The Term Structure of Interest Rates as a Random Field

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Abstract

Forward rate dynamics are modeled as a random field. In contrast to multi-factor models, random field models offer a parsimonious description of term structure dynamics, while eliminating the self-inconsistent practice of recalibration. Several examples of random field models are presented. The form of the drift of the instantaneous forward rate process necessary to preclude arbitrage under the risk neutral measure is obtained. Forward measures are characterized, and used to price a bond option when the forward volatility structure depends upon the square root of the current spot rate.

Approaches to modeling term structure dynamics have grown tremendously in sophistication over the past twenty years. The first generation of models introduced finite-factor processes with constant coefficients, and then defined the spot rate as a function of these factors.[1] Unfortunately, this class of models does not permit consistency with the current term structure. The second generation of models either assigned explicit time dependence to the spot rate process, or modeled the forward rate process directly, using the initial forward rate curve as an input.[2] Although this class of models allows one to fit the current yield curve, it does not permit consistency with term structure innovation. That is, typically there does not exist a possible realization of the N-Brownian motions $\{dz_i(t)\}|_{i=1}^{N}$ (in an N-factor model) that is consistent with the innovation of the 120 strips that comprise the riskless yield curve. Therefore, practitioners need to continuously recalibrate parameters of their model in order to fit the new term structure. But within the frameworks of this class of models, those parameters are supposed to be deterministic, not stochastic processes that need to be constantly updated. Hence, this second class of models is also inconsistent with empirical data.

To overcome this difficulty, we model the term structure of interest rates as a random field. Limiting his scope to Gaussian random fields (deterministic volatility structures), Kennedy (1994) obtains the form of the drift of the instantaneous forward rate process necessary to preclude arbitrage under the risk neutral measure. Here, we generalize his results to include non-Gaussian random fields.

Recalibration is unnecessary for random field models. In analogy with an N-factor APT (Ross 1976) model, where the return of each security depends upon N market risks *and* an idiosyncratic risk, a random field model attributes to each instantaneous forward an innovation which is imperfectly correlated with the innovation of any linear combination of other instantaneous forwards. This feature is quite distinct from N-factor term structure models, where in theory any security can be perfectly hedged (instantaneously) by purchasing a portfolio of N additional assets. But the analogy between modeling stocks and modeling bonds with idiosyncratic risk is not exact: there is no natural ordering for stocks, whereas forwards can be ordered by their maturity date, producing a relatively smooth yield curve. From casual observation it is clear that adding arbitrary idiosyncratic risk to the dynamics of each instantaneous forward rate is unreasonable; the correlation between the fluctuations of two forwards should go to unity as the difference in their maturities vanishes. Indeed, when this restriction

[1]See, for example, Vasicek (1977), Brennan and Schwartz (1979), Cox, Ingersoll and Ross (1985b) Constantinides (1992), and Longstaff and Schwartz (1992).

[2]See, for example, Cox, Ingersoll and Ross (1985b), Ho and Lee (1986), Black, Derman, and Toy (1990), Black and Karasinski (1991), Heath, Jarrow, and Morton (1990, 1992), and Constantinides (1992).

is imposed, random field models predict that the best hedging instrument for a given bond is another one of similar maturity.

Since instantaneous forwards form a continuum, in general, random field models are infinite-factor models. We emphasize, though, that random field models accommodate both finite- and infinite-factor models. In particular, we demonstrate below that all (continuous) finite-factor models are special cases of the random field models presented here.[3]

In addition to eliminating the inconsistent practice of recalibration, there are many other advantages to modeling the term structure as a random field. First, random fields offer a much more parsimonious description of term structure dynamics than their multi-factor counterparts. Indeed, in a simple model proposed below, only one additional parameter needs to be estimated to convert a one-factor model into a random field. This parameter measures how quickly the correlation of the innovations between two forwards drop as a function of difference in time to maturity. If this parameter is identically zero, then perfect correlation is obtained, and the random field model then reduces back to the one-factor model.

As another advantage, the random field framework naturally accounts for the fact that the best hedging instrument for a bond is another one of similar maturity. This is in stark contrast to finite factor models. For example, a three-factor model predicts that a 30-year bond can be perfectly hedged by an appropriate position in 1-month, 2-month, and 3-month bills. Of course, pension fund managers realize that such hedging strategies would fail in practice. Therefore, they typically hedge interest rate risk by estimating the duration of short-, middle-, and long-term commitments separately.[4] From a random field perspective, such hedging strategies are clearly superior compared to, say, using only short term bonds to hedge interest rate risk of the entire portfolio. However, such a strategy leads to arbitrary distinction between short-, middle-, and long-term maturities. This arbitrariness in turn leads to a suboptimal portfolio choice from, say, a mean-variance efficient frontier analysis.

The rest of the paper is as follows. Section 1 derives the result of HJM (1990, 1992) in a form more suitable for generalization to random fields, and offers some simple examples of random field models. Section 2 derives the form of the drift function of the instantaneous forward rate (under the risk neutral measure) necessary to preclude arbitrage. In Section 3, forward measures are characterized, and used to price a bond option when the volatility

[3]It is straightforward to add jump processes to the analysis, but for simplicity, we ignore such processes here. See, for example, Björk et al (1996).

[4]Ho (1992) attempts to capture the idea of hedging separately 'key rate durations' at several different maturities. However, his approach ignores the correlation between different key-rate durations, and these correlations are an important input for determining optimal portfolio choice. Indeed, Heaney and Cheng (1984) use the entire covariance structure to determine optimal mean-variance portfolios.

structure is proportional to the square root of the current spot rate. We conclude in Section 4.

1 Prologue

Before introducing random fields, we rederive the result of HJM (1990, 1992) in a format better suited for our purposes.

Assume that the risk-neutral dynamics at time s of a discount bond maturing at time T can be written as

$$\frac{dP^T(s)}{P^T(s)} = r(s)\,ds - \sum_{i=1}^{N} \sigma^{i,T}(s)\,dZ_Q^i(s)\,, \tag{1}$$

where, the $\{dZ_Q^i(s)\}\Big|_{i=1}^{N}$ are independent Brownian motions. In general, the volatility structures $\sigma^{i,T}(s)$ themselves can be stochastic. Define the instantaneous forward rates as

$$f^T(s) \equiv -\frac{\partial}{\partial T}\log P^T(s)\,. \tag{2}$$

Using Ito's Lemma, the forward rate dynamics can be obtained:

$$df^T(s) = \sum_{i=1}^{N} \sigma^{i,T}(s)\,\sigma_T^i(s)\,ds + \sum_{i=1}^{N} \sigma_T^i(s)\,dZ_Q^i(s)\,, \tag{3}$$

where we have defined

$$\sigma_T^i(s) \equiv \frac{\partial}{\partial T}\left[\sigma^{i,T}(s)\right]\,. \tag{4}$$

Hence, as first noted by HJM (1990, 1992), the drift of the instantaneous forward rates is completely specified by the volatility structure:

$$\begin{aligned} \mu_T^Q(s) &\equiv \sum_{i=1}^{N} \sigma^{i,T}(s)\,\sigma_T^i(s) \\ &= \sum_{i=1}^{N} \left[\int_s^T du\,\sigma_u^i(s)\right]\sigma_T^i(s)\,, \end{aligned} \tag{5}$$

where we have used the fact that $\sigma^{i,s}(s) = 0$, as necessary, to insure that $P^s(s) = 1$. Since the Brownian motions are independent, the covariance between forward rate innovations is

$$df^{T_1}(s)\,df^{T_2}(s) = \sum_{i=1}^{N} \sigma_{T_1}^i(s)\,\sigma_{T_2}^i(s)\,ds\,. \tag{6}$$

For our purposes, it is convenient to rewrite the forward rate dynamics as

$$df^T(s) = \mu_T^Q(s)\,ds + \sigma_T(s)\,dZ_Q^T(s)\,, \tag{7}$$

where

$$\sigma_T(s) \equiv \sqrt{\sum_{i=1}^{N} \left[\sigma_T^i(s)\right]^2}\,, \tag{8}$$

and $dZ_Q^T(s)$ is a Brownian motion. Note that

$$\left[df^T(s)\right]^2 = \left[\sigma_T(s)\right]^2 \, ds = \sum_{i=1}^{N} \left[\sigma_T^i(s)\right]^2 \, ds\,, \tag{9}$$

consistent with Equation 6 for $T_1 = T_2 = T$. More generally, define the correlation function $c(\cdot,\cdot,\cdot)$ via

$$dZ_Q^{T_1}(s)\, dZ_Q^{T_2}(s) \equiv c(s, T_1, T_2)\, ds\,. \tag{10}$$

It follows from Equation 7 that

$$df^{T_1}(s)\, df^{T_2}(s) = \sigma_{T_1}(s)\, \sigma_{T_2}(s)\, c(s, T_1, t_2)\, ds\,. \tag{11}$$

A necessary condition for Equations 6 and 11 to be consistent is

$$\sigma_{T_1}(s)\, \sigma_{T_2}(s)\, c(s, T_1, T_2) = \sum_{i=1}^{N} \sigma_{T_1}^i(s)\, \sigma_{T_2}^i(s)\,. \tag{12}$$

Comparing this with Equation 5, the drift of the instantaneous forward can also be written in the form

$$\mu_T^Q(s) = \int_s^T du\, \sigma_T(s)\, \sigma_u(s)\, c(s, T, u)\,. \tag{13}$$

We emphasize that this description is identical to the HJM (1990, 1992) result. However, this form is easier to generalize to random fields. The reason why is straightforward: rather than introduce Brownian motions with an arbitrary index i, this framework attributes to each instantaneous forward a Brownian motion $dZ_Q^T(s)$ which is indexed by the maturity of the forward T. Since the HJM (1990, 1992) framework is limited to a finite number of factors N, the correlation matrix $c(\cdot,\cdot,\cdot)$ between any $(N+M)$ subset of forwards is singular for any positive integer M. Below, we generalize the HJM (1990, 1992) result to include covariance structures which are non-singular for any finite number of forwards.[5]

Before continuing, it is important to distinguish between an infinite-factor model and a finite-factor model defined on an infinite state space. It well known that, in general, term structure dynamics may need to be defined on an infinite state space. Indeed, as demonstrated by HJM (1990, 1992), spot rate dynamics may depend upon the entire history of the yield

[5]In fact, it is non-singular even in the continuum sense, as demonstrated by Heaney and Cheng (1984).

curve.[6] This fact spurred investigation into characterizing the class of term structure models that are supported on a finite state space.[7] We emphasize that this search is not relevant for random fields, since the state space must be at least as large as the number of factors. Hence, the claim of Kennedy (1994, pg. 254) that the spot rate can be written as a one-factor Markov process cannot be correct. Indeed, within a random field model, it is impossible to price the continuum of securities in terms of a finite set of state variables.[8] Conversely, regardless of the number of state variables needed to produce a dynamical model which is jointly-Markov, a correlation matrix of (M+N) instantaneous forward rates must be singular in an N-factor HJM model, where M is any positive integer.

Heaney and Cheng (1984) were the first to investigate optimal portfolio selection in the presence of a continuum of bond securities. It is well known, from a mean-variance framework with a finite number of securities, that the optimal portfolio of weights $\mathbf{x}$ satisfy the first-order condition

$$\mathbf{\Sigma}\,\mathbf{x} = \frac{\mathbf{1}}{\mathbf{2}}\left(\lambda_{\mathbf{1}}\,\mu + \lambda_{\mathbf{2}}\,\mathbf{1}\right)\;, \tag{14}$$

where $\mathbf{\Sigma}$ is the covariance matrix of security returns, λ_1 and λ_2 are Lagrange multipliers, μ is a vector of expected returns, and $\mathbf{1}$ is a vector of ones. The method of solution for the finite-security case typically involves inverting the covariance matrix. However, such inversion in the infinite-factor case cannot be done in a straightforward manner. Rather, the solution is written in terms of the eigenvectors and eigenvalues of the infinite-factor covariance function. The use of eigenvector analysis basically converts the non-denumerably infinite problem into a countably-infinite problem. The eigenvalue approach also provides the technical conditions that must be satisfied for a solution to exist. For example, the solution must be square-integrable, and hence, defined on a Hilbert space.

Random fields have been studied extensively in the mathematics literature, and typically afford results that are straightforward generalizations of concepts found in finite-factor stochastic calculus. For example, martingale representation fits prominently in the theory. The Feynman-Kac formula generalizes in a straightforward manner, expressing an equivalence between the

[6]Indeed, it was the non-Markovian nature of the spot rate which lead Musiela (1992) to propose modeling the yield curve as a stochastic partial differential equation, which is a special case of a random field.

[7]See, for example, Carverhill (1994), Ritchken and Sankarsubramanian (1995), and Inui and Kijima (1998).

[8]Kennedy (1994) introduces a model where he correctly determines $\mathrm{E}\left[dr_t|\,\{r_s\}\right] = m\,dt$, $\mathrm{Var}\left[dr_t|\,\{r_s\}\right] = s^2\,dt\;,\;\; \forall s \in (-\infty, t)$. However, for the spot rate process to be of the form $dr = m\,dt + s\,dz$, one must show that, given *all* information available at time t: $\mu\,dt \equiv \mathrm{E}\left[dr_t|\,\mathcal{F}_t\right] = m\,dt$, $\sigma^2\,dt \equiv \mathrm{Var}\left[dr_t|\,\mathcal{F}_t\right] = s^2\,dt$. However, this is not the case. Indeed, if his answer were correct, then all bond prices, and hence all forward rates, would be perfectly correlated. But note that his prescription leads to the correlation $\mathrm{Corr}\left[f^{T_1}(s)\,f^{T_2}(s)\right] = \sqrt{\frac{\exp(-\lambda\max(T_1,T_2))}{\exp(-\lambda\min(T_1,T_2))}}$, which is less than unity.

infinite dimensional stochastic differential equation and the infinite-dimensional Fokker-Planck equation, better known as the Kolmogorov field equation. More importantly for this paper, we use the random-field generalization of Girsanov's Theorem below to prove the main result of the paper. Proofs of such theorems can be found, for example, in Da Prato and Zabczyk (1992). Whereas a Brownian motion is parameterized by a single time parameter $dZ(s)$, random fields have a minimum of two parameters, $dZ^T(s)$, in which case their dynamics can be specified on a Brownian sheet, as in Adler (1981). We note that the Brownian sheet description is valid only if the forward rates are jointly Markov. In this case, term structure dynamics can be described as a 'random string', that is, with one 'spatial' dimension. If one wishes to model forward rate dynamics as a function of both current and past forward rates, it would be necessary to increase the size of the space from a 'string' to a two-dimensional 'drumhead'.[9]

1.1 Models of Correlation Structures

To generate a sample path of a Brownian motion $dZ(s)$, one would discretize time, and select from a random number generator a single number for each time interval $\triangle$. This number would be generated from a normal population $\Delta Z \equiv Z(s+\triangle) - Z(\triangle) \sim \Phi\,[0, \triangle]$. However, to generate a random field, an entire function $\Delta Z^T(s)$ must be generated for each time interval $\triangle$. The simplest models to generate a random field are those that are well known in spectral analysis, namely, the auto-regressive and moving-average processes.

The continuous-time analogue of an AR(1) process is the Ornstein-Uhlenbeck process. Realizations of $dZ^s(s)$ are generated from the normal distribution

$$dZ^s(s) \sim \Phi\,[0\,,\, ds]\ ,$$

and the rest of the Brownian field at time s is generated by

$$dZ^T(s) = dZ^s(s)\, e^{-\rho(T-s)} + \sqrt{2\rho}\int_{u=s}^{T} dz^s(u)\, e^{-\rho(T-u)}\,, \tag{15}$$

where the $\{dz\}$ satisfy

$$\mathrm{E}\,[dz^s(u)] = 0\,, \qquad\qquad \mathrm{Cov}\,[dz^s(u_1)\,,\, dz^s(u_2)] = \begin{cases} ds\, du_1 & \text{if } (u_1 = u_2) \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Note that the $\{dz\}$ are processes new to the finance literature. Whereas Brownian motions have dimension $|dZ| \sim T^{\frac{1}{2}}$, the new processes have dimension $|dz| \sim T^1$. As with traditional

[9]The technique of increasing the state space to permit a non-Markovian 'delay' system to be rewritten as a Markovian system is described in Da Prato and Zabczyk (1992).

Ito calculus, the correlation structure has rigorous mathematical interpretation only after appropriate integrations are performed.

The definition of $dZ^T(s)$ has been appropriately normalized so that the unconditional variance equals ds:

$$\begin{aligned} \text{Var}\left[dZ^T(s)\right] &= ds\, e^{-2\rho(T-s)} + ds\, \left[1 - e^{-2\rho(T-s)}\right] \\ &= ds\,. \end{aligned}$$

The correlation is

$$\text{Cor}\left[dZ^{T_1}(s)\,,\, dZ^{T_2}(s)\right] \equiv \frac{\text{Cov}\left[dZ^{T_1}(s)\,,\, dZ^{T_2}(s)\right]}{\sqrt{\text{Var}\left[dZ^{T_1}(s)\right]}\,\sqrt{\text{Var}\left[dZ^{T_2}(s)\right]}} = e^{-\rho|T_1-T_2|}\,.$$

Note that if $\rho = 0$, then the Brownian field reduces to the 1-factor case, since the correlation between innovations of any two forwards becomes unity. In this case, the superscript is superfluous. Generalization to Brownian fields which reduce to N-factor models when $\rho_i = 0$, $i = (1, 2, \cdots, N)$ is straightforward. For example, for arbitrary functions $f(\cdot)$, $g(\cdot)$, define

$$dZ^T(s) \equiv \frac{f(T)\, dZ_1^T(s) + g(T)\, dZ_2^T(s)}{\sqrt{f(T)^2 + g(T)^2}}\,, \tag{17}$$

where both $dZ_1^{(\cdot)}(\cdot)$ and $dZ_2^{(\cdot)}(\cdot)$ are generated as in Equation 15, and are mutually independent,

$$dZ_1^{T_1}(s)\, dZ_2^{T_2}(s) = 0 \qquad \forall\{T_1, T_2\}\,. \tag{18}$$

Then, the correlation function is

$$\text{Cor}\left[dZ^{T_1}(s)\,,\, dZ^{T_2}(s)\right] = ds\, \frac{f(T_1)\, f(T_2)\, e^{-\rho_1|T_1-T_2|} + g(T_1)\, g(T_2)\, e^{-\rho_2|T_1-T_2|}}{\sqrt{f(T_1)^2 + g(T_1)^2}\,\sqrt{f(T_2)^2 + g(T_2)^2}}\,.$$

When $\rho_1 = \rho_2 = 0$, the correlation structure reduces to a two-factor model.

Below, we model the instantaneous forward rates as

$$df^T(s) = \mu_T(s)\, ds + \sigma_T(s)\, dZ^T(s)\,.$$

Kennedy (1997) demonstrates that for forward rates to be modeled as a Gaussian, Markov, and stationary random field, the volatility structure must be of the form $\sigma_T(s) = \sigma\, e^{-\kappa(T-s)}$, and the correlation structure must be $\text{Cor}\left[dZ^{T_1}(s)\,,\, dZ^{T_2}(s)\right] = e^{-\rho|T_1-T_2|}$. Note that such a model is characterized by only three parameters.

The Ornstein Uhlenbeck process is continuous but not differentiable. Empirical observation may suggest that a smoother random field process may be more appropriate. From a practical standpoint, bond prices via strips exist at only three-month intervals, so one must use some technique to construct a forward rate curve from these discrete data points. For example, the maximum smoothness criterion of Adams and Van Deventer (1994) generates forward rate curves which are four-times differentiable. Hence, if one uses the Adams and Van-Deventer (1994) approach, consistency would imply a correlation structure which is also four-times differentiable.

One way to generate a process that is once-differentiable is to integrate over an OU process. For example, define the process,

$$dV^T(s) = \sqrt{2\rho^2} \int_{-\infty}^{T} du\, dZ^u(s)\, e^{-\rho(T-u)} \,,$$

which can be written in the more useful form

$$dV^T(s) = e^{-\rho(T-s)} dV^s(s) + \sqrt{2\rho^2} \int_{s}^{T} du\, dZ^u(s)\, e^{-\rho(T-u)} \,.$$

The correlation structure is

$$\text{Cor}\left[dV^{T_1}(s)\,,\, dV^{T_2}(s)\right] = e^{-\rho\tau}\, [1 + \rho\tau] \,.$$

where $\tau = |T_1 - T_2|$. Note that by a Taylor expansion,

$$\begin{aligned} \text{Cor}\left[dV^{T_1}(s)\,,\, dV^{T_2}(s)\right] &= (1 - \rho\tau + \frac{1}{2}(\rho\tau)^2 + \cdots)(1 + \rho\tau) \\ &= = 1 - \frac{1}{2}(\rho\tau)^2 + \cdots \end{aligned}$$

the term linear in $|T_1 - T_2|$ vanishes, implying that the correlation structure is differentiable. The relationship between differentiability of the underlying process and the differentiability of the correlation structure is well known from spectral analysis. See, for example, Priestley (1981).

Additional integration procedures produce correlation structures which are even smoother. Such structures may be necessary for a given model to be meaningful. For example, there is empirical support for the existence of a curvature-reducing component $\frac{\partial^2}{\partial T^2} f^T(s)$ in the drift of forward rate dynamics.[10] To model this case, it is necessary to consider forward rate curves which are at least twice-differentiable in T. One more integration

$$dW^T(s) = \sqrt{\frac{4\rho^2}{3}} \int_{-\infty}^{T} du\, dV^u(s)\, e^{-\rho(T-u)} \,,$$

[10] See, for example, Bouchaud et al (1998).

generates the correlation structure

$$\mathrm{Cor}\left[dW^{T_1}(s)\,,\,dW^{T_2}(s)\right] = e^{-\rho\tau}\left[1+\rho\tau+\frac{1}{3}(\rho\tau)^2\right]\,.$$

A Taylor expansion on this structure reveals that both the linear and cubic terms vanish.

Besides the continuous-time versions of auto-regressive processes, a random field can also be generated from moving average processes. For example, define

$$dZ^T(s) = \frac{1}{\sqrt{L}}\int_{u=(T-L)}^{T} dz^s(u)\,, \tag{19}$$

where the $\{dz\}$ are defined in Equation 16. Such a process generates a covariance structure

$$dZ^{T_1}(s)\,dZ^{T_2}(s) = ds\left[1-\frac{|T_1-T_2|}{L}\right]\mathbf{1}_{(L>|T_1-T_2|)}\,. \tag{20}$$

Note that the correlation is zero when the difference in maturities is greater than L. Empirically, Cannabaro (1995) finds that the long- and short-end yields still have correlations of about 0.5, so L would have to be significantly longer than thirty years to fit the empirical data well. As with the Ornstein-Uhlenbeck process, the correlation structure is not differentiable. However, as above, smoother structures can be generated by integrating over such a process.

2 Random Fields Under Equivalent Martingale Measures

Pearson and Sun (1994) empirically test the Cox, Ingersoll, and Ross (1985b) one factor term structure model. As stated previously, all finite-factor term structure models taken literally are incompatible with empirical observation. To overcome this difficulty, they add 'disturbances' to the bond prices, and assume these disturbances are independent. Such independence is reasonable if the noise is due to nonsynchronous trading or quotation errors. But if these disturbances are actually due to model misspecification, then within their model, instantaneous forwards of similar maturity will have highly correlated disturbance terms. The random field model of interest rates inherently accounts for these 'disturbance terms', making the ad-hoc assumption of Pearson and Sun unnecessary.

In general, $(N+1)$ bonds are needed to construct a perfect hedge in an N-factor economy (see, for example, Cox, Ingersoll, and Ross (1981)). This hedge permits the risk-neutral measure to be identified. But a random field model generally implies the existence of an infinite number of economic factors, and hence no riskless portfolio can be constructed if only bonds of different maturities are used. Thus rather than identifying the risk neutral measure, here we *assume* its existence and write down the dynamics of the instantaneous forwards under this measure. As in HJM (1990, 1992), the forwards are treated as fundamental.

Define the risk-neutral forward rate dynamics as

$$df^T(s) \equiv \mu_T^Q(s)\, ds + \sigma_T(s)\, dZ_Q^T(s)\,. \tag{21}$$

with correlation structure

$$dZ_Q^{T_1}(s)\,,\, dZ_Q^{T_2}(s) \equiv c\,(s, T_1, T_2)\ ds.$$

Generalizing the theorem of Kennedy (1994) to non-Gaussian fields, we claim:

Proposition 1 *The following statements are equivalent:*

(a) $P^T(s) = E_s^Q\left[e^{-\int_s^T du\, r(u)}\right]$.

(b) The discounted bond-price process

$$X^T(s) = e^{-\int_0^s du\, r(u)} P^T(s)$$

is a martingale under the risk neutral measure.

(c) $\mu_T^Q(s) = \sigma_T(s) \int_s^T du\, \sigma_u(s)\, c(s, T, u)$.

Proof: We show $a \to b \to c \to a$.

$(a \to b)$: Multiplying both sides of (a) by $e^{-\int_0^s du\, r(u)}$, we obtain

$$X^T(s) = \mathrm{E}_s^Q\left[e^{-\int_0^T du\, r(u)}\right] .$$

Thus for $\forall v \in [0, s]$,

$$\begin{aligned} \mathrm{E}_v^Q X^T(s) &= \mathrm{E}_v^Q\, \mathrm{E}_s^Q \left[e^{-\int_0^T du\, r(u)}\right] , \\ &= \mathrm{E}_v^Q \left[e^{-\int_0^T du\, r(u)}\right] , \\ &= X^T(v)\,, \end{aligned}$$

where the law of iterated expectations has been used. Thus under the risk neutral measure the discounted-bond price process is a martingale. □

$(b \to c)$: Differentiating the definition of the discounted price process, we find

$$\frac{dX^T(s)}{X^T(s)} = \frac{dP^T(s)}{P^T(s)} - r(s)\, ds\,.$$

Since X is a martingale under Q, and hence the drift of $\frac{dX}{X}$ is zero, the drift of $\frac{dP}{P}$ must equal $r\,ds$. Hence, under the risk neutral measure, the instantaneous expected rate of return for the bond is the risk free rate.

By definition,

$$P^T(s) = e^{-\int_s^T du\, f^u(s)} \, .$$

Differentiating, we obtain

$$\frac{dP^T(s)}{P^T(s)} = r(s)\,ds - \int_s^T du\, df^u(s) + \frac{1}{2}\left[\int_s^T du\, df^u(s)\right]^2 , \tag{22}$$

where $r(s) \equiv f^s(s)$. Since the instantaneous expected rate of return is the risk free rate, the drift of the instantaneous forward must satisfy

$$\int_s^T du\, \mu_u^Q(s) = \frac{1}{2}\left[\int_s^T du\, df^u(s)\right]^2 .$$

Taking a derivative with respect to T, we obtain

$$\begin{aligned} \mu_T^Q(s) &= \left[\int_s^T du\, df^u(s)\right] df^T(s) \\ &= \int_s^T du\, \sigma_u(s)\, dZ_Q^u(s)\, \sigma_T(s)\, dZ_Q^T(s) \\ &= \sigma_T(s) \int_s^T du\, \sigma_u(s)\, c(s,T,u) \qquad \square \end{aligned}$$

$(c \to a)$: Define the dynamics of the instantaneous forward rates as

$$df^u(v) = \mu_u^Q(v)\, dv + \sigma_u(v)\, dZ_Q^u(v) \, , \tag{23}$$

with

$$\mu_u^Q(v) = \sigma_u(v) \int_v^u dt\, \sigma_t(v)\, c(v,u,t) \ .$$

Integrating the forward rate dynamics $\int_{v=s}^u$, the spot rate satisfies

$$r(u) = f^u(s) + \int_s^u dv\, \mu_u^Q(v) + \int_s^u dZ_Q^u(v)\, \sigma_u(v) \, . \tag{24}$$

Hence,

$$\begin{aligned} \mathrm{E}_s^Q\left[e^{-\int_s^T du\, r(u)}\right] &= e^{-\int_s^T du\, f^u(s)}\, \mathrm{E}_s^Q\left[e^{-\int_s^T du \int_s^u dv\, \mu_u^Q(v) - \int_s^T du \int_s^u dZ_Q^u(v)\, \sigma_u(v)}\right] \\ &= P^T(s)\, \mathrm{E}_s^Q\left[e^{-\int_s^T dv \int_v^T du\, \mu_u^Q(v) - \int_{v=s}^T \int_{u=v}^T dZ_Q^u(v)\, du\, \sigma_u(v)}\right] , \end{aligned}$$

where the last step follows from the identity

$$\int_s^T du \int_s^u dv\, F(u,v) = \int_s^T dv \int_v^T du\, F(u,v) \qquad\qquad \forall F(\cdot\,,\,\cdot)\,.$$

Girsanov's theorem implies that the expectation on the right hand side is equal to unity.[11] We can demonstrate this by writing the right hand side as a path integral, where we consider finite Δ, and define N and n through $N\Delta = T$, $n\Delta = s$. At the end of the problem, we take the limit $\Delta \to 0$, $(N-n) \to \infty$, keeping $(N-n)\Delta = (T-s)$.[12]

Using the law of iterated expectations, one obtains

$$\mathrm{E}_s^Q\left[e^{-\int_s^T dv \int_v^T du\, \mu_u^Q(v) - \int_{v=s}^T \int_v^T dZ_Q^u(v)\, du\, \sigma_u(v)}\right] = \lim_{\Delta\to 0} \prod_{j=n}^{N-1} \mathrm{E}_{v=j\Delta}^Q\left[e^{-\Delta \int_v^T du\, \mu_u^Q(v) - \int_v^T du\, \sigma_u(v)\, dZ_Q^u(v)}\right].$$

Here, it is necessary to write out the (non-commuting) expectations increasing in time from left to right, and then perform the expectations from right to left. Noting

$$\begin{aligned}
\mathrm{E}_v^Q\left[\int_v^T du\, \sigma_u(v)\, dZ_Q^u(v)\right] &= 0 \\
\frac{1}{2}\,\mathrm{Var}_v^Q\left[\int_v^T ds\, \sigma_u(v)\, dZ_Q^u(v)\right] &= \Delta \int_v^T \sigma_u(v) \int_v^u dw\, \sigma_w(v) c(v,u,w) \\
&= \Delta \int_v^T du\, \mu_u^Q(v)\,,
\end{aligned}$$

we see that each term individually is unity. Continuing the operation from right to left, the entire expectation is thus found to be unity. Hence

$$\mathrm{E}_s^Q\left[e^{-\int_s^T du\, r(u)}\right] = P^T(s) \qquad\qquad \square$$

Note that if $dZ_Q^T(s) = dZ_Q(s)\ \forall T$, then $c\,(s, T_1, T_2) = 1$ and the one-factor HJM (1990, 1992) result obtains. Since the HJM approach incorporates all (continuous) traditional term structure models which specify the underlying spot rate dynamics, and since the HJM approach has been shown to be a special case of the random field model, it follows that the random field model incorporates all previous (continuous) term structure models.

3 Identifying the Forward Measure

The existence of a risk neutral measure implies that derivative securities can be priced by arbitrage. Under the risk neutral measure, the current price of a security $V(s)$ can be obtained

[11] See, for example, De Prato and Zabczyk (1992).

[12] Path integrals have a rigorous mathematical foundation. See, for example, Feynman and Hibbs (1965). For convergence theorems on path integrals, see Fujiwara (1982).

from its value at maturity date T via

$$V(s) = \mathrm{E}_s^Q \left[e^{-\int_s^T du\, r(u)}\, V(T) \right] .$$

Under this measure, the ratio $\frac{V(s)}{B(s)}$ is a martingale, where the rolling-over money market account

$$B(s) \equiv e^{\int_0^s du\, r(u)}$$

has been chosen as numeraire.

Often, it is more convenient to use as numeraire a discount bond with some maturity T. This defines the T-forward measure. For example, consider a portfolio comprised of α shares of the money market fund, and θ_u shares of discount bond maturing at date u:

$$V(s) = \alpha\, B(s) + \int_s^\infty du\, \theta_u\, P^u(s) .$$

The instantaneous change in portfolio value is

$$dV = \alpha\, dB(s) + \int_s^\infty du\, \theta_u\, dP^u(s) .$$

Using the bond $P^T(s)$ as numeraire, we define $V^*(s) \equiv \frac{V(s)}{P^T(s)}$. From Equation 22, the bond price process is

$$\frac{dP^T(s)}{P^T(s)} = r(s)\, ds - \int_s^T du\, \sigma_u(s)\, dZ_Q^u(s) . \tag{25}$$

Applying Ito's lemma to the definition of $V^*(s)$, we find

$$\begin{aligned} dV^*(s) &= \frac{\alpha\, B(s)}{P^T(s)} \int_s^T dv\, \sigma_v(s) \left[dZ_Q^v(s) + ds \int_s^T dw\, \sigma_w(s)\, c(s,v,w) \right] \\ &\quad + \int_s^\infty du\, \frac{\theta_u\, P^u(s)}{P^T(s)} \int_u^T dv\, \sigma_v(s) \left[dZ_Q^v(s) + ds \int_s^T dw\, \sigma_w(s)\, c(s,v,w) \right] . \end{aligned}$$

Hence, the T-forward measure has been identified:

$$dZ_{R(T)}^v(s) = dZ_Q^v(s) + \, ds \int_s^T dw\, \sigma_w(s)\, c(s,v,w) .$$

Under this measure, V^* is a martingale, and therefore

$$V(v) = P^T(v)\, \mathrm{E}_v^{R(T)} \left[\frac{V(s)}{P^T(s)} \right] . \tag{26}$$

For the special case that $T = s$, Equation 26 reduces to

$$V(v) = P^s(v)\, \mathrm{E}_v^{R(s)} \left[V(s) \right] . \tag{27}$$

Also note that the T-forward rate dynamics is a martingale under the T-forward measure:

$$\begin{aligned} df^T(s) &= \sigma_T(s) \left(dZ_Q^T(s) + ds \int_s^T dw\, \sigma_w(s)\, c(s,T,w) \right) \\ &\equiv \sigma_T(s)\, dZ_{R(T)}^T(s) . \end{aligned}$$

3.1 Pricing Bond Options

The payoff of a European option with maturity s, whose underlying asset is a discount bond with maturity T, is

$$C(s,s,T) = \left(P^T(s) - K\right)\, \mathbf{1}_{(P^T(s)>K)}\,. \qquad (28)$$

This can be thought of as a portfolio of two securities:

$$V_1(s,s,T) = P^T(s)\, \mathbf{1}_{(P^T(s)>K)}\,, \qquad V_2(s,s,T) = -K\, \mathbf{1}_{(P^T(s)>K)}\,.$$

Applying Equation 26 to V_1 and Equation 27 to V_2, the price of the option at time v can be written in the form

$$\begin{aligned} C(v,s,T) &= P^T(v)\, \mathrm{E}_v^{R(T)}\left[1_{(\log P^T(s)>\log K)}\right] - K\, P^s(v)\, \mathrm{E}_v^{R(s)}\left[1_{(\log P^T(s)>\log K)}\right] \\ &= P^T(v)\, \mathrm{E}_v^{R(T)}\left[1_{(-\log K > \int_s^T du\, f^u(s))}\right] - K\, P^s(v)\, \mathrm{E}_v^{R(s)}\left[1_{(-\log K > \int_s^T du\, f^u(s))}\right]. \end{aligned} \qquad (29)$$

Kennedy (1994) models the volatility and correlation structure as deterministic. In this case, $\left(\int_s^T du\, f^u(s)\right)$ is distributed normally under both forward measures, and the bond-option price takes a form similar to Black and Scholes (1972). But for any model where forward rates are not normally distributed, solving the expectations is a much more formidable task. However, progress can be made by following the approach of Heston (1993). Writing the indicator function $\mathbf{1}_{(\cdot)}$ in integral form, the bond option price can be written as

$$\begin{aligned} C(v,s,T) &= P^T(v)\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty d\lambda\, \mathrm{Re}\left[\frac{e^{-i\lambda \log K}\, G(\lambda,T)}{i\,\lambda}\right]\right\} \\ &\quad - K\, P^s(v)\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty d\lambda\, \mathrm{Re}\left[\frac{e^{-i\lambda \log K}\, G(\lambda,s)}{i\,\lambda}\right]\right\}, \end{aligned} \qquad (30)$$

where the characteristic function of $\log P^T(s)$ under different forward measures is defined as

$$\begin{aligned} G(\lambda,\cdot) &\equiv \mathrm{E}_v^{R(\cdot)}\left[e^{i\lambda \log P^T(s)}\right] \\ &= \mathrm{E}_v^{R(\cdot)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right]. \end{aligned} \qquad (31)$$

The implication of Equation 30 is that, if the characteristic function can be written in closed-form, then so can the bond-option price. The proposed method of solution is to use a path integral approach together with iterated expectations.[13] First, the characteristic function is written as

$$\mathrm{E}_v^{R(\cdot)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right] = \mathrm{E}_v^{R(\cdot)}\, \mathrm{E}_{v+\Delta}^{R(\cdot)} \cdots \mathrm{E}_{s-\Delta}^{R(\cdot)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right].$$

[13]This method is very useful for obtaining bond prices and characteristic functions for all of the well known models, including Vasicek (1977), Cox, Ingersoll, and Ross (1985B), and Beaglehole/Tenney (1991) and Constantinides (1992). See Goldstein (1997).

Note that for sufficiently small Δs,

$$[f^u(s)|\mathcal{F}_{s-\Delta}] \sim \Phi\left[f^u(s-\Delta) + \Delta\,\mu_u^{R(\cdot)}(s-\Delta)\,,\Delta\,\sigma_u^2(s-\Delta)\right]$$

is normally distributed. Thus,

$$\begin{aligned} \mathrm{E}_{(s-\Delta)}^{R(\cdot)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(s-\Delta)}\, e^{-i\lambda\,\Delta\int_s^T du\,\mu_u^{R(\cdot)}(s-\Delta)} \\ &\quad \times e^{-\Delta\,\frac{\lambda^2}{2}\int_s^T du\,\sigma_u(s-\Delta)\int_s^T dw\,\sigma_w(s-\Delta)\,c(s-\Delta,u,w)} \end{aligned}$$

If this operation maintains a form that permits each successive expectation to be obtained, then the characteristic function can be obtained. We use this approach to demonstrate

Proposition 2 *Given the forward rate volatility structure*

$$\sigma_u(s) = \sigma\,B'(u-s)\,\sqrt{r(s)}\,,$$

and a deterministic correlation structure, then the characteristic function,

$$G(\lambda,x) \equiv E_v^{R(x)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] = e^{-i\lambda\int_s^T du\, f^u(v)+\int_v^s dw\, h_x(s-w)\, f^w(v)}\,,$$

is linear in the current forward rates $f^{(\cdot)}(\nu)$. Defining $h_x(t) = -i\lambda$ for $t < 0$, the function $h_x(t)$, for $t > 0$, is determined implicitly via

$$\begin{aligned} h_x(s-v) &= \sigma^2\int_v^T du\, h_x(s-u)\,B'(u-v)\int_x^u dw\,B'(w-v)\,c(v,u,w) \\ &+ \frac{\sigma^2}{2}\int_v^T du\, h_x(s-u)\,B'(u-v)\int_v^T dw\, h_x(s-w)\,B'(w-v)\,c(v,u,w)\,. \quad (32) \end{aligned}$$

The price of a European bond option with strike price K is then obtained from Equation 30.

Proof: See Appendix A. □

Note that, even for the one-factor case ($c(\cdot,\cdot,\cdot) = 1$), this model permits consistency with both the current term structure of interest rates, and the current term structure of volatility through calibration of $B'(\cdot)$. In particular, $B'(\cdot)$ can be calibrated to match the 'volatility hump' that has been observed empirically.[14] However, even in the infinite-factor case, since $B'(\cdot)$ is a deterministic function, it would have to be continuously recalibrated to maintain consistency with the term structure of volatility.[15] Hence, although this random field model presented permits consistency with all forward rate innovation, it is not general enough to

[14]See, for example, Amin and Morton (1994), Amin and Ng (1997).

[15]I thank an anonymous referee for pointing out this fact.

permit consistency with volatility innovation. However, this can also be rectified. For example, one model where both the forward rates and their volatilities are inputs, and innovations of both are specified by a random field is:

$$\begin{aligned} df^T(s) &= \mu_T^Q(s)\, ds + \sqrt{V^T(s)}\, dZ_Q^T(s)\,, \\ dV^T(s) &= \kappa_{(T-s)} \left(\theta_{(T-s)} - V^T(s)\right)\, ds + \sigma_{(T-s)} \sqrt{V^T(s)}\, dW_Q^T(s)\,. \end{aligned} \tag{33}$$

where

$$dZ_Q^{T_1}(s)\, dW_Q^{T_2}(s) = c^*\left(s, T_1, T_2\right) \qquad \forall \{T_1\,, T_2\}\,. \tag{34}$$

Unfortunately, such a model does not seem to possess a closed-form solution.

4 Conclusion

Random field models of the term structure are generalizations of the finite-factor models which dominate the financial economics literature. Yet in contrast with finite-factor models, random field models are consistent with both the current yield curve and term structure innovation. In addition, a random field framework offers models that are much more parsimonious than their multi-factor counterparts. Finally, in contrast to finite-factor models, random field models predict that the best hedging instrument for a bond is another bond of similar maturity.

As the last example in the previous section implies, random field models can be made sufficiently flexible to provide consistency with all marketable prices and their innovations. In such a framework, no securities would be left to price by arbitrage. Whether or not this is a 'good' thing is an empirical issue. But note that, if all security prices are inputs of a model, then term structure dynamics under the original measure is what becomes important. That is, rather than looking for arbitrage opportunities, an agent will search for a portfolio which maximizes some objective function. As stated previously, Heaney and Cheng (1984) provide a general framework for determining mean-variance efficient portfolios. One model of potential interest might include a 'curvature reducing' term in the drift of forward rates. Note that a discretized version of a curvature reducing term

$$\frac{\partial^2}{\partial T^2} f^T(s) \approx \left(\frac{1}{\Delta}\right)^2 \left[f^{T+dT}(s) - 2\, f^T(s) + f^{T-dT}(s)\right]$$

is linear in the forward rates. This implies that, if the volatility structure is deterministic, such a model is a generalized Ornstein-Uhlenbeck process, and thus has a closed-form solution.

Whereas N-factor models predict that long term bonds can be perfectly hedged with an appropriate position of N- short term bonds, a random field model predicts that a better hedging instrument for a long term bond is another of similar maturity. Interestingly, a finite factor model which accounts for parameter uncertainty also comes to this same conclusion. Whether or not a random field model and a finite factor model which accounts for parameter uncertainty select similar optimal portfolios to hold might be an interesting question for future research. Certainly, the random field model would be considerably more parsimonious.

5 Appendix A

Here we demonstrate that if the forward rate volatility structure is of the form

$$\sigma_u(w) = \sigma\, B'(u-w)\, \sqrt{r(w)}\,,$$

and the correlation structure is deterministic, then the characteristic function of $\log P^T(s)$ under the x-forward measure,

$$G(\lambda, x) = \mathrm{E}_v^{R(x)}\left[e^{i\lambda \log P^T(s)}\right] = \mathrm{E}_v^{R(x)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right]\,,$$

takes a form linear in the current forward rates,

$$\mathrm{E}_v^{R(s)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right] = e^{-i\lambda \int_s^T du\, f^u(v) + \int_v^s dw\, h_x(s-w)\, f^w(v)}\,,$$

where the function $h_x(\cdot)$ is specified below.

Note that the stochastic forward rate dynamics are

$$\begin{aligned} df^u(w) &= dw\, \sigma^2\, r(w)\, B'(u-w) \int_w^u dt\, B'(t-w)\, c(w,u,t) + \sigma\, B'(u-w)\, \sqrt{r(w)}\, dZ_Q^u(w) \\ &= dw\, \sigma^2\, r(w)\, B'(u-w) \int_x^u dt\, B'(t-w)\, c(w,u,t) + \sigma\, B'(u-w)\, \sqrt{r(w)}\, dZ_{R(s)}^u(w) \end{aligned}$$

under the x-forward measure. The proposed method of solution is to write the expectation as a path integral and use iterated expectations:

$$\mathrm{E}_v^{R(s)}\left[e^{i\lambda \log P^T(s)}\right] = \mathrm{E}_v^{R(s)}\, \mathrm{E}_{v+\Delta}^{R(\cdot)} \cdots\, \mathrm{E}_{s-\Delta}^{R(\cdot)}\left[e^{-i\lambda \int_s^T du\, f^u(s)}\right]\,.$$

Note that for sufficiently small Δ, $[f^u(s)|\mathcal{F}_{s-\Delta}]$ is normally distributed, with

$$\begin{aligned} \mathrm{E}_{(s-\Delta)}^{R(s)}\left[-i\lambda \int_s^T du\, f^u(s)\right] &= -i\lambda \int_s^T du\, \Big[f^u(s-\Delta) + \Delta\, \sigma^2\, r(s-\Delta) \times \\ &\qquad B'(u-s+\Delta) \int_x^u dt\, B'(t-s+\Delta)\, c(s-\Delta, u, t)\Big]\,, \\ \frac{1}{2}\,\mathrm{Var}_{(s-\Delta)}^{R(s)}\left[-i\lambda \int_s^T du\, f^u(s)\right] &= -\Delta \frac{(\lambda\,\sigma)^2}{2}\, r(s-\Delta) \int_s^T du\, B'(u-s+\Delta) \times \\ &\qquad \int_s^T dw\, B'(w-s+\Delta)\, c(s-\Delta, u, w)\,. \end{aligned}$$

It is well known that, for any normally distributed random variable $\tilde{Y} \sim \Phi\left[\mu_Y\,,\, \sigma_Y^2\right]$, we have

$$\mathrm{E}\left[e^{\tilde{Y}}\right] = e^{\mu_Y + \frac{1}{2}\sigma_Y^2}\,.$$

Thus, we can write

$$
\begin{aligned}
\mathrm{E}_{s-\Delta}^{R(s)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(s-\Delta)+\Delta\, h_x(\Delta)\, r(s-\Delta)} \\
&= e^{-i\lambda\int_s^T du\, f^u(s-\Delta)+\Delta\, h_x(\Delta)\, f^{s-\Delta}(s-\Delta)}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
h_x(\Delta) &= -i\lambda\sigma^2\int_s^T du\, B'(u-s+\Delta)\int_x^u dw\, B'(w-s+\Delta)\, c(s-\Delta,u,w) \\
&\quad -\lambda^2\sigma^2\int_s^T du\, B'(u-s+\Delta)\int_s^u dw\, B'(w-s+\Delta)\, c(s-\Delta,u,w)\,.
\end{aligned}
$$

Indeed, each successive iteration produces a function that is linear in the forward rates. To specify the functional form of $h_x(\cdot)$, we write

$$
\begin{aligned}
\mathrm{E}_{v+\Delta}^{R(x)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(v+\Delta)+\int_{v+\Delta}^s du\, h_x(s-u)\, f^u(v+\Delta)} \\
&= e^{\int_{v+\Delta}^T du\, h(s-u)\, f^u(v+\Delta)}
\end{aligned}
$$

where for compactness we have defined $h_x(s-u) = -i\lambda$ for values of u such that $s < u < T$. By iterating back one step, we find an implicit function for $h_x(\cdot)$:

$$
\begin{aligned}
\mathrm{E}_{v}^{R(s)}\left[e^{-i\lambda\int_s^T du\, f^u(s)}\right] &= e^{-i\lambda\int_s^T du\, f^u(v)+\int_{v+\Delta}^s du\, h(s-u)\, f^u(v)+\Delta\, r(v)\, h_x(s-v)} \\
&= e^{-i\lambda\int_s^T du\, f^u(v)+\int_v^s du\, h_x(s-u)\, f^u(v)}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
h_x(s-v) &= \sigma^2\int_v^T du\, h_x(s-u)\, B'(u-v)\int_s^u dw\, B'(w-v)\, c(v,u,w) \\
&+ \frac{\sigma^2}{2}\int_v^T du\, h_x(s-u)\, B'(u-v)\int_v^T dw\, h_x(s-w)\, B'(w-v)\, c(v,u,w)\,.
\end{aligned}
$$

References

[1] Adams, K.J., and D.R. Van Deventer, 1994 "Fitting Yield Curves and Forward Rate Curves with Maximum Smoothness," *Journal of Fixed Income,* 4, 52-62.

[2] Adler, R.J., 1981, *The Geometry of Random Fields*, Wiley, NY.

[3] Amin, K., and A. Morton, 1994, "Implied Volatility Functions in Arbitrage-Free Term Structure Models, *Journal of Financial Economics*, 35, 141-180.

[4] Amin, K., and V. Ng, 1997, "Inferring Future Volatility from the Information in Implied Volatility in Eurodollar Options: A New Approach", *Review of Financial Studies*, 10, 333-367.

[5] Beaglehole, D. and M. Tenney, 1991,"Corrections and Additions to 'A Nonlinear Equilibrium Model of the Term Structure of Interest Rates," *Journal of Financial Economics,* 32, 345-353.

[6] Björk, T., G. Di Masi, Y. Kabanov, and W. Runggaldier, 1996, "Towards a General Theory of Bond Markets", Forthcoming in *Finance and Stochastics.*

[7] Bouchaud, J.P., R. Cont, N. El-Karoui, M. Potters, and N. Sagna, 1998, "Phenomenology of the Interest Rate Curve", working paper.

[8] Black, F., E. Derman, and W Toy, 1990, "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options", *Financial Analysts Journal*, 46, 33-39.

[9] Black, F., and P. Karasinski, 1991, "Bond and Option Pricing when Short Rates are Lognormal", *Financial Analysts Journal* 47, 52-59.

[10] Black, F. and M. Scholes, 1972, "The Valuation of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-654.

[11] Brennan, M., and E. Schwartz, 1979, "A Continuous Time Approach to the Pricing of Bonds," *Journal of Banking and Finance,* 3, 133-155.

[12] Canabarro, E., 1995, "Where do One-Factor Interest Rate Models Fail?", *Journal of Fixed Income,* 5, 31-52.

[13] Carverhill, A., 1994, "When is the Short Rate Markovian?", *Mathematical Finance,* 4, 305-312.

[14] Constantinides, G. M., 1992, "A theory of the Nominal Term Structure of Interest Rates," Review of Financial Studies, 5, 531-552.

[15] Cox, J., J. Ingersoll, and S. Ross, 1981, "The Relation between Forward Prices and Futures Prices," *Journal of Financial Economics,* 9, 321-346.

[16] Cox, J., J. Ingersoll, and S. Ross, 1985b, "A Theory of the Term Structure of Interest Rates," *Econometrica,* 53, 385-408.

[17] Da Prato, G., and J. Zabczyk, 1992, *Stochastic Equations in Infinite Dimensions*, University Press, Cambridge.

[18] Feynman, R.P., and A.R. Hibbs, 1965, *Quantum Mechanics and Path Integrals,* McGraw-Hill, New York.

[19] Fujiwara, D., 1982, "Remarks on the convergence of Feynman path integrals," in A.V. Balakrishnan and M. Thoma (eds.) *Theory and Application of Random Fields: Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin, Germany.

[20] Goldstein, R.S., 1997, "Identifying the class of term structure models possessing closed-form solutions for bond and bond-option prices: an expectation approach," working paper, Ohio State University.

[21] Heaney, W.J., and P.L. Cheng, 1984, "Continuous Maturity Diversification of Default-Free Bond Portfolios and a Generalization of Efficient Diversification," *Journal of Finance,* 4, 1101-1117.

[22] Heath, D.C., R.A. Jarrow, and A. Morton, 1990 "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation," Journal of Financial and Quantitative Analysis, 25, 419-440.

[23] Heath, D.C., R.A. Jarrow, and A. Morton, 1992 "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," Econometrica, 60, 77-105.

[24] Heston, S.I., 1993, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies,* 6, 327-342.

[25] Ho, T.S.Y., 1992, Key Rate Durations: Measures of Interest Rate Risks, *J. of Fixed Income,* 2, 29-44.

[26] Ho, T.S.Y., and S.B. Lee, 1986 "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance,* 41, 1011-1029.

[27] Inui, K., and K Masaaki, (1998), "A Markovian Framework in Multi-Factor Heath-Jarrow-Morton models", *Journal of Financial and Quantitative Analysis*, 33, 423-440.

[28] Kennedy, D.P., 1994, "The Term Structure of Interest Rates as a Gaussian Random Field," *Mathematical Finance,* 4, 247-258.

[29] Kennedy, D.P., 1997, "Characterizing Gaussian Models of the Term Structure of Interest Rates," forthcoming in *Mathematical Finance.*

[30] Longstaff, F., and E. Schwartz, 1992, "Interest Rate Volatility and the Term Structure of Interest Rates: A Two-Factor General Equilibrium Model," *Journal of Finance,* 47, 1259-1282.

[31] Musiela, M., 1993, "Stochastic PDE's and Term Structure Models," Working Paper.

[32] Pearson, N.D., and T.S. Sun, (1994) "Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model," *Journal of Finance* 49, 1279-1304.

[33] Priestley, M.B., 1981, *Spectral Analysis and Time Series*, Academic Press, London.

[34] Ritchken, P., and L. Sankarsubramanian, 1995, "Volatility Structures of Forward Rates and the Dynamics of the Term Structure", *Mathematical Finance*, 5, 55-72.

[35] Ross, S.A., 1976a "Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory,* 13, 341-360.

[36] Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics,* 5, 177-188.