Aggregation of Preferences for Skewed Asset Returns*

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Abstract

This paper characterizes individual demand functions as well as equilibrium demand and risk premiums in the presence of skewness risk. We show a three-fund separation theorem where agents hold the risk-free asset, the market portfolio and a so-called skewness portfolio. The skewness portfolio is the portfolio that gives the optimal hedge of the squared market portfolio; it contributes to the skewness risk premium through co-variation with the squared market returns and supports a stochastic discount factor that is quadratic in the market return. However, a tracking error portfolio of the squared market return also contributes to pricing and enters quadratically into the stochastic discount factor together with the cross-sectional variance of skew-preferences.

Keywords

skewness, portfolio, stochastic discount factor, polynomials, market return

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1 Introduction

Empirical studies documented that securities’ returns are not normally distributed; the seminal work of Harvey and Siddique (2000) showed that skewness risk is an important component in the risk-premium. This renewed interest in the compensation of skewness risks and led to an active literature stream\(^1\). This stream typically assumes that the aggregation of preferences leads to stochastic discount factors that are polynomials in the market return. However, it is well known that the aggregation of preferences in incomplete markets leads to a representative agent where the weight of each individual is stochastic; thus the stochastic discount factor may actually depend on individual securities via unspanned powers of the market return.

Our paper studies in detail the aggregation of preferences with skewed returns. We derive individual demand and prove a three-fund separation theorem: agents hold the risk-free asset, the market portfolio and a new so-called skewness portfolio, in proportions that reflect their preferences for variance and skewness risk. The skewness portfolio is the portfolio that provides the optimal hedge to the squared market return. We show that an asset’s skewness risk is priced as long as the co-skewness of this asset with the market portfolio as well as the aggregate skew-tolerance do not vanish. The equilibrium contribution to the risk-premium of individual stocks is driven by their covariance with the squared market return; this confirms the common practice of using an SDF that is quadratic in the market return. In line with this, we link our results to the beta pricing relationships proposed by Harvey and Siddique (2000).

A tracking error shows up in a hedging the squared market return with the available securities; we show that, beyond average skew-tolerance, this leads to an additional lower order contribution to the risk premium on pentosis (fifth moment) due to the cross-sectional variance and skewness of investor’s skew-tolerances. This matches a well-known result about incomplete markets that investors’ heterogeneity and in particular the cross-sectional variance of investors characteristics may matter in equilibrium pricing, see e.g. Constantinides and Duffie (1996). This also shows that aggregation of skewness risk for pricing is far more complicated than thought: the standard

\(^1\)For example, Dittmar (2002) and Barone-Adesi et al. (2004) study how skewness risk is priced; Chang et al. (2006) test whether Fama-French factors proxy for skewness and higher moments.
approach of studying polynomials of market returns (based on a postulated representative agent) misses this additional, priced tracking portfolio.

Our paper makes several contributions. First, we use the spirit of Samuelson (1970), but depart from it in two directions: first, we allow for a richer pattern of risk premiums in order to analyze the pricing of skewness risk; second we do not intend to interpret his small noise expansion parameter as we only are interested in structural properties of risk-premiums and portfolio holdings.

Second, we confirm Samuelson (1970) that the classical CAPM two fund separation theorem continues to hold as a first approximation; in addition we show that agents add the skewness portfolio in a second order approximation.

Finally, we clarify the common practice of studying SDF that are polynomials in the market return. Most empirical studies looked at skewness extensions of the CAPM which add the squared market return as a factor; they justify this extension on ad-hoc Taylor-series expansion for agents’ utility function that is truncated at the third-order term, see, e.g., Kraus and Litzenberger (1976), Barone-Adesi (1985), Dittmar (2002). We provide a rigorous foundation for this common practice and show that it is warranted in complete markets, that is if the squared market return can be hedged perfectly. In incomplete markets however, we show that an additional priced factor comes up that depends on the cross-sectional variance of investors’ skew-tolerances. It is well known about incomplete markets that the utility functions aggregate into a single so-called representative agent through stochastic weights, see, e.g. Magill and Quinzii (1996); our structural result therefore has the interpretation that these stochastic weights matter for pricing.

The outline of our paper is as follows. The next section discusses why and how asymmetry risk should be priced in equilibrium. In the following section we discuss in greater detail skewness demand in equilibrium, present fund separation theorem in the presence of skewness risk and study linear pricing relationships based on quadratic pricing factors. Finally, in section 4 we show that the aggregation in incomplete markets leads a risk factor that is unspanned by polynomial SDFs in the market return. The paper concludes with section 5. Throughout, we only present the ideas in the main text and refer all underlying calculations to the appendix.
2 Why should skewness risk be priced in equilibrium?

Our starting point is a given cross section of asset returns $R_i, i = 1, ..., n$. Here, $R_i$ stands for the (gross) return from investing $1$ in risky security $i = 1, ..., n$, while $R_f$ will denote the gross return on the riskless (safe) security. Our analysis is in the spirit of Samuelson (1970), i.e. to understand the price of higher order moments in equilibrium we study returns and corresponding moments through a family of economies indexed by a single scaling parameter $\sigma$. More precisely, we factorize the joint probability distribution of the vector $R = (R_i)_{1 \leq i \leq n}$. We assume the vector $[R - E(R)]/\sigma$ has a given probability distribution for all $\sigma > 0$, i.e. the distribution of asset returns is the same in all economies of interest, up to scale-location. Since we focus on second and third moments, we will define the squared matrices $\Sigma = (\Sigma_{ij})_{1 \leq i,j \leq n}$, and $\Gamma_i$ for $1 \leq i \leq n$ with the properties

\[
\begin{align*}
\text{Var}(R) &= E \left[ \left( (R - E(R))(R - E(R))^T \right) \right] = \sigma^2 \Sigma, \\
\text{Skew}_i(R) &= E \left[ \left( (R - E(R))(R - E(R))^T (R_i - E(R_i)) \right) \right] = \sigma^3 \Gamma_i.
\end{align*}
\]

The matrix $\Sigma$ is assumed to be positive and symmetric. In other words, we consider a family of financial markets indexed by $\sigma > 0$ where the global variance and skewness structures $\Sigma$ and $\Gamma = (\Gamma_i)_{1 \leq i \leq n}$ are invariant to changes in the parameter $\sigma$. The scale parameter $\sigma$ allows us to describe the pattern of risk premiums from their series expansions:

\[
E(R_i) - R_f = \sigma^2 \sum_{j=0}^{\infty} a_{ij} \sigma^j
\]

Samuelson (1970) assumes $a_{ij} = 0$ for all $j > 0$; this, however, appears overly restrictive for a thorough analysis of the price of risk coming from higher order moments beyond variance. In addition, Samuelson (1970) is interested in small scale parameters $\sigma$; as he intends to motivate the use of mean-variance analysis, he also puts forward an interpretation in continuous time, where the scale parameter $\sigma$ corresponds to a small interval of time $\Delta t$. No such restrictions are involved in our setting defined by (1) and (2). This setting is actually general, as it only maintains smoothness assumptions about investors’ preferences that are necessary to ensure risk premiums are analytical functions of the scale parameter $\sigma$, when standardized by $\sigma^2$. Standardization by
means that risk aversion has a second-order nature, as defined by Segal and Spivak (1990). First order risk aversion is beyond the scope of this paper.

We will actually set the focus in this paper on the first few coefficients $a_{i0}$, $a_{i1}$ and part of $a_{i2}$ and $a_{i3}$, so that we even do not really assume that risk premiums are analytical functions. There are then two interpretations of our results: either we adopt the assumption of analytical functions and describe general properties of the market; or we may consider, under less restrictive assumptions, that our results are accurate descriptions only for small $\sigma$, see Samuelson (1970) and more recently Judd and Guu (2001).

The risk premium coefficients will be determined by the equilibrium of financial markets, populated by $S$ (heterogeneous) investors $s = 1, ..., S$; each investor $s$ maximizes expected utility $E[u_s(W_s)]$ of her terminal wealth $W_s$ obtained by investing some initial wealth $q_s$ in a portfolio. Throughout, their portfolio is defined in terms of shares of wealth invested $\theta_s = (\theta_{is})_{1 \leq i \leq n}$; this gives terminal wealth

$$W_s = q_s \left[ R_f + \sum_{i=1}^{n} \theta_{is} (R_i - R_f) \right].$$

The demand $\theta_s$ of investor $s$ for risky assets solves the first order conditions of her expected utility maximization:

$$E[u'_s(W_s)(R_i - R_f)] = 0, i = 1, ..., n. \tag{4}$$

It is well known that risk premiums implied by (4) are non-zero because the marginal utility $u'_s(W_s)$ is not constant. It is common to set the focus on its first derivative $u''_s(W_s)$ as this leads to a mean-variance analysis. However, there is no reason not to be interested in higher order derivatives, and as such, in higher order moments of returns. Samuelson (1970) characterizes the impact of these higher order moments on the demand for risky assets through a series expansion of the optimal portfolio $\theta_s = (\theta_{is})_{1 \leq i \leq n}$, i.e. of the solution of (4) associated to the expansion (2):

$$\theta_{is} = \sum_{j=0}^{\infty} \theta_{isj} \sigma^j.$$

Our focus is on the demand side of the financial markets and should be completed when the offer of financial assets is also be seen as a function of the scale parameter $\sigma$. However, such
an extension would simply complicate notation without modifying the economic interpretation of the result. It is therefore convenient for us to assume that the total market capitalization \( \sum_{s=1}^{S} \theta_{is} q_s \) of any asset \( i = 1, \ldots, n \) is a given number, i.e. it is the same for all economies indexed by \( \sigma > 0 \) we consider throughout.

We assume that the total offer of risky assets is a fixed vector \( \xi = (\xi_i)_{1 \leq i \leq n} \) of shares of the total amount \( S\bar{q} = \sum_{s=1}^{S} q_s \) of invested wealth and we have the following market clearing conditions for each risky asset \( i = 1, \ldots, n \) and all \( j > 0 \):

\[
\sum_{s=1}^{S} \theta_{is0} q_s = \xi_i S\bar{q}, \quad \text{and}
\]

\[
\sum_{s=1}^{S} \theta_{isj} q_s = 0.
\]  

Samuelson’s key insight that we extend in this paper is the following: for any investor \( s \), the first \( K \) coefficients \( \theta_{isj}, j = 0, \ldots, K - 1 \), only depend on the first \((K + 1)\) derivatives of the utility functions \( u_s, s = 1, \ldots, S \), computed at the level of wealth \( q_s R_f \). (This level can be seen as the terminal wealth resulting from investing the initial wealth \( q_s \) only in the safe asset). This permits successive determination of demand and market clearing starting from terms of order zero and going up one order at a time.

Equations (5) and (6) characterize the equilibrium between supply and demand of risky assets. Since the supply does not depend on \( \sigma \), all the higher terms in the series expansion must be zero, hence (6). The market return is then, with obvious notations:

\[
R_M = R_f + \sum_{i=1}^{n} \xi_i (R_i - R_f) = R_f + \xi^\top (R - R_f \epsilon_n). \tag{7}
\]

It is then easy to check that the traditional mean-variance pricing in the sense of the Sharpe-Lintner CAPM is encapsulated in the first coefficient \( a_{i0} \) of the series expansion (2):

\[
\sigma^2 a_{i0} = \frac{1}{\bar{\tau}} \text{Cov}(R_i, R_M), \tag{8}
\]

where \( \bar{\tau} \) stands for the average risk tolerance. Here, individual and average risk tolerance coefficients are defined by

\[
\tau_s = -\frac{u'_s(q_s R_f)}{q_s u''_s(q_s R_f)}, \quad \bar{\tau} = \frac{1}{S\bar{q}} \sum_{s=1}^{S} \tau_s q_s. \tag{9}
\]
It is worth noting that heterogeneity across investors’ risk tolerances \( \tau_s \) \((s = 1, \ldots, S)\) may come either from heterogeneity of their utility functions \( u_s \) or from heterogeneity of their invested wealth \( q_s \). In both cases we find here the well-known result that heterogeneity does not really matter for mean-variance pricing; only the average risk tolerance coefficient \( \bar{\tau} \) matters.

It was the key insight of Samuelson (1970), that an approximation of the risk premium \( (E(R_i) - R_f) \) by the first term \( \sigma^2 a_{i0} \) would actually provide an asset pricing model that matches the traditional Sharpe-Lintner CAPM. Thus, it is necessary to study at least an approximation with an expansion order higher by one to understand the price of skewness risk. It has been well known since Kraus and Litzenberger (1976) that it is co-skewness of an asset and not skewness that enters in its risk premium. Here, co-skewness is defined by

\[
c_i = \xi^\top \Gamma \xi = \frac{1}{\sigma^3} \text{Cov} \left[ (R_M - E(R_M))^2, R_i \right].
\]

(10)

Note that – by comparison with standard notations – we have rescaled the co-skewness coefficient in order to get a quantity independent of the scale parameter \( \sigma \). As usual, co-skewness represents the contribution of asset \( i \) to the skewness of the market return:

\[
\sigma^3 \sum_{i=1}^{n} \xi_i c_i = E \left[ (R_M - E(R_M))^3 \right].
\]

We show that in equilibrium the market price of skewness risk is

\[
\sigma^3 a_1 = -\bar{\rho}^2 \sigma^3 = -\bar{\rho}^2 \text{Cov} \left[ R, (R_M - E(R_M))^2 \right],
\]

(11)

where individual \( (\rho_s) \) and average \( (\bar{\rho}) \) skew tolerance coefficients are defined by

\[
\rho_s = \frac{u'_s(q_s R_f) u'''_s(q_s R_f)}{2 (u''_s(q_s R_f))^2}, \bar{\rho} = \frac{\sum_{s=1}^{S} q_s \tau_s \rho_s}{\sum_{s=1}^{S} q_s \tau_s}.
\]

Note that for homogeneous investors (same utility function \( u = u_s \), same wealth \( q = \bar{q} \) invested), the risk premium on skewness risk would simply be \( a_{i1} = -u'''(qR_f)q^2/(2u'(qR_f))c_i \). However, our result displays the aggregate impact on the skewness premium in case of investors heterogeneity. Approximating the risk premium \( (E(R_i) - R_f) \) by its first two terms \( \sigma^2 [a_{i0} + \sigma a_{i1}] \) is similar to extending the CAPM to an asset pricing model that takes into account skewness.
risk. We expect the third derivative of utility functions to be non-negative (see Kimball’s concept of prudence) and thus, an asset with positive co-skewness may display a risk premium smaller than the one predicted by standard CAPM. Skewness risk is priced in equilibrium as soon as the properly defined average skew tolerance is non-zero and some assets display a non-zero co-skewness (while the market return may well be symmetric).

3 Portfolio separation and pricing factors with skewness risk

3.1 Demand for skewness risk

We have seen that the Sharpe-Lintner CAPM gives a correct picture of actual risk premiums insofar as we focus on the first term $\sigma^2 a_{i0}$ of their series expansion. The same applies to the equilibrium demand for risky assets, i.e. a standard mean-variance approach is sufficient insofar as we focus on the first term $\theta_{s0}$ of the series expansion for the demand of investor $s$:

$$\theta_{s0} = \frac{\tau_s}{\sigma^2} \xi.$$  

(12)

This is the celebrated two-fund separation theorem, where all investors hold the same portfolio of risky assets defined by portfolio weights $\xi$. By analogy with the CAPM relationship (8), let us study an additional fund by portfolio weights $\xi^{sk}$ leading to a return $R^{sk} = R_f + \sum_{i=1}^{n} \xi^{sk}_i (R_i - R_f)$ with the property that:

$$\sigma^3 a_{i1} = -\frac{\rho}{\tau^2} Cov(R_i, R^{sk}).$$  

(13)

Note that this equation defines $\xi^{sk}$ as a unique solution of a linear system of $n$ equations with non-singular matrix $Var(R)$. We actually show that if $c$ stands for the co-skewness vector, then $a_1$ is given by (11) and then

$$\xi^{sk} = \sigma \Sigma^{-1} c.$$

This new portfolio plays the same role for pricing skewness risk that the market portfolio played to price (co-)variance risk. Therefore, throughout this paper we refer to it as the skewness portfolio. Not surprisingly, it also appears in investors demand for risky assets, i.e. we find the
following analogue of formula (12):

$$\sigma \theta_{s1} = \frac{\tau_s (\rho_s - \bar{\rho})}{\bar{\tau}^2} \xi^{sk}. \quad (14)$$

Equations (12, 14) establish a three-fund theorem: All investors hold the same two portfolios of risky assets (so-called mutual funds) defined by portfolio weights $\xi$ and $\xi^{sk}$, respectively. The only difference in actual holdings comes from wealth heterogeneity (ceteris paribus, the amount invested is proportional to wealth), heterogeneity in risk tolerances (the amount invested is proportional to risk tolerance at this wealth level) and, regarding the second mutual fund, heterogeneity in skew tolerances (the amount invested is proportional to relative spread between the skew tolerance of investor $s$ and the average one). Thus, an investor with stronger or weaker preferences for skewness than the average must underdiversify with respect to mean-variance efficiency.

Recall that the wealth weighted sum of $s$ in relation to the wealth weighted sum of $s$ gives the average skew tolerance coefficient $\bar{\rho}$. Therefore, although investors’ holdings in the skewness portfolio may be different from zero, aggregate demand of it vanishes\(^2\).

It is worth elaborating in more detail why investors will have non-zero positions in the skewness portfolio when their preferences for skewness depart from the average skewness preference. Comparing (11) and (13) shows that for all assets $i = 1, ..., n$:

$$\sigma \text{Cov}(R_i, R^{sk}) = \text{Cov} \left[ R_i, (R_M - E(R_M))^2 \right]. \quad (15)$$

In other words, we have the following decomposition for the squared market return:

$$(R_M - E(R_M))^2 = R^{sk} + T^{sk}, \quad (16)$$

where the tracking error $T^{sk}$ is uncorrelated with all traded asset returns $R_i, i = 1, ..., n$. Up to an additive constant, the skewness portfolio return $R^{sk}$ can therefore be interpreted as an affine projection of the squared (demeaned) market return on the traded asset returns. In other words, the skewness portfolio is the mutual fund that is best able to track the squared market return;

\(^2\)Essentially this is due to our simplifying assumption that the offer of risky assets does not depend on the scale parameter $\sigma$. Changing this assumption may introduce higher order dependence.
thus, demand for the skewness portfolio is induced by the demand for tracking the squared market return\(^3\).

In complete markets, the squared market return can be replicated through the traded assets and the tracking error disappears. In incomplete markets, however, the squared market return will typically not be in the asset span. The variance \(\text{Var}(T^{sk})\) of the tracking error can then be interpreted as a measure of market incompleteness. The bigger this variance, the less accurate is the feasible hedge of the squared market return that a representative agent would like to hold for the sake of skewness preferences (see e.g. Dittmar (2002)). This interpretation in terms of market incompleteness will be confirmed in the next section.

### 3.2 Quadratic pricing factors

A convenient way to describe an asset pricing model is through a random variable called Stochastic Discount Factor (henceforth SDF), see e.g. Cochrane (2001). It is well known that the Sharpe-Lintner CAPM is tantamount to an SDF that is affine w.r.t. the market return. To accommodate some widely documented departures from the CAPM, empirical asset pricing may consider higher-order polynomials in the market return as a candidate SDF. For instance a quadratic SDF should accommodate pricing of skewness, see, among others, Harvey and Siddique (2000) and Dittmar (2002). It is well-known that the quadratic SDF leads to a linear pricing relationship for excess returns:

\[
E_t[r_{it+1}] = \lambda_{1t} \text{Cov}_t[r_{it+1}, r_{Mt+1}] - \lambda_{2t} \text{Cov}_t[r_{it+1}, r^2_{Mt+1}],
\]

where \(r_{it+1}, r_{Mt+1}\) stands for excess returns on asset \(i\) and the market, respectively.

With a conditional viewpoint, the important insight of our analysis in section 2 is the identification of the SDF, leading us to the sensitivities:

\[
\lambda_{1t} = \frac{1}{\tau} + 2 \frac{\bar{p}}{\tau^2} E_t[r_{Mt+1}], \quad \lambda_{2t} = \frac{\bar{p}}{\tau^2}. \tag{17}
\]

The price of skewness is contained in the sensitivity \(\lambda_{2t}\): when it is zero we are back in a mean-variance pricing world, while \(\lambda_{2t} \neq 0\) leads to mean-variance-skewness pricing. Note that

\(^3\)In line with this reasoning, a careful examination of option markets would show that preferences for skewness introduce non-zero demand in asymmetric assets like options (see Judd and Leisen (2010)).
\( \lambda_{2t} \) is something like a structural invariant, only time-varying through the value of preference parameters computed from the derivatives of the utility function at \( R_{ft} \). Ultimately, estimating the size of \( \lambda_{2t} \) is at the core of empirical studies, see, e.g. Harvey and Siddique (2000).

It is tempting to apply the linear pricing relationship (17) to the squared market return; to do so, let us assume that \( r_{Mt+1}^2 \) corresponds to a portfolio available in the market. (We discussed in the previous section that \( R_{Mt+1}^2 \) and thus \( r_{Mt+1}^2 \) may not be a portfolio in incomplete markets.)

We denote by \( \eta_t = E_t[m_{t+1}r_{Mt+1}^2] \) the price and by \( \tilde{r}_{Mt+1} = r_{Mt+1}^2/\eta_t - R_{ft} \) the excess return on the squared market portfolio; applying the linear pricing relationship (17) to this excess return gives

\[
E_t[r_{Mt+1}^2] - \eta_t R_{ft} = \lambda_{1t} \text{Cov}_t(r_{Mt+1}, r_{Mt+1}^2) - \lambda_{2t} \text{Cov}_t(r_{Mt+1}^2, r_{Mt+1}^2).
\]

The linear pricing relationship (17) can be applied to the market excess return; this gives a system of two equations that can be resolved:

\[
\begin{align*}
\lambda_{1t} &= \frac{\text{Var}_t(r_{Mt+1})E_t[r_{Mt+1}^2] - \text{Cov}_t(r_{Mt+1}, r_{Mt+1}^2)(E_t[r_{Mt+1}^2] - \eta_t R_{ft})}{\text{Var}_t(r_{Mt+1})\text{Var}(r_{Mt+1}^2) - (\text{Cov}_t(r_{Mt+1}, r_{Mt+1}^2))^2}, \\
\lambda_{2t} &= \frac{-\text{Var}_t(r_{Mt+1})(E_t[r_{Mt+1}^2] - \eta_t R_{ft}) - \text{Cov}_t(r_{Mt+1}, r_{Mt+1}^2)E_t[r_{Mt+1}^2]}{\text{Var}_t(r_{Mt+1})\text{Var}(r_{Mt+1}^2) - (\text{Cov}_t(r_{Mt+1}, r_{Mt+1}^2))^2}.
\end{align*}
\]

One may be tempted to set \( \eta_t = 0 \); this characterization of \( \lambda_{1t}, \lambda_{2t} \) then coincides with formulas (7b, 7c) put forward by Harvey and Siddique (2000). However, this appears to be at odds with a no-arbitrage condition, since \( \eta_t = E_t \left[ m_{t+1}r_{Mt+1}^2 \right] \) is the price of a strictly positive payoff and should therefore be strictly positive.

4 Unspanned SDF in polynomials of market return

So far we approximated the risk premium \( (E(R_t) - R_f) \) through the first two terms \( \sigma^2(a_{i0} + \sigma a_{i1}) \) of its expansion (2); in this approximation, equation (17) states that a second-order polynomial in the market return is a sufficient statistic to define the SDF, in accordance with the representative agent model of Dittmar (2002). It takes a careful investigation of the fourth term \( \sigma^4a_{i3} \) of the expansion (2) to understand that polynomials in the market return do not capture completely the pricing impact of investors’ heterogeneity in incomplete markets.
In general, the market return is not a sufficient statistic in incomplete markets because the standard definition of co-pentosis, namely \( \text{Cov}[(R_M - E(R_M))^4, R_i] \) does not capture the pricing effect of \( \text{Cov}[(R^{sk} - E(R^{sk}))^2, R_i] \), because a random tracking error \((T^{sk})^2\) in the difference \([(R_M - E(R_M))^4 - (R^{sk})^2]\) makes these two covariance risks different. (Here, \( R^{sk} \) is the skewness portfolio return defined in the previous section and \( T^{sk} \) is the tracking error introduced in equation (16).)

Let us define the cross-coskewness
\[
\xi_i^{sk} = \frac{1}{\sigma^3} \text{Cov}[(R^{sk} - E(R^{sk}))^2, R_i]
\]

While co-skewness represents the contribution of asset \( i \) to the skewness of the market return and while the traditional market beta represents the contribution of asset \( i \) to the variance of the market return, the cross-coskewness represents (up to a proportionality factor) the contribution of asset \( i \) to the skewness of the return on the skewness portfolio:
\[
\sigma^3 \sum_{i=1}^{n} \xi_i^{sk} c_i^{sk} = E[(R^{sk} - E[R^{sk}])^3].
\]

Note that aggregating the cross-skewness coefficients with weights of the skewness portfolio is sensible since the investors’ holdings in skewness portfolio sum to zero at the aggregate market level. However, since each individual investor may hold a mix of market portfolio and skewness portfolio, variance of each of them matters and warrants the risk premiums proportional to respective contributions to respective variances.

The risk premium associated to cross co-skewness is actually showing up in the following expression of the fourth term \( a_3 \) of the risk premium expansion. We show in appendix that for each asset \( i = 1, \ldots, n \), \( a_{i3} \) is a linear combination of \( \text{Cov}[R_i, R_M] \), \( \text{Cov}[R_i, (R_M - E(R_M))^2] \), \( \text{Cov}[R_i, (R_M - E(R_M))^4] \), but also \( \text{Cov}[R_i, (R^{sk} - E(R^{sk}))^2] \). In other words, the pricing kernel must be a linear combination of not only \( R_M, (R_M)^2, (R_M)^3, (R_M)^4 \) \(( (R_M)^3 \) would actually show up in \( a_2 \)) but also involves \((R^{sk})^2\). Note that \( R^{sk} \) itself is not needed for polynomial-SDF spanning since it is spanned by the couple \((R_M, (R_M)^2)\). By contrast, \((R^{sk})^2\) is needed and as explained above, corresponding betas coefficients provide a decomposition of the skewness of the skewness portfolio. There is no need of beta coefficients for decomposing the variance of the
skewness portfolio: they are encapsulated in linear combinations of Sharpe-Lintner betas and Kraus-Litzenberger co-skewness coefficients.

We show that the price of the pricing factor \((R_{sk})^2\) (or equivalently the risk sensitivity of this factor in a SDF decomposition extending the quadratic one of section 3.2) is proportional to:

\[
\bar{\rho}^3 \left\{ \frac{Var(\rho)}{\bar{\rho}^2} + \frac{Sk(\rho)}{\bar{\rho}^3} \right\},
\]

(18)

where:

\[
Var(\rho) = \frac{\sum_{s=1}^S q_s \tau_s (\rho_s - \bar{\rho})^2}{\sum_{s=1}^S q_s \tau_s}, \quad Sk(\rho) = \frac{\sum_{s=1}^S q_s \tau_s (\rho_s - \bar{\rho})^3}{\sum_{s=1}^S q_s \tau_s}.
\]

In other words, the risk sensitivity of the additional factor \((R_{sk})^2\) is typical of implications of market incompleteness. Formula (18) first reminds us that this market incompleteness matters for pricing due to a non-zero average skew-tolerance \(\bar{\rho}\) so that the skewness portfolio is a mimicking portfolio for a priced factor defined by squared market return. Second, in the same way a non-zero cross-sectional variance of consumption may weight the pricing factor due to market incompleteness (see e.g. Constantinides and Duffie (1996)), the cross sectional dispersion of skew tolerances is the cause for a priced factor \((R_{sk})^2\) (or equivalently a priced squared tracking error on the squared market return). This cross sectional dispersion of skewness tolerances shows up in two ways:

1. First, through the squared coefficient of variation \(Var(\rho)/(\bar{\rho})^2\) of skew-tolerances \(\rho_s, s = 1, \ldots, S\).

2. Second, through the standardized cross-sectional skewness \(Sk(\rho)/(\bar{\rho})^3\) of skew-tolerances \(\rho_s, s = 1, \ldots, S\).

These two possibly non-zero measures of cross-sectional dispersion explain that, when both market incompleteness and heterogeneity of preferences are at stake, an approach to preference for higher order moments based on a representative agent as in Dittmar (2002) will miss some pricing factors by setting the focus only on polynomial in the market return. The point made above regarding preference for skewness could of course be easily extended to any higher order moment.
5 Conclusion

This paper showed a three-fund separation theorem, i.e. agents hold the risk-free asset, the market portfolio and a skewness portfolio. In complete markets, the skewness portfolio is the portfolio that replicates the squared market portfolio, the skewness risk premium is driven by co-variation with the squared market such that a pricing is characterized by a stochastic discount factor that is a quadratic polynomial in the market return. In incomplete markets, however, the skewness portfolio is the orthogonal projection of the market return on available assets leading to a tracking error portfolio. That portfolio contributes to pricing in addition to the skewness portfolio and its squared value enters into the stochastic discount factor. Empirical studies of skewness risk based on stochastic discount factors that are polynomials in the market return miss this factor.

Appendix

By assumption, the n-dimensional vector \( Y = \frac{R - E(R)}{\sigma} \) has a given distribution, invariant to changes in \( \sigma > 0 \), with mean 0 and variance \( \Sigma \). We will repeatedly use the following decomposition of excess returns:

\[
r_i = R_i - R_f = \sigma Y_i + E(R_i) - R_f = \sigma Y_i + \sigma^2 \sum_{j=0}^{\infty} a_{ij}\sigma^j \quad (A-19)
\]

For our derivations in the appendix, we introduce the concepts of co-variation, more precisely co-skewness, co-kurtosis and co-pentosis of asset \( i \) with a portfolio \( \theta \) as follows:

\[
\begin{align*}
b_i(\theta) &= \frac{1}{\sigma^2} Cov \left( R_i, (\theta^\top (R - ER)) \right) = Cov \left( Y_i, (\theta^\top Y) \right) \\
c_i(\theta) &= \frac{1}{\sigma^3} Cov \left( R_i, (\theta^\top (R - ER))^2 \right) = Cov \left( Y_i, (\theta^\top Y)^2 \right) \\
d_i(\theta) &= \frac{1}{\sigma^4} Cov \left( R_i, (\theta^\top (R - ER))^3 \right) = Cov \left( Y_i, (\theta^\top Y)^3 \right) \\
f_i(\theta) &= \frac{1}{\sigma^5} Cov \left( R_i, (\theta^\top (R - ER))^4 \right) = Cov \left( Y_i, (\theta^\top Y)^4 \right)
\end{align*}
\]
We will also denote vectors as column matrices as follows:

\[
\begin{align*}
a_j &= (a_{ij})_{1 \leq i \leq n}, j = 0, 1, \ldots \\
b(\theta) &= (b_i(\theta))_{1 \leq i \leq n} = \Sigma \theta \\
c(\theta) &= (c_i(\theta))_{1 \leq i \leq n}, d(\theta) = (d_i(\theta))_{1 \leq i \leq n}, f(\theta) = (f_i(\theta))_{1 \leq i \leq n}
\end{align*}
\]

For \( \theta = \xi \), we consider marketwise quantities as follows:

\[
\begin{align*}
b &= \Sigma \xi, c = c(\xi), d = d(\xi), f = f(\xi) \\
Sk &= E[(\xi^Y)^3], \xi_0^k = \Sigma^{-1}c
\end{align*}
\]

Note that the skewness portfolio \( \xi^k \) defined in main text is nothing but \( \sigma \xi_0^k \).

We first solve the agent’s utility maximization problem for any given investor \( s = 1, \ldots, S \).

### A.1 Individual demand for given risk premiums

For any given value \( \sigma > 0 \) of the scale parameter, we want to describe the solution \( \theta_s(\sigma) = (\theta_{ts}(\sigma))_{1 \leq t \leq n} \) of the agent’s utility maximization problem as characterized by the first order conditions:

\[
E[u_s'(W_s(\theta(\sigma)))r_i] = 0, \forall i = 1, \ldots, n, \quad (A-20)
\]

\[
W_s(\theta(\sigma)) = q_s \left\{ R_f + \sum_{i=1}^{n} \theta_{ts}(\sigma)r_i \right\}
\]

Assuming for sake of expositional simplicity that the utility function \( u_s \) is analytical, \( h_{ts}(\sigma) = u_s'(W_s(\theta(\sigma))) \cdot r_i \) is an analytical function of the scale parameter \( \sigma \) with a zero constant term since the limit of the net return \( r_i \) when \( \sigma \) goes to zero is zero. Thus, we write:

\[
h_{ts}(\sigma) = \sum_{j=1}^{\infty} h_{tsj}\sigma^j
\]

The first order conditions (A-20) then read for \( i = 1, \ldots, n \):

\[
E(h_{tsj}) = 0, \forall j = 1, 2, 3, \ldots. \quad (A-21)
\]
We compute the series coefficients $h_{isj}$ by multiplying the expansion (A-19) with the series expansion of $u'_s(W_s(\theta(\sigma)))$. To get it, we first write a Taylor series around $q_s R_f$:

\[
\begin{align*}
    h_{isj} &= u'_s(q_s R_f) + u''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{ks} r_k \right) \\
    &\quad + \frac{1}{2} u'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{ks} r_k \right)^2 \\
    &\quad + \frac{1}{6} u''''(q_s R_f) q_s^3 \left( \sum_{k=1}^{n} \theta_{ks} r_k \right)^3 \\
    &\quad + \frac{1}{24} u^{(5)}(q_s R_f) q_s^4 \left( \sum_{k=1}^{n} \theta_{ks} r_k \right)^4 + \ldots
\end{align*}
\]

Now, we deduce from (A-19) the series expansion of portfolio returns:

\[
\begin{align*}
    \sum_{k=1}^{n} \theta_{ks}(\sigma) r_k &= \sigma \sum_{k=1}^{n} \theta_{ks0} Y_k + \sigma^2 \sum_{k=1}^{n} (\theta_{ks1} Y_k + \theta_{ks0} a_{k0}) \\
    &\quad + \sigma^3 \sum_{k=1}^{n} (\theta_{ks2} Y_k + \theta_{ks1} a_{k0} + \theta_{ks0} a_{k1}) \\
    &\quad + \sigma^4 \sum_{k=1}^{n} (\theta_{ks3} Y_k + \theta_{ks2} a_{k0} + \theta_{ks1} a_{k1} + \theta_{ks0} a_{k2}) + \ldots
\end{align*}
\]

Jointly, (A-22) and (A-23) imply:

\[
\begin{align*}
    u'_s(W_s(\theta)) &= u'_s(q_s R_f) + \sigma u''(q_s R_f) q_s \left( \sum_{k=1}^{n} \theta_{ks0} Y_k \right) \\
    &\quad + \sigma^2 \left\{ u''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{ks1} Y_k + \theta_{ks0} a_{k0}) \right) + \frac{1}{2} u'''(q_s R_f) q_s^2 \left( \sum_{k=1}^{n} \theta_{ks0} Y_k \right)^2 \right\} \\
    &\quad + \sigma^3 \left\{ u''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{ks2} Y_k + \theta_{ks1} a_{k0} + \theta_{ks0} a_{k1}) \right) + \frac{1}{6} u''''(q_s R_f) q_s^3 \left( \sum_{k=1}^{n} \theta_{ks0} Y_k \right)^3 \right\} \\
    &\quad + \sigma^4 \left\{ u''(q_s R_f) q_s \left( \sum_{k=1}^{n} (\theta_{ks3} Y_k + \theta_{ks2} a_{k0} + \theta_{ks1} a_{k1} + \theta_{ks0} a_{k2}) \right) + \frac{1}{24} u^{(5)}(q_s R_f) q_s^4 \left( \sum_{k=1}^{n} \theta_{ks0} Y_k \right)^4 \right\} + \ldots
\end{align*}
\]
Multiplying this with the expansion (A-19) of $r_i$ and collecting terms, this now implies that:

\[
\begin{align*}
    h_{i,1} &= u'_s(q_sR_f)Y_i \\
    h_{i,2} &= u'_s(q_sR_f)a_{i0} + u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} \theta_{k=0}Y_kY_i \right) \\
    h_{i,3} &= u'_s(q_sR_f)a_{i1} + u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} \theta_{k=0}Y_ka_{i0} \right) \\
        &\quad + \left\{ u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} (\theta_{k=1}Y_k + \theta_{k=0}a_{k0}) \right) + \frac{1}{2} u'''_s(q_sR_f)q_s^2 \left( \sum_{k=1}^{n} \theta_{k=0}Y_k \right)^2 \right\} Y_i \\
    h_{i,4} &= u'_s(q_sR_f)a_{i2} + u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} (\theta_{k=2}Y_k + \theta_{k=1}a_{k0} + \theta_{k=0}a_{k1}) \right) \\
        &\quad + \left\{ u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} (\theta_{k=2}Y_k + \theta_{k=1}a_{k0} + \theta_{k=0}a_{k1}) \right) + \frac{1}{6} u'''_s(q_sR_f)q_s^3 \left( \sum_{k=1}^{n} \theta_{k=0}Y_k \right)^3 \right\} Y_i \\
    h_{i,5} &= u'_s(q_sR_f)a_{i3} + u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} \theta_{k=0}Y_k \right) a_{i2} \\
        &\quad + \left\{ u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} (\theta_{k=1}Y_k + \theta_{k=0}a_{k0}) \right) + \frac{1}{2} u'''_s(q_sR_f)q_s^2 \left( \sum_{k=1}^{n} \theta_{k=0}Y_k \right)^2 \right\} a_{i1} \\
        &\quad + \left\{ u''_s(q_sR_f)q_s \left( \sum_{k=1}^{n} (\theta_{k=2}Y_k + \theta_{k=1}a_{k0} + \theta_{k=0}a_{k1}) \right) + \frac{1}{6} u'''_s(q_sR_f)q_s^3 \left( \sum_{k=1}^{n} \theta_{k=0}Y_k \right)^3 \right\} a_{i0} \\
        &\quad + \left\{ \frac{1}{2} u''''_s(q_sR_f)q_s^2 \left( \sum_{k=1}^{n} (\theta_{k=3}Y_k + \theta_{k=2}a_{k0} + \theta_{k=1}a_{k1} + \theta_{k=0}a_{k2}) \right)^2 + \frac{1}{24} u^{(5)}(q_sR_f)q_s^4 \left( \sum_{k=1}^{n} \theta_{k=0}Y_k \right)^4 \right\} Y_i \end{align*}
\]

Now that we have characterized the first five terms in the expansion of $h_{is}$ for each $i = 1, \ldots, n$, we study in our final step the expansion of the first order conditions $E(h_{isj}) = 0$ for $j = 1, \ldots, 5$ and all $i = 1, \ldots, n$.

Recall that $E(Y_i) = 0$ for $i = 1, \ldots, n$ by definition such that $E(h_{is1}) = 0$ for $i = 1, \ldots, n$. Up to order five, the informative first order conditions are as follows:
1. $E(h_{i,s2}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'_s(q_sR_f)a_{i0} + u''_s(q_sR_f)q_s \sum_{k=1}^n \theta_{k,s0} \text{Cov}(Y_k, Y_i)$$

In vector-matrix notation this reads:

$$0 = u'_s(q_sR_f)a_{i0} + u''_s(q_sR_f)q_s \Sigma \theta_{s0} \text{ i.e. } \theta_{s0} = \tau_s \Sigma^{-1} a_0$$

(A-24)

where:

$$\tau_s = -\frac{u'_s(q_sR_f)}{q_s u''_s(q_sR_f)}$$

stands for the relative risk tolerance.

2. The condition $E(h_{i,s3}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'_s(q_sR_f)a_{i1} + u''_s(q_sR_f)q_s \sum_{k=1}^n \theta_{k,s1} \text{Cov}(Y_k, Y_i) + \frac{1}{2} u''_s(q_sR_f)q_s^2 \text{Cov} \left( \left( \sum_{k=1}^n \theta_{k,s0} Y_k \right)^2, Y_i \right)$$

Using the definition of co-skewness with a portfolio that we introduced at the beginning of this appendix, this reads in vector-matrix notation:

$$0 = u'_s(q_sR_f)a_{i1} + u''_s(q_sR_f)q_s \Sigma \theta_{s1} + \frac{1}{2} u''_s(q_sR_f)q_s^2 c(\theta_{s0})$$

This gives:

$$\theta_{s1} = -\frac{u''_s(q_sR_f)q_s}{2u''_s(q_sR_f)} \Sigma^{-1} c(\theta_{s0}) - \frac{u'_s(q_sR_f)}{u''_s(q_sR_f)q_s} \Sigma^{-1} a_1$$

(A-25)

where:

$$\rho_s = \frac{u'_s(q_sR_f)u''_s(q_sR_f)}{2[u''_s(q_sR_f)]^2}$$

is the skew tolerance.

3. The condition $E(h_{i,s4}) = 0$ gives for $i = 1, ..., n$:

$$0 = u'_s(q_sR_f)a_{i2} + u''_s(q_sR_f)q_s(\theta_{s0}^2 a_0 + \frac{1}{2} u''_s(q_sR_f)q_s^2 \text{Var}(\theta_{s0}^2Y) a_{i0}$$

$$+ u'_s(q_sR_f)q_s \text{Cov}(\theta_{s2}^2 Y, Y_i) + \frac{1}{6} u''_s(q_sR_f)q_s^3 \text{Cov} \left( \left( \theta_{s0}^2 Y \right)^3, Y_i \right)$$

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Using the definition of co-kurtosis with a portfolio that we introduced at the beginning of this appendix, this reads in vector-matrix notation:

\[
0 = u'_s(q_s R_f) a_2 + u''_s(q_s R_f) q_s (\theta_{s0}^A a_0) a_0 + \frac{1}{2} u'''_s(q_s R_f) q_s^2 \text{Var} \left( \theta_{s0}^A Y \right) a_0 \\
+ u''_s(q_s R_f) q_s \Sigma \theta_{s2} + \frac{1}{6} u'''_s(q_s R_f) q_s^3 d(\theta_{s0}).
\]

This gives:

\[
q_s \Sigma \theta_{s2} = q_s \tau_s a_2 - (q_s \theta_{s0}^A a_0) a_0 + \frac{q_s \rho_s}{\tau_s^2} \text{Var} \left( \theta_{s0}^A Y \right) a_0 - \frac{q_s \kappa_s}{\tau_s^2} d(\theta_{s0}) 
\tag{A-26}
\]

where:

\[
\kappa_s = \frac{[u'_s(q_s R_f)]^2 u''_s(q_s R_f)}{6[u''_s(q_s R_f)]^3}
\]

is the kurtosis tolerance.

4. Finally, the condition \( E(h_{is5}) = 0 \) gives for \( i = 1, ..., n \):

\[
0 = u'_s(q_s R_f) a_3 + u''_s(q_s R_f) q_s (\theta_{s0}^A a_0) a_1 + \frac{1}{2} u'''_s(q_s R_f) q_s^2 \text{Var} \left( \theta_{s0}^A Y \right) a_1 \\
+ u''_s(q_s R_f) q_s (\theta_{s1}^A a_0 + \theta_{s0}^A a_1) a_0 + \frac{1}{6} u'''_s(q_s R_f) q_s^3 E \left[ (\theta_{s0}^A Y)^3 \right] a_0 \\
+ u''_s(q_s R_f) q_s \text{Cov} (\theta_{s1}^A Y, Y_i) + \frac{1}{2} u'''_s(q_s R_f) q_s^2 \text{Cov} \left[ (\theta_{s1}^A Y + \theta_{s0}^A a_0)^2, Y_i \right] \\
+ \frac{1}{24} u'''_s[q_s^5 (q_s R_f) q_s^4 \text{Cov} \left[ (\theta_{s1}^A Y)^4, Y_i \right]
\]

Using the definition of co-pentosis with a portfolio that we introduced at the beginning of this appendix, this reads in vector-matrix notation:

\[
0 = u'_s(q_s R_f) a_3 + u''_s(q_s R_f) q_s (\theta_{s0}^A a_0) a_1 + \frac{1}{2} u'''_s(q_s R_f) q_s^2 \text{Var} \left( \theta_{s0}^A Y \right) a_1 \\
+ u''_s(q_s R_f) q_s (\theta_{s1}^A a_0 + \theta_{s0}^A a_1) a_0 + \frac{1}{6} u'''_s(q_s R_f) q_s^3 E \left[ (\theta_{s0}^A Y)^3 \right] a_0 \\
+ u''_s(q_s R_f) q_s \Sigma \theta_{s3} + \frac{1}{2} u'''_s(q_s R_f) q_s^2 \text{Cov} \left[ Y, (\theta_{s1}^A Y + \theta_{s0}^A a_0)^2 \right] + \frac{1}{24} u'''_s[q_s^5 (q_s R_f) q_s^4 f(\theta_{s0})
\]

which gives:

\[
q_s \Sigma \theta_{s3} = q_s \tau_s a_3 - q_s (\theta_{s0}^A a_0) a_1 + \frac{q_s \rho_s}{\tau_s^2} \text{Var} \left( \theta_{s0}^A Y \right) a_1 - q_s (\theta_{s1}^A a_0 + \theta_{s0}^A a_1) a_0 \\
- \frac{q_s \kappa_s}{\tau_s^2} E \left[ (\theta_{s0}^A Y)^3 \right] a_0 + \frac{q_s \rho_s}{\tau_s^2} \text{Cov} \left[ Y, (\theta_{s1}^A Y + \theta_{s0}^A a_0)^2 \right] + \frac{q_s \kappa_s}{\tau_s^2} f(\theta_{s0}) 
\tag{A-27}
\]

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where:
\[ \pi_s = \frac{[u'_s(q_s R_f)]^3 u_s^{[5]}(q_s R_f)}{24[u'_s(q_s R_f)]^4} \]
is the pentos tolerance.

### A.2 Market clearing conditions

Market clearing conditions are:

\[ \sum_{s=1}^{S} q_s \theta_{s0} = S \bar{q} \xi \quad \text{and} \quad \sum_{s=1}^{S} q_s \theta_{sj} = 0, j = 1, 2, 3. \]

1. Plugging in the value of \( \theta_{s0} \) given by (A-24), the first market clearing condition gives:

\[ \sum_{s=1}^{S} q_s \tau_s \xi^{-1} a_0 = S \bar{q} \xi \]

that is:

\[ a_0 = \frac{1}{\bar{\tau}} b, \theta_{s0} = \frac{\tau_s}{\bar{\tau}} \xi \]

where:

\[ \bar{\tau} = \frac{1}{S} \sum_{s=1}^{S} q_s \tau_s \]

2. Plugging in the value of \( \theta_{s1} \) given by (A-25), the second market clearing condition gives:

\[ \sum_{s=1}^{S} q_s \tau_s \xi^{-1} \left[ c(\theta_{s0}) \frac{\rho_s}{\tau_s^2} + a_1 \right] = 0 \]

with, from (A-28):

\[ c(\theta_{s0}) = \left( \frac{\tau_s}{\bar{\tau}} \right)^2 c \]

We deduce:

\[ a_1 = -\frac{\bar{\rho}}{(\bar{\tau})^2} c, \theta_{s1} = \frac{\tau_s}{\bar{\tau}^2}(\rho_s - \bar{\rho}) \xi^{-1} c = \frac{\tau_s}{\bar{\tau}^2}(\rho_s - \bar{\rho}) \xi^{sk} \]

where:

\[ \bar{\rho} = \frac{\sum_{s=1}^{S} q_s \tau_s \rho_s}{\sum_{s=1}^{S} q_s \tau_s} \]

3. Plugging in the value of \( \theta_{s2} \) given by (A-26), the third market clearing condition gives:

\[ 0 = a_2 \sum_{s=1}^{S} q_s \tau_s - \sum_{s=1}^{S} \left( q_s \theta_{s0} a_0 - \frac{q_s \rho_s}{\tau_s} Var(\theta_{s0} Y) a_0 + \frac{q_s \kappa_s}{\tau_s^2} d(\theta_{s0}) \right) \]
Noting that
\[ \theta_{s0} = \frac{\tau_s}{\bar{\tau}} \xi, \quad a_0 = \frac{1}{\bar{\tau}} b = \frac{1}{\bar{\tau}} \Sigma \xi, \quad d(\theta_{s0}) = \left( \frac{\tau_s}{\bar{\tau}} \right)^3 d, \]
we get
\[ 0 = a_2 \sum_{s=1}^{S} q_s \tau_s - \sum_{s=1}^{S} \left( q_s \frac{\tau_s}{(\bar{\tau})^2} (\xi^\perp \Sigma \xi) a_0 - \frac{q_s \rho_s \tau_s}{(\bar{\tau})^2} (\xi^\perp \Sigma \xi) a_0 + \frac{q_s \xi_s \tau_s}{(\bar{\tau})^3} d \right), \]
that is
\[ a_2 = (1 - \bar{\rho}) \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^3} b + \bar{\kappa} \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^3} d, \]
where
\[ \bar{\kappa} = \frac{\sum_{s=1}^{S} q_s \tau_s \xi_s}{\sum_{s=1}^{S} q_s \tau_s}. \]

4. Plugging in the value of \( \theta_{s3} \) given by (A-27), the fourth market clearing condition gives:
\[ 0 = a_3 \sum_{s=1}^{S} q_s \tau_s + \sum_{s=1}^{S} \left( -q_s (\theta_{s0}^\perp a_0) a_1 + \frac{q_s \rho_s}{\tau_s} Var \left( \theta_{s0}^\perp Y \right) a_1 - q_s (\theta_{s1}^\perp a_0 + a_{s0}) a_0 \right) \]
Recalling that:
\[ \theta_{s0} = \frac{\tau_s}{\bar{\tau}} \xi, \quad a_0 = \frac{1}{\bar{\tau}} \Sigma \xi, \quad a_1 = -\frac{\bar{\rho}}{(\bar{\tau})^2} c; \]
\[ \theta_{s1} = \frac{\tau_s}{\bar{\tau}} (\rho_s - \bar{\rho}) \Sigma^{-1} c, \quad f(\theta_{s0}) = \left( \frac{\tau_s}{\bar{\tau}} \right)^4 f \]
we get
\[ 0 = a_3 \sum_{s=1}^{S} q_s \tau_s + \sum_{s=1}^{S} \left( q_s \tau_s \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^2} \bar{\rho} c - q_s \tau_s \rho_s \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^2} \bar{\rho} c - q_s \tau_s (\rho_s - \bar{\rho}) \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^2} b + q_s \tau_s \rho_s \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^2} b 
- q_s \tau_s \kappa_s \frac{S k}{(\bar{\tau})^4} b + \frac{q_s \tau_s}{(\bar{\tau})^2} Cov \left( y, \left( (\frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^2})^2 + \frac{\xi^\perp \Sigma \xi}{(\bar{\tau})^2} \right) \right) + q_s \tau_s \pi_s b \right) \]
that is
\[ a_3 = \frac{[(\bar{\rho})^2 - \bar{\rho} - 2 Var(\rho)] c(\xi^\perp \Sigma \xi)}{(\bar{\tau})^4} + (\bar{\rho} - \bar{\rho}, \frac{S k}{(\bar{\tau})^4} b \right)
- \frac{Cov \left( y, \left( (\xi^\perp \Sigma \xi)^2 \right) \right)}{(\bar{\tau})^4} [Sk(\rho) + \bar{\rho} Var(\rho)] - \frac{\pi}{(\bar{\tau})^4} f \]
where
\[ \pi = \frac{\sum_{s=1}^{S} q_s \tau_s \pi_s}{\sum_{s=1}^{S} q_s \tau_s}, \quad Var(\rho) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \bar{\rho})^2}{\sum_{s=1}^{S} q_s \tau_s}, Sk(\rho) = \frac{\sum_{s=1}^{S} q_s \tau_s (\rho_s - \bar{\rho})^3}{\sum_{s=1}^{S} q_s \tau_s}. \]
In terms of higher moments of returns, one may want to reread formula (A-29) for risk premium as:

\[
a_3 = \frac{(\bar{\rho})^2 - \bar{\rho} - 2\text{Var}(\rho)}{(\bar{\tau})^4} \text{Var}(\xi^4Y)\text{Cov}[Y, (\xi^4Y)^2] + \frac{\bar{k} - \bar{\rho}}{(\bar{\tau})^4} E[(\xi^3Y)^3] \text{Cov}[Y, (\xi^3Y)] \\
- \frac{\text{Sk}(\rho) + \bar{\rho}\text{Var}(\rho)}{(\bar{\tau})^4} \text{Cov}[Y, (\xi^3Y^2)^2] - \frac{\bar{\pi}}{(\bar{\tau})^4} \text{Cov}[Y, (\xi^3Y^4)]
\]

References


