we discretize time and use iterated expectations. We look at the last expectation in the bond price formula:

$$\int_{-\infty}^{\infty} dx_{(N-1)} \int_{-\infty}^{\infty} dy_{(N-1)} \, \Pi\left[x_{(N-1)}, y_{(N-1)} | x_{(N-2)}, y_{(N-2)}\right] \, e^{-\Delta t \, r\left(x_{(N-1)}, y_{(N-1)}\right)} \, .$$

Since everything is normally distributed in the limit $\Delta t \to 0$, it will typically be convenient to write this as

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_{(N-1)} \;\; \Pi\left[x_{(N-1)} | x_{(N-2)}, y_{(N-2)}\right] \\ &\qquad\qquad \times \int_{-\infty}^{\infty} dy_{(N-1)} \, \Pi\left[y_{(N-1)} | x_{(N-1)}, x_{(N-2)}, \, y_{(N-2)}\right] \, e^{-\Delta t r\left(x_{(N-1)}, y_{(N-1)}\right)} \, , \end{aligned}$$

and use the projection theorem to obtain

$$\Pi\left[y_{(N-1)} | x_{(N-1)}, x_{(N-2)}, y_{(N-2)}\right] \, .$$

We look for models where successive expectations remain invariant in form. Many models have already been analyzed. For example, the model of Longstaff and Schwartz (1992) uses

$r = x + y$, and then models both x and y as square root processes. Duffie and Kan (1994) discuss to the extent that such a model can be generalized. Once again, these types of models maintain an affine structure in the state variables.

The multi-factor quadratic/bilinear model is investigated by Constantinides (1992) and Beaglehole and Tenney (1991, 1992). Again, the expectation approach demonstrates that the state variable dynamics must be limited to multivariate Ornstein-Uhlenbeck processes to obtain closed form expressions.

We mention that Longstaff and Schwartz (1992) eventually eliminate the state variables x and y and rewrite the dynamics in terms of the spot rate r and variance V. Thus, they have a model with stochastic volatility. Another model with stochastic volatility is

$$\begin{aligned} dr &= \kappa(\theta - r)\,dt + \sigma\,dW^Q \\ d\sigma &= \lambda(\phi - \sigma)\,dt + \nu\,dZ^Q\,, \end{aligned}$$

with $dW^Q\,dZ^Q = \rho\,dt$. Again, it is straightforward to demonstrate that the bond price formula is of the form

$$P^T(s) = e^{A(T-s)+B(T-s)\,r_s+C(T-s)\,\sigma_s+D(T-s)\,\sigma^2(s)}\,,$$

with the coefficients satisfying

$$\begin{aligned} A' &= \kappa\theta B + \lambda\phi C + \frac{\nu^2}{2}C^2 + \nu^2 D \\ B' &= 1 - \kappa B \\ C' &= -\lambda C + 2D\lambda\phi + 2\nu^2 CD + \nu\rho BC \\ D' &= 2\nu^2 D^2 - 2\lambda D + \frac{1}{2}B^2 + 2BD\nu\rho\,. \end{aligned}$$

5 Infinite Factor Models

One difficulty with finite factor models is that they need to be constantly recalibrated to remain consistent with the current term structure. Another problem is that a 3-factor model, for example, implies that one can (instantaneously) hedge a 20-year bond with a 3-, 4-, and 5-month bill. To overcome these difficulties, random field, or infinite factor models have been introduced by Kennedy (1994, 1997) and Goldstein (1997). One can specify the forward rate dynamics under the risk neutral measure as

$$df^T(s) = \mu_T(s)\,ds + \sigma_T(s)\,dZ_Q^T(s)\,.$$

The superscript on the Brownian motion implies that, corresponding to each maturity date T there is a unique innovation that is not a linear combination of any finite number of other

Brownian motions. Hence, associated with each maturity date in the continuum is an additional factor, somewhat analogous to idiosyncratic risk associated with each security in a CAPM framework. The random field is characterized by the correlations between the innovations

$$dZ_Q^{T_1}(s)\, dZ_Q^{T_2}(s) = ds\, c(s, T_1, T_2)\,.$$

Generalizing the result of Heath, Jarrow, and Morton (1990, 1992), the drift $\mu_T(s)$ is completely specified by the volatility structure $\sigma_T(s)$ and the correlation structure $c(s, T_1, T_2)$.

Here, we demonstrate that if the forward rate volatility structure is of the form

$$\sigma_u(w) = \sqrt{\sigma^2\, r_w + \gamma}\; B'(u-w)\,,$$

and the correlation structure $c(\cdot,\cdot,\cdot)$ is deterministic, then the characteristic function of $\log P^T(s)$ under FRAM(T)

$$G(\lambda) = \mathrm{E}_v^{R(T)}\left[e^{i\lambda \log P^T(s)}\right] = \mathrm{E}_v^{R(T)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right]\,,$$

takes a form linear in the current forward rates,

$$\mathrm{E}_v^{R(T)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right] = e^{-i\lambda \int_s^T du\, f^u(v) + \int_v^s du\, h_T(s-u)\, f^u(v) + A_T(s-v)}\,,$$

where the functions $h_T(\cdot)$ and $A_T(\cdot)$ are specified below. The characteristic function takes the same form under FRAM(s), but is specified by different functions $h_s(\cdot)$ and $A_s(\cdot)$. Both solutions are necessary to price the bond option via Equation 3.

The forward rate process is

$$\begin{aligned} df^u(w) &= dw\,(\sigma^2\, r_w + \gamma)\, B'(u-w) \int_w^u dt\, B'(t-w)\, c(w,u,t) + \sqrt{\sigma^2\, r_w + \gamma}\, B'(u-w)\; dZ_Q^u(w) \\ &= -dw\,(\sigma^2\, r_w + \gamma)\, B'(u-w) \int_u^T dt\, B'(t-w)\, c(w,u,t) + \sqrt{\sigma^2\, r_w + \gamma}\, B'(u-w)\, dZ_{R(T)}^u(w) \end{aligned}$$

under FRAM(T). If the correlation structure is stationary,

$$c(w,u,t) = c(u-w, t-w)\,,$$

then it is convenient to define

$$D(u-w, T-w) \equiv \int_u^T dt\, B'(t-w)\, c(u-w, T-w)\,,$$

producing the dynamics

$$df^u(w) = -dw\,(\sigma^2\, r_w + \gamma)\, B'(u-w)\, D(u-w, T-w) + \sqrt{\sigma^2\, r_w + \gamma}\, B'(u-w)\, dZ_{R(T)}^u(w)\,.$$

The proposed method of solution is to write the expectation as a path integral and use iterated expectations:

$$\mathrm{E}_v^{R(T)}\left[e^{i\lambda \log P^T(s)}\right] = \mathrm{E}_v^{R(T)}\,\mathrm{E}_{(v+\Delta)}^{R(T)}\cdots\mathrm{E}_{(s-\Delta)}^{R(T)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right].$$

Note that for sufficiently small Δ, $\left[f^u(s)|\mathcal{F}_{(s-\Delta)}\right]$ is normally distributed, with

$$\mathrm{E}_{(s-\Delta)}^{R(T)}\left[-i\lambda\int_s^T du\, f^u(s)\right] = -i\lambda\int_s^T du\left[f^u(s-\Delta)-\Delta\,(\sigma^2\, r_{(s-\Delta)}+\gamma)\,B'(u-s+\Delta)\,D(u-s+\Delta, T-s+\Delta)\right]$$

and

$$\frac{1}{2}\,\mathrm{Var}_{(s-\Delta)}^{R(T)}\left[-i\lambda\int_s^T du\, f^u(s)\right] = -\Delta\frac{\lambda^2}{2}\,(\sigma^2\, r_{(s-\Delta)}+\gamma)\int_s^T du\, B'(u-s+\Delta)\,D(u-s+\Delta, T-s+\Delta)$$

Thus, we can write

$$\begin{aligned}\mathrm{E}_{(s-\Delta)}^{R(T)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(s-\Delta)+h(\Delta)\, r_{(s-\Delta)}+A(\Delta)}\\ &= e^{-i\lambda\int_s^T du\, f^u(s-\Delta)+h(\Delta)\, f^{(s-\Delta)}(s-\Delta)+A(\Delta)}.\end{aligned}$$

Indeed, each successive iteration produces a function that is linear in the forward rates. To specify the functional forms of $h(\cdot)$ and $A(\cdot)$, we write

$$\begin{aligned}\mathrm{E}_{(v+\Delta)}^{R(T)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(v+\Delta)+\int_{v+\Delta}^s du\, h(s-u)\, f^u(v+\Delta)}\, e^{A(s-v-\Delta)}\\ &\equiv e^{\int_{v+\Delta}^T du\, h(s-u)\, f^u(v+\Delta)}\, e^{A(s-v+\Delta)}\end{aligned}$$

where for compactness we have defined $h(s-u) = -i\lambda$ for values of u such that $s < u < T$. By iterating back one step, we find an implicit function for $h(\cdot)$:

$$\begin{aligned}\mathrm{E}_v^{R(T)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(v)+\int_{v+\Delta}^s du\, h(s-u)\, f^u(v)+\Delta\, r(v)\, h(s-v)+A(s-v)}\\ &= e^{-i\lambda\int_s^T du\, f^u(v)+\int_v^s du\, h(s-u)\, f^u(v)+A(s-v)},\end{aligned}$$

where

$$\begin{aligned}h(s-v) &= -\sigma^2\int_v^T du\, h(s-u)\, B'(u-v)\, D(u-v, T-v)\\ &\quad+\frac{\sigma^2}{2}\int_v^T du\, h(s-u)\, B'(u-v)\int_v^T dw\, h(s-w)\, B'(w-v)\, c(v,u,w),\\ A'(s-v) &= \frac{\gamma}{\sigma^2}\, h(s-v),\end{aligned}$$

with initial condition $A(0) = 0$. Once the values of $h(\cdot)$ and $A(\cdot)$ have been obtained, one need only calculate an integral to obtain the bond price.

6 Conclusion

The examples above have been limited to term structure models, but it is straightforward to generalize the approach for pricing other assets. For example, stochastic volatility models for option pricing have previously used the fact that the characteristic function may be easier to obtain than the probability density directly. The above analysis implies that the square-root variance model of Heston (1993) and the Ornstein-Uhlenbeck model of Stein and Stein (1991) and Ball and Roma (1994) may be the only stochastic volatility models with simple characteristic functions.

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