Pricing Kernels with Stochastic Skewness and Volatility Risk

Online Appendix

In this online appendix, I first provide a brief comparison of the small noise expansion series and the Taylor expansion series. I then use separably Taylor expansion series and the small noise expansion series to derive the pricing kernel in a one-market model with two dates, and the pricing kernel in a two-market model with three dates. I thereafter, investigate whether one obtains a pricing kernel that depends on the market co-skewness and the market volatility in a long run risk model and also in a model where the representative investor chooses his optimal allocation in the presence of the market equity and the variance swap contracts. The remaining part of the appendix contains additional tables and figures used in the paper.
Appendix A: A Brief Comparison of the Small Noise Expansion and the Taylor Expansion Series

To approximate the investor’s utility $u[W]$ with Taylor expansion series, the usual route is to assume that $W - EW$ is small

$$|W - E(W)| < \delta \text{ for } \delta > 0,$$  \hspace{1cm} (A1)

and expand $u[.]$ around the expected value $E(W)$. Similarly to the Taylor expansion series, the small noise expansion framework would assume

$$W - E(W) = \epsilon Y$$  \hspace{1cm} (A2)

and drive $\epsilon$ toward zero without making any restrictions on the distribution of $Y$. A simple comparison of both approaches in (A1) and (A2) shows that under the restriction

$$|Y| < \delta/\epsilon,$$  \hspace{1cm} (A3)

the small noise expansion series and the Taylor expansion series are equivalent. The small noise expansion approach drives $\epsilon$ toward 0. In order words, $\epsilon \to 0$ is equivalent to

$$\epsilon = \frac{1}{2^n},$$  \hspace{1cm} (A4)

where $n$ is very large. Replacing (A4) in (A2) produces

$$Y = 2^n (W - E[W]).$$  \hspace{1cm} (A5)

The random variable $Y$ represents the innovation in the aggregate wealth scaled by a large number. The probability density function of $Y$ and $W - E[W]$ are proportional.

In this paper, I consider the small noise approximation of innovations in returns

$$R_{kr+1} - E_{i}R_{kr+1} = \epsilon Y_{kr+1} \text{ and } R_{kr+2} - E_{i+1}R_{kr+2} = \epsilon Y_{kr+2}$$  \hspace{1cm} (A6)

and derive the pricing kernel in a one-period model with two dates (see Appendix C), and the pricing kernel in a two-period model with three dates (see Appendix D).
Appendix B: Pricing Kernel Obtained with the Standard Taylor Series in a Two-Period Model with Three Dates

I assume that the representative agent maximizes his expected utility in a two-period model with three dates

\[ V_t = \max_{\{\omega_t\}} E_t \left( \max_{\{\omega_{t+1}\}} E_{t+1} \left( u\left[ W_{t+2} \right] \right) \right) , \]  

(B1)

where the investor’s wealth is

\[ W_{t+2} = (R_f + \omega_t R_{t+1}) (R_f + \omega_{t+1} R_{t+2}) \]  

(B2)

and \( R_{\tau+1}^e = (R_{\tau+1} - R_f), \tau = t, t+1 \). The initial wealth \( W_t = 1 \). I use the first-order condition of (B1) with respect to \( \omega_t \)

\[ E_t \left( u' \left[ W_{t+2} \right] (R_{kt+1} - R_f) \right) = 0 , \]  

(B3)

and use the decomposition \( \text{Cov}_t (X, Y) = E_t (XY) - E_t (X) E_t (Y) \) to expand (B3) to derive the risk premium at time \( t \)

\[ E_t (R_{kt+1} - R_f) = -\text{Cov}_t \left( E_t \left( u' \left[ W_{t+2} \right] \right) \frac{u' \left[ W_{t+2} \right]}{E_t \left( u' \left[ W_{t+2} \right] \right)} , R_{kt+1} \right) . \]  

(B4)

I show in subsection B.1 and B.2 that the static Taylor expansion series and the recursive Taylor expansion series of the investor’s marginal utility produce pricing kernels where the co-skewness risk and the volatility risk factor have identical prices.

B.1. Static Taylor Expansion Series

The standard Taylor expansion series of \( u' \left[ W_{t+2} \right] \) around \( (\omega_t R_{t+1}, \omega_{t+1} R_{t+2}) = (R_f, R_f) \) produces

\[ u' \left[ W_{t+2} \right] = u' \left[ R_f^2 \right] + u'' \left[ R_f^2 \right] (\omega_t R_{t+1} - R_f) + u''' \left[ R_f^2 \right] (\omega_{t+1} R_{t+2} - R_f) \]  

\[ + \frac{1}{2} u''' \left[ R_f^2 \right] (\omega_t R_{t+1} - R_f)^2 + \frac{1}{2} u''' \left[ R_f^2 \right] (\omega_{t+1} R_{t+2} - R_f)^2 . \]  

(B5)
I replace (B5) in the risk premium (B4) and show

\[
E_t (R_{lt+1} - R_f) = -\frac{u''[R_f^2]}{E_t u' [W_{t+2}]} Cov_t (\omega_t R_{t+1}, R_{kt+1}) - \frac{u''[R_f^2]}{E_t u' [W_{t+2}]} Cov_t (\omega_t+1 R_{t+2}, R_{kt+1}) 
\]

(B6)

\[
-\frac{u''[R_f^2]}{2E_t u' [W_{t+2}]} Cov_t (\omega_t R_{t+1} - R_f)^2, R_{kt+1}) 
\]

\[
-\frac{u''[R_f^2]}{2E_t u' [W_{t+2}]} Cov_t (\omega_t+1 R_{t+2} - R_f)^2, R_{kt+1}) .
\]

I notice that

\[
Cov_t (R_{Mt+2} - R_f)^2, R_{kt+1}) = E_t \left( (R_{Mt+2} - R_f)^2 R_{kt+1} \right) - \left( E_t (R_{Mt+2} - R_f)^2 \right) \left( E_t (R_{kt+1}) \right) 
\]

(B7)

\[
= E_t \left( R_{kt+1} \left( E_{t+1} (R_{Mt+2} - R_f)^2 \right) \right) - \left( E_t (E_{t+1} (R_{Mt+2} - R_f)^2) \right) \left( E_t (R_{kt+1}) \right) 
\]

\[
= Cov_t \left( E_{t+1} (R_{Mt+2} - R_f)^2, R_{kt+1} \right) .
\]

Since \( \omega_{t+1} R_{t+2} = R_{Mt+2} \), \( Cov_t (R_{Mt+2}, R_{kt+1}) = Cov_t (E_{t+1} R_{Mt+2}, R_{kt+1}) \), and \( \sigma_{Mt+1}^2 \approx E_{t+1} (\omega_{t+1} R_{t+2} - R_f)^2 \), the pricing kernel consistent with (B6) is

\[
\mathcal{M}_{t+1} = \frac{1}{R_f} + G_1 (R_{Mt+1} - E_{t+1} R_{Mt+1}) + G_1 (E_{t+1} R_{Mt+2} - E_t R_{Mt+2}) 
\]

(B8)

\[
+ G_2 \left( (R_{Mt+1} - R_f)^2 - E_{t} (R_{Mt+1} - R_f)^2 \right) + G_2 \left( \sigma_{Mt+1}^2 - E_{t} \sigma_{Mt+1}^2 \right)
\]

with

\[
G_1 = \frac{u''[R_f^2]}{R_f E_t u' [W_{t+2}]} \quad \text{and} \quad G_2 = \frac{u''[R_f^2]}{2R_f E_t u' [W_{t+2}]} .
\]

The pricing kernel in (B8) resembles the pricing kernel with co-skewness and volatility risk derived in Proposition 1, except that the prices of co-skewness and volatility risk in (B8) are identical.

**B.2. Recursive Taylor Expansion Series**

I notice that \( \mathcal{W}_{t+2} \) can be written as \( \mathcal{W}_{t+2} = \mathcal{W}_{t+1} (\omega_{t+1} R_{t+2}) \) and use Taylor expansion series recursively to derive the pricing kernel.

First, at time \( t+1 \), I notice that \( \mathcal{W}_{t+1} = \omega_t R_{t+1} \) and use Taylor series to expand the investor’s marginal
utility $u'[W_{t+2}]$ around $\omega_{r+1}R_{r+2} = R_f$ as follows

$$u'[W_{t+2}] = u'[R_fW_{t+1}] + u''[R_fW_{t+1}] W_{t+1} (\omega_{r+1}R_{r+2} - R_f) + \frac{1}{2}u'''[R_fW_{t+1}] (W_{t+1})^2 (\omega_{r+1}R_{r+2} - R_f)^2.$$  \hfill (B9)

I denote by $h(W_{t+1})$ the right hand side of (B9), and hence

$$u'[W_{t+2}] = h(W_{t+1}).$$  \hfill (B10)

Second, at time $t$, I expand (B9) around $W_{t+1} = R_f$

$$h(W_{t+1}) = h(R_f) + \frac{1}{1!}h'(R_f)(W_{t+1} - R_f) + \frac{1}{2!}h''(R_f)(W_{t+1} - R_f)^2,$$  \hfill (B11)

where

$$h'(W_{t+1}) = \frac{\partial (h(W_{t+1}))}{\partial W_{t+1}},$$

$$h''(W_{t+1}) = \frac{\partial^2 (h(W_{t+1}))}{\partial^2 W_{t+1}}.$$

When $W_{t+1} = R_f$, I use (B10) and (B9) to obtain

$$h(R_f) = u'[R_f^2] + u''[R_f^2] R_f (\omega_{r+1}R_{r+2} - R_f) + \frac{1}{2}u'''[R_f^2] R_f^2 (\omega_{r+1}R_{r+2} - R_f)^2.$$  \hfill (B12)

and

$$h'(R_f) = R_f u''[R_f^2] + \left(u''[R_f^2] + R_f u'''[R_f^2]\right) (\omega_{r+1}R_{r+2} - R_f) + \left(R_f u'''[R_f^2] + \frac{1}{2}R_f^3 u'''[R_f^2]\right) (\omega_{r+1}R_{r+2} - R_f)^2.$$  \hfill (B13)

and

$$h''(R_f) = R_f^2 u'''[R_f^2] + \left(R_f^3 u'''[R_f^2] + 2R_f u'''[R_f^2]\right) (\omega_{r+1}R_{r+2} - R_f) + \frac{1}{2}R_f^2 u'''[R_f^2] R_f^2 (\omega_{r+1}R_{r+2} - R_f)^2 + \left(R_f^2 u'''[R_f^2] (\omega_{r+1}R_{r+2} - R_f)^2 + u'''[R_f^2] R_f^2 (\omega_{r+1}R_{r+2} - R_f)^2\right).$$  \hfill (B14)
I replace (B12), (B13), and (B14) in (B11) and obtain

\[
\begin{align*}
  u' [W_{t+2}] & = \left( u' [R_f^2] + u'' [R_f^2] R_f (\omega_{t+1} R_{t+2} - R_f) + \frac{1}{2} u''' [R_f^2] R_f^2 (\omega_{t+1} R_{t+2} - R_f)^2 \right) \\
  & + \frac{1}{1!} \left( R_f u'' [R_f^2] + \frac{u'' [R_f^2] + R_f u'' [R_f^2]}{2} (\omega_{t+1} R_{t+2} - R_f) \right) (\omega_{t+1} R_{t+2} - R_f) \\
  & + \frac{1}{2!} \left( R_f^2 u''' [R_f^2] + \frac{R_f u''' [R_f^2] + 2 R_f u''' [R_f^2] + R_f^2 u''' [R_f^2]}{2} (\omega_{t+1} R_{t+2} - R_f) \right) (\omega_{t+1} R_{t+2} - R_f)^2 \\
  & + u'' [R_f^2] (\omega_{t+1} R_{t+2} - R_f)^2 + u''' [R_f^2] R_f^2 (\omega_{t+1} R_{t+2} - R_f)^2.
\end{align*}
\]  

Replacing (B15) in the risk premium (B4) allows me to show that the contributions of the market, the co-skewness, and the volatility factors to the risk premium on asset \(k\) are respectively

\[
-Cov_t \left( \frac{\frac{1}{2} R_f u'' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right),
\]

\[
-Cov_t \left( \frac{\frac{1}{2} R_f^2 u''' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right),
\]

and

\[
-Cov_t \left( \frac{\frac{1}{2} R_f^2 u''' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right).
\]

Therefore, neglecting the remaining cross-product terms in (B15) allows to write the risk premium on asset \(k\) as

\[
E_t (R_{kt+1} - R_f) = -Cov_t \left( \frac{\frac{1}{2} R_f u'' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right) -Cov_t \left( \frac{\frac{1}{2} R_f^2 u''' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right)
\]

\[
-Cov_t \left( \frac{\frac{1}{2} R_f^2 u''' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right)
\]

\[
-Cov_t \left( \frac{\frac{1}{2} R_f^2 u''' [R_f^2]}{E_t u' [W_{t+2}]} R_{kt+1} \right)
\]
Since, \( \omega R_{t+1} = R_{Mt+1} \), (B16) simplifies to

\[
E_t (R_{kt+1} - R_f) = - \frac{R_f u'' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \text{Cov}_t \left( R_{Mt+1}, R_{kt+1} \right)
- \frac{1}{2!} \frac{R_j^2 u''' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \text{Cov}_t \left( (R_{Mt+1} - R_f)^2, R_{kt+1} \right)
- \frac{1}{2!} \frac{R_j^2 u''' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \text{Cov}_t \left( (R_{Mt+2} - R_f)^2, R_{kt+1} \right).
\]

(B17)

I use (B7), the approximation \( E_t+1 (R_{Mt+2} - R_f)^2 \approx \sigma_{Mt+1}^2 \), and rewrite the risk premium (B17) as

\[
E_t (R_{kt+1} - R_f) = - \frac{R_f u'' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \text{Cov}_t \left( R_{Mt+1}, R_{kt+1} \right)
- \frac{1}{2!} \frac{R_j^2 u''' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \text{Cov}_t \left( (R_{Mt+1} - R_f)^2, R_{kt+1} \right)
- \frac{1}{2!} \frac{R_j^2 u''' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \text{Cov}_t \left( \sigma_{Mt+1}^2, R_{kt+1} \right).
\]

(B18)

Since the co-skewness and the market volatility factors have identical prices, \( - \frac{1}{2!} \frac{R_j^2 u''' \left[ R_j^2 \right]}{E_t u' \left[ W_{t+2} \right]} \), it is straightforward to show that, the co-skewness factor and the market volatility factor have the same factor loading in the pricing kernel specification.
Appendix C: Pricing Kernel Derived in a One-Period Model with Two Dates Using the Small Noise Expansion Series

In this section, I show that when the small noise assumption (A6) is used, existing asset pricing models can be derived in a one-period model with two dates. I consider the representative agent’s optimization in a one-period model with two dates

$$\max_{\omega} E_t (u' [W_{t+1}]),$$

(C1)

where the investor’s wealth is

$$W_{t+1} = W_t (R_f + \omega_t R_{t+1})$$

and

$$R_{t+1} = R_{t+1} - R_f.$$ (C2)

Without loss of generality, I assume that $W_t = 1$. The first-order conditions for (C1) are

$$E_t (u'[W_{t+1}] (R_{kt+1} - R_f)) = 0, \text{ for } k = 1, 2, \ldots$$ (C3)

and the expected excess return can be obtained by applying the covariance decomposition $Cov_t (X, Y) = E_t (XY) - E_t (X) E_t (Y)$ to (C3):

$$E_t (R_{kt+1} - R_f) = -Cov_t \left( \frac{u' [W_{t+1}]}{E_t u' [W_{t+1}]}, R_{kt+1} \right).$$ (C4)

I use the small noise assumption

$$R_{kt+1} - E_t R_{kt+1} = \varepsilon Y_{kt+1},$$ (C5)

and write

$$W_{t+1} - E_t W_{t+1} = (\omega_t Y_{t+1}) \varepsilon,$$ (C6)

where $Y_{t+1}$ is a vector whose components are $Y_{kt+1}$ for $k = 1, 2, \ldots$. The Taylor expansion series of $u' [W_{t+1}]$ around $\varepsilon = 0$ produces

$$u' [W_{t+1}] = \lim_{\varepsilon \to 0} u'[W_{t+1}] + \lim_{\varepsilon \to 0} \left( u'[W_{t+1}] \right)^{[1]} \varepsilon + \ldots + \frac{1}{Q!} \lim_{\varepsilon \to 0} \left( u'[W_{t+1}] \right)^{[Q]} \varepsilon^Q,$$ (C7)

where $\left( u'[W_{t+1}] \right)^{[j]}$ represents the $j$th derivative of $u' [W_{t+1}]$ with respect to $\varepsilon$. I notice that the term
\[ \lim_{\varepsilon \to 0} u'[W_{t+1}] \] is constant and use (C7) to express the risk premium (C4) as

\[ E_t (R_{kt+1} - R_f) = \varepsilon^2 a_{kt}[\varepsilon], \quad (C8) \]

with

\[ a_{kt}[\varepsilon] = \sum_{j=1}^{Q} \frac{1}{(j-1)!} Cov_t \left( -\lim_{\varepsilon \to 0} \frac{1}{j} u'[W_{t+1}] \right) \varepsilon^{j-1}. \quad (C9) \]

(C9) simplifies to

\[ a_{kt}[\varepsilon] = \sum_{j=1}^{Q} \frac{1}{(j-1)!} a_{kt}^{[j-1]}[0] \varepsilon^{j-1}. \quad (C10) \]

where

\[ a_{kt}^{[j-1]}[0] = Cov_t \left( -\lim_{\varepsilon \to 0} \frac{1}{j} u'[W_{t+1}] \right) \varepsilon^{j-1}. \quad (C11) \]

For \( j = 1, 2, \) and 3,

\[ \left( u'[W_{t+1}] \right)^{[1]} = u''[W_{t+1}] W_{t+1}, \quad (C12) \]

\[ \left( u'[W_{t+1}] \right)^{[2]} = u'''[W_{t+1}] (W_{t+1}')^2 + u''[W_{t+1}] W_{t+1}'', \quad (C13) \]

\[ \left( u'[W_{t+1}] \right)^{[3]} = u''''[W_{t+1}] (W_{t+1}')^3 + 3 u'''[W_{t+1}] W_{t+1}' W_{t+1}'' + u''[W_{t+1}] W_{t+1}''', \quad (C14) \]

where

\[ W_{t+1}' = \frac{\partial W_{t+1}}{\partial \varepsilon}, \quad W_{t+1}'' = \frac{\partial^2 W_{t+1}}{\partial \varepsilon^2}, \quad \text{and} \quad W_{t+1}''' = \frac{\partial^3 W_{t+1}}{\partial \varepsilon^3}, \quad (C15) \]

and

\[ \lim_{\varepsilon \to 0} W_{t+1}' = \omega_t Y_{t+1} \] and \( \lim_{\varepsilon \to 0} W_{t+1}'' = \lim_{\varepsilon \to 0} W_{t+1}''' = 0. \quad (C16) \]

I ignore high-order terms \( Q > 3 \) in (C8)-(C9) and use (C12)-(C14) to show that

\[ E_t (R_{kt+1} - R_f) = \varepsilon^2 a_{kt}[\varepsilon] = \varepsilon^2 \left( a_{kt}^{[0]}[0] + \frac{1}{1!} a_{kt}^{[1]}[0] \varepsilon + \frac{1}{2!} a_{kt}^{[2]}[0] \varepsilon^2 \right), \quad (C17) \]
which simplifies to

\[
E_t (R_{kt+1} - R_f) = \frac{1}{\varphi} \text{Cov}_t ((\omega_t R_{t+1}^+), R_{kt+1}) - \frac{\rho}{\varphi^2} \text{Cov}_t \left( ((\omega_t R_{t+1}^+) - E_t (\omega_t R_{t+1}^+))^2, R_{kt+1} \right) + \frac{\kappa}{2\varphi^3} \text{Cov}_t \left( ((\omega_t R_{t+1}^+) - E_t (\omega_t R_{t+1}^+))^3, R_{kt+1} \right),
\]

with

\[
\varphi = -u' [R_f], \quad \rho = -\frac{\varphi^2 u'' [R_f]}{2 u' [R_f]}, \quad \text{and} \quad \kappa = \frac{\varphi^3 u''' [R_f]}{3 u' [R_f]},
\]

I recall that the risk premium on the risky asset can be expressed as the negative of the covariance of the pricing kernel with the return on the risky asset

\[
E_t (R_{kt+1} - R_f) = -R_f \text{Cov}_t (M_{t+1}, R_{kt+1}),
\]

and I compare the risk premium \((C18)\) to \((C20)\) and recover the functional form of the pricing kernel \(M_{t+1}^+\):

\[
M_{t+1}^+ = \frac{1}{R_f} - \frac{1}{\varphi R_f} r_{Mt+1} + \frac{\rho}{\varphi^2 R_f} (r_{Mt+1}^2 - E_t r_{Mt+1}^2) - \frac{\kappa}{2\varphi^3 R_f} (r_{Mt+1}^3 - E_t r_{Mt+1}^3),
\]

where

\[
r_{Mt+1} = R_{Mt+1} - E_t R_{Mt+1} \quad \text{where} \quad R_{Mt+1} = \bar{\sigma}_t R_{t+1} \quad \text{and} \quad \bar{\sigma}_t = \mathcal{W}_t \omega_t.
\]

When \(\rho = \kappa = 0\), \((C21)\) reduces to the pricing kernel in the CAPM model. When \(\kappa = 0\), \((C21)\) reduces to the pricing kernel in the market co-skewness models of \(\varphi\) and \(\nu\). When \(\rho \neq 0\) and \(\kappa \neq 0\), \((C21)\) reduces to the pricing kernel in \(\varphi\). The price of the market co-skewness factor is \(-\frac{\rho}{\varphi^2 R_f} < 0\), while the price of risk of the market co-kurtosis is \(\frac{\kappa}{2\varphi^3 R_f} > 0\).

To summarize, the small noise expansion produces existing pricing kernels in a one-period model with two dates. The pricing kernel is a polynomial function of the market return. In Appendix D, I show that when the time interval is extended to a two-period model with three dates, the pricing kernel as opposed to \((C21)\) is a function of the market return, the market volatility, the market skewness, and the market kurtosis. I show that the prices of these risk factors are related to risk-aversion, skewness preference, kurtosis preference, and high-order preferences. These prices are not to be confused with the price of risk of powers of the market return.
Appendix D: Pricing Kernels in a Two-Period Model with Three Dates Using the Small Noise Expansion Series

D.1. Expected Return Decomposition

As opposed to Appendix C, I derive the pricing kernel when the representative agent maximizes his expected utility over a two-period \([t, t+2]\) interval with three dates \(t, t+1,\) and \(t+2\). I define \(R^e_{t+1} = R_{t+1} - R_f\) as the vector of excess return on the risky assets. The representative agent optimization problem is

\[
\max_{\{\omega_t\}} E_t \left( \max_{\{\omega_{t+1}\}} E_{t+1} \left( u \left[ \mathcal{W}_{t+2} \right] \right) \right) \tag{D1}
\]

with

\[
\mathcal{W}_{t+2} = \mathcal{W}_t \left( R_f + \omega_t R^e_{t+1} \right) \left( R_f + \omega_{t+1} R^e_{t+2} \right). \tag{D2}
\]

where

\[
R_{kt+1} - E_t R_{kt+1} = \varepsilon Y_{kt+1} \text{ and } R_{kt+2} - E_{t+1} R_{kt+2} = \varepsilon Y_{kt+2}. \tag{D3}
\]

Without loss of generality, I assume that \(\mathcal{W}_t = 1\) and solve (D1) in two steps. I first solve

\[
V_{t+1} = \max_{\{\omega_{t+1}\}} E_{t+1} \left( u \left[ \mathcal{W}_{t+2} \right] \right). \tag{D4}
\]

Second, given \(V_{t+1}\) (in D4), I solve

\[
\max_{\{\omega_t\}} E_t \left( V_{t+1} \right). \tag{D5}
\]

**STEP 1:** The first-order conditions of (D4) are

\[
E_{t+1} \left( u' \left[ \mathcal{W}_{t+2} \right] \left( R_{kt+2} - R_f \right) \right) = 0, \text{ for } k = 1, 2, \ldots \tag{D6}
\]

where

\[
R_{kt+2} - E_{t+1} R_{kt+2} = \varepsilon Y_{kt+2}. \tag{D7}
\]

I apply the covariance definition, \(Cov_{t+1} (X, Y) = E_{t+1} (XY) - E_{t+1} (X) E_{t+1} (Y)\), to (D6) and show that the
risk premium on the risky asset is

\[ E_{t+1} (R_{kt+2} - R_f) = -\text{Cov}_{t+1} \left( \frac{u' [\mathcal{W}_{t+2}]}{E_{t+1} u' [\mathcal{W}_{t+2}], R_{kt+2}} \right). \]  

(D8)

The Taylor expansion series of \( u' [\mathcal{W}_{t+2}] \) around \( \varepsilon = 0 \) produces

\[ u' [\mathcal{W}_{t+2}] = \sum_{j=0}^{Q} \frac{1}{j!} \left( \lim_{\varepsilon \to 0} \left( u' [\mathcal{W}_{t+2}] \right)^{[j]} \right) \varepsilon^j, \]

(D9)

where \( \left( u' [\mathcal{W}_{t+2}] \right)^{[j]} \) represents the \( j \)th derivative of \( u' [\mathcal{W}_{t+2}] \) with respect to \( \varepsilon \). I replace (D9) in (D8) and obtain

\[ E_{t+1} (R_{kt+2} - R_f) = \varepsilon^2 a_{kt+1} [\varepsilon], \]

(D10)

where

\[ a_{kt+1} [\varepsilon] = \sum_{j=1}^{Q} \frac{1}{(j-1)!} a_{kt+1}^{[j-1]} [0] \varepsilon^{j-1} \]

(D11)

and

\[ a_{kt+1}^{[j-1]} [0] = \text{Cov}_{t+1} \left( \frac{1}{j} \left( u' [\mathcal{W}_{t+2}] \right)^{[j]} \right) \frac{\partial \varepsilon}{\partial a_{kt+1}}. \]

(D12)

For \( j = 1, 2, \) and \( 3, \)

\[ 
\begin{align*}
(u' [\mathcal{W}_{t+2}])^{[1]} &= u'' [\mathcal{W}_{t+2}] \mathcal{W}_{t+2}' , \\
(u' [\mathcal{W}_{t+2}])^{[2]} &= u'' [\mathcal{W}_{t+2}] \left( \mathcal{W}_{t+2}' \right)^2 + u'' [\mathcal{W}_{t+2}] \mathcal{W}_{t+2}'' , \\
(u' [\mathcal{W}_{t+2}])^{[3]} &= u''' [\mathcal{W}_{t+2}] \left( \mathcal{W}_{t+2}' \right)^3 + 3u'' [\mathcal{W}_{t+2}] \mathcal{W}_{t+2}' \mathcal{W}_{t+2}'' + u'' [\mathcal{W}_{t+2}] \mathcal{W}_{t+2}'' , \\
(u' [\mathcal{W}_{t+2}])^{[4]} &= u''' [\mathcal{W}_{t+2}] \left( \mathcal{W}_{t+2}' \right)^4 + 6u''' [\mathcal{W}_{t+2}] \left( \mathcal{W}_{t+2}' \right)^2 \mathcal{W}_{t+2}'' + u'' [\mathcal{W}_{t+2}] \mathcal{W}_{t+2}'' .
\end{align*}
\]

where

\[ 
\begin{align*}
\mathcal{W}_{t+2}' &= \frac{\partial \mathcal{W}_{t+2}}{\partial \varepsilon} , \\
\mathcal{W}_{t+2}'' &= \frac{\partial^2 \mathcal{W}_{t+2}}{\partial^2 \varepsilon} , \\
\mathcal{W}_{t+2}''' &= \frac{\partial^3 \mathcal{W}_{t+2}}{\partial^3 \varepsilon} , \\
\mathcal{W}_{t+2}'''' &= \frac{\partial^4 \mathcal{W}_{t+2}}{\partial^4 \varepsilon} .
\end{align*}
\]

(D13) \( \text{to} \) (D18)
and

\[
\begin{align*}
\lim_{\varepsilon \to 0} \frac{\partial W_{t+2}}{\partial \varepsilon} &= R_f (\omega_t Y_{t+1}) + R_f (\omega_{t+1} Y_{t+2}), \\
\lim_{\varepsilon \to 0} \frac{\partial^2 W_{t+2}}{\partial^2 \varepsilon} &= 2 (\omega_t Y_{t+1}) (\omega_{t+1} Y_{t+2}), \\
\lim_{\varepsilon \to 0} \frac{\partial^3 W_{t+2}}{\partial^3 \varepsilon} &= 0, \\
\lim_{\varepsilon \to 0} \frac{\partial^4 W_{t+2}}{\partial^4 \varepsilon} &= 0.
\end{align*}
\]

Combining (D10) and (D7) produces

\[
R_{kt+2} = R_f + \varepsilon^2 a_{kt+1} [e] + \varepsilon Y_{kt+2}.
\]

I use assumption (D7) and then expand (D12) for \(j = 1, \ldots, 4\), ignoring the cross-product terms to show:

\[
\begin{align*}
a_{[0]}^{[0]} &= \frac{1}{\varrho} R_{f} \text{Cov}_{t+1} (\omega_{t+1} Y_{t+2}, Y_{kt+2}), \\
a_{[1]}^{[0]} &= -\frac{\rho}{\varrho^2} R_{f}^2 \text{Cov}_{t+1} \left( (\omega_{t+1} Y_{t+2})^2, Y_{kt+2} \right), \\
a_{[2]}^{[0]} &= \frac{\kappa}{\varrho^3} R_{f}^3 \text{Cov}_{t+1} \left( (\omega_{t+1} Y_{t+2})^3, Y_{kt+2} \right), \\
a_{[3]}^{[0]} &= -\frac{\delta}{\varrho^4} R_{f}^4 \text{Cov}_{t+1} \left( (\omega_{t+1} Y_{t+2})^4, Y_{kt+2} \right),
\end{align*}
\]

with

\[
\varrho = -\frac{u' [R_{f}^2]}{u'' [R_{f}^2]}, \quad \rho = \frac{\varrho^2 u'' [R_{f}^2]}{2 u' [R_{f}^2]}, \quad \kappa = \frac{\varrho^3 u''' [R_{f}^2]}{3 u' [R_{f}^2]}, \quad \delta = \frac{\varrho^4 u'''' [R_{f}^2]}{4 u' [R_{f}^2]}. \tag{D28}
\]

**STEP 2:** Now, I assume (D23) where \(a_{[j-1]}^{[0]} [0]\) is defined in (D24)–(D27) and solve (D5)

\[
\max_{\{\omega_t\}} E_t (V_{t+1}). \tag{D29}
\]

The first-order conditions are

\[
E_t \left( u' [W_{t+2}] (R_{kt+1} - R_f) \right) = 0, \tag{D30}
\]

where

\[
R_{kt+1} - E_t R_{kt+1} = \varepsilon Y_{kt+1} \tag{D31}
\]
and

$$R_{kt+2} = R_f + \varepsilon^2 a_{kt+1} [\varepsilon] + \varepsilon Y_{kt+2}. \quad (D32)$$

I recall that the proof of (D32) is given in **STEP 1**. I apply the definition of the covariance, $Cov_t(X, Y) = E_t(XY) - E_t(X)E_t(Y)$, to (D30) and show that the risk premium on the risky asset is

$$E_t (R_{kt+1} - R_f) = -Cov_t \left( \frac{u'[W_{t+2}]}{E_t u'[W_{t+2}]} , R_{kt+1} \right). \quad (D33)$$

The Taylor expansion series of $u'[W_{t+2}]$ around $\varepsilon = 0$ produces

$$u'[W_{t+2}] = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \lim_{\varepsilon \to 0} \left( u'[W_{t+2}] \right)^{[j]} \right) \varepsilon^j, \quad (D34)$$

where $\left( u'[W_{t+2}] \right)^{[j]}$ represents the $j$th derivative of $u'[W_{t+2}]$ with respect to $\varepsilon$. I replace (D34) in (D33) and show that

$$E_t (R_{kt+1} - R_f) = \varepsilon^2 a_{kt} [\varepsilon], \quad (D35)$$

with

$$a_{kt} [\varepsilon] = \sum_{j=1}^{\infty} \frac{1}{(j-1)!} a_{kt}^{[j-1]} [0] \varepsilon^{j-1} \quad (D36)$$

and

$$a_{kt}^{[j-1]} [0] = Cov_t \left( \lim_{\varepsilon \to 0} \frac{1}{j} \left( u'[W_{t+2}] \right)^{[j]} , Y_{kt+1} \right). \quad (D37)$$

I notice that

$$W_{t+2} = \left( R_f + \omega_t R_{t+1}^c \right) \left( R_f + \omega_{t+1} R_{t+2}^c \right). \quad (D38)$$

Since I show in **STEP 1** that

$$R_{kt+2} = R_f + \varepsilon^2 a_{kt+1} (\varepsilon) + \varepsilon Y_{kt+2}. \quad (D39)$$

The investor’s wealth (D38) can be expanded as follows

$$W_{t+2} = \left( R_f + \omega_t R_{t+1}^c \right) \left( R_f + \omega_{t+1} \left( \varepsilon^2 a_{t+1} (\varepsilon) + \varepsilon Y_{t+2} \right) \right). \quad (D40)$$
Therefore,

\[
\begin{align*}
\lim_{\varepsilon \to 0} \frac{\partial \mathcal{W}_{t+2}^j}{\partial \varepsilon} &= R_f (\omega, Y_{t+1}) + R_f (\omega_{t+1} Y_{t+2}), \quad (D41) \\
\lim_{\varepsilon \to 0} \frac{\partial^2 \mathcal{W}_{t+2}^j}{\partial \varepsilon^2} &= 2R_f \omega_{t+1} a_{t+1} (0) + 2 (\omega, Y_{t+1}) (\omega_{t+1} Y_{t+2}), \quad (D42) \\
\lim_{\varepsilon \to 0} \frac{\partial^3 \mathcal{W}_{t+2}^j}{\partial \varepsilon^3} &= 6R_f \omega_{t+1} a_{t+1}' (0) + 6 (\omega, Y_{t+1}) (\omega_{t+1} a_{t+1} (0)), \quad (D43) \\
\lim_{\varepsilon \to 0} \frac{\partial^4 \mathcal{W}_{t+2}^j}{\partial \varepsilon^4} &= 24 (\omega, Y_{t+1}) (\omega_{t+1} a_{t+1}' (0)) + 12R_f \omega_{t+1} a_{t+1}'' (0). \quad (D44)
\end{align*}
\]

I use equations (D13)–(D17) and (D41)–(D44) to expand \(a_{kt}^{j-1}[0]\), ignore the cross-product terms, and show:

\[
\begin{align*}
a_{kt}^{[0]}[0] &= \frac{R_f}{\varrho} \text{Cov}_t (\omega, Y_{t+1}, Y_{kt+1}), \quad (D45) \\
a_{kt}^{[1]}[0] &= -\frac{\rho}{\varrho^2} R_f^2 \text{Cov}_t (\omega, Y_{t+1})^2, Y_{kt+1}) + \frac{(1-\rho)}{\varrho^2} R_f^2 \text{Cov}_t ((\omega_{t+1} Y_{t+2})^2, Y_{kt+1}), \quad (D46) \\
a_{kt}^{[2]}[0] &= \frac{\kappa}{\varrho^3} R_f^3 \text{Cov}_t ((\omega, Y_{t+1})^3, Y_{kt+1}) + \frac{(\kappa - 2\rho)}{\varrho^3} R_f^3 \text{Cov}_t ((\omega_{t+1} Y_{t+2})^3, Y_{kt+1}), \quad (D47) \\
a_{kt}^{[3]}[0] &= -\frac{\delta}{\varrho^3} R_f^4 \text{Cov}_t (\omega, Y_{t+1})^4, Y_{kt+1}) - \frac{(\delta - 3\kappa)}{\varrho^3} R_f^4 \text{Cov}_t ((\omega_{t+1} Y_{t+2})^4, Y_{kt+1}). \quad (D48)
\end{align*}
\]

I combine (D36) and (D31) and show

\[
R_{kt+1} = R_f + \varepsilon^2 a_{kt} [\varepsilon] + \varepsilon Y_{kt+1}, \quad (D49)
\]

where \(a_{kt} [\varepsilon]\) is defined in (D36) with \(a_{kt}^{[j-1]}[0]\) defined in (D45)–(D48).

**D.2. Pricing Kernels**

In this section, I use the relation between the pricing kernel and the risk premium on risky assets

\[
E_t (R_{kt+1} - R_f) = -R_f \text{Cov}_t (\mathbf{M}_{kt+1}, R_{kt+1}) \quad (D50)
\]

to recover the functional form of the pricing kernel.

**Pricing Kernel with Co-skewness and Volatility Risk When Q=1**
I derive the pricing kernel when the order in (D36) is $Q = 1$. I recall that, from (D49) and (D23),

$$R_{kt+1} = R_f + \varepsilon^2 a_{kt} [e] + \varepsilon Y_{kt+1} \text{ and } R_{kt+2} = R_f + \varepsilon^2 a_{kt+1} [e] + \varepsilon Y_{kt+2}. \tag{D51}$$

When $Q = 1$, the risk premium (D35) is

$$E_t (R_{kt+1} - R_f) = \frac{1}{\varphi} R_f \text{Cov}_t (\omega_t, Y_{kt+1}) \varepsilon^2 - \frac{\rho}{\varphi^2} R_f^2 \text{Cov}_t \left( \left( \omega_t, Y_{kt+1} \right)^2, Y_{kt+1} \right) \varepsilon^3 \tag{D52}$$

$$+ \left( 1 - \frac{\rho}{\varphi^2} \right) R_f \text{Cov}_t \left( \left( \omega_t + Y_{kt+1} \right)^2, Y_{kt+1} \right) \varepsilon^3,$$

which simplifies to

$$E_t (R_{kt+1} - R_f) = \frac{1}{\varphi} R_f \text{Cov}_t \left( r_{Mt+1}, R_{kt+1} \right) - \frac{\rho}{\varphi^2} R_f^2 \text{Cov}_t \left( \sigma^2_{Mt+1}, R_{kt+1} \right) \tag{D53}$$

$$+ \left( 1 - \frac{\rho}{\varphi^2} \right) \text{Cov}_t \left( \left( r_{Mt+2} \right)^2, R_{kt+1} \right),$$

where

$$r_{Mt+1} = R_{Mt+1} - E_t R_{Mt+1}, R_{Mt+1} = \varpi_t R_t + 1, \text{ and } \varpi_t = \lim_{\varepsilon \to 0} \mathcal{W}_t \omega_t \text{ with } \tau = t, t + 1. \tag{D54}$$

I notice that

$$\text{Cov}_t \left( \left( r_{Mt+2} \right)^2, R_{kt+1} \right) = \text{Cov}_t \left( E_t \left( \left( r_{Mt+2} \right)^2, R_{kt+1} \right) \right) = \text{Cov}_t \left( \sigma^2_{Mt+1} - E_t \sigma^2_{Mt+1} \right), \tag{D55}$$

where $\sigma^2_{Mt+1}$ is the variance of the market return. I compare (D53) to (D50) and recover the pricing kernel

$$\mathcal{M}_{t+1} = \frac{1}{R_f} - \frac{1}{\varphi} r_{Mt+1} + \frac{\rho}{\varphi^2} R_f \left( \sigma^2_{Mt+1} - E_t \sigma^2_{Mt+1} \right) - \left( 1 - \frac{\rho}{\varphi^2} \right) \left( \sigma^2_{Mt+1} - E_t \sigma^2_{Mt+1} \right). \tag{D56}$$

### Pricing Kernel with Co-skewness and Volatility Risk When $Q=2$

I derive the pricing kernel when the order in (D36) is $Q = 2$. The risk premium (D35) is

$$E_t (R_{kt+1} - R_f) = \frac{1}{\varphi} R_f \text{Cov}_t \left( r_{Mt+1}, R_{kt+1} \right) - \frac{\rho}{\varphi^2} R_f^2 \text{Cov}_t \left( \sigma^2_{Mt+1}, R_{kt+1} \right) \tag{D57}$$

$$+ \left( 1 - \frac{\rho}{\varphi^2} \right) \text{Cov}_t \left( \sigma^2_{Mt+1}, R_{kt+1} \right) + \frac{\kappa}{2\varphi^3} R_f^3 \text{Cov}_t \left( r_{Mt+1}^3, R_{kt+1} \right)$$

$$+ \left( \kappa - 2\rho \right) \text{Cov}_t \left( s_{Mt+1}, R_{kt+1} \right).$$
where \( s_{Mt+1} = E_{t+1} [r^3_{Mt+1}] \) is the skewness of the market return. I compare (D57) to (D50) and recover the pricing kernel

\[
M_{t+1} = \frac{1}{R_f} - \frac{1}{\theta} r^2_{Mt+1} + \frac{\rho}{\theta^2} R_f \left( r^2_{Mt+1} - E_t r^2_{Mt+1} \right) - \frac{\kappa}{2 \theta^3} R_f^2 \left( r^3_{Mt+1} - E_t r^3_{Mt+1} \right)
\]

\[\text{(D58)}\]

\[\text{Pricing Kernel with Co-skewness and Volatility Risk When Q=3}\]

I derive the pricing kernel when the order in (D36) is \( Q = 3 \). The risk premium (D35) is

\[
E_t (R_{kt+1} - R_f) = \frac{1}{\theta} R_f Cov_t (r_{Mt+1}, R_{kt+1}) - \frac{\rho}{\theta^2} R_f^2 Cov_t (r^2_{Mt+1}, R_{kt+1})
\]

\[\text{(D59)}\]

\[\text{where } k_{Mt+1} = E_{t+1} [r^4_{Mt+1}] \text{ is the kurtosis of the market return. I compare (D59) to (D50) and recover the pricing kernel}\]

\[
M_{t+1} = \frac{1}{R_f} - \frac{1}{\theta} r^2_{Mt+1} + \frac{\rho}{\theta^2} R_f \left( r^2_{Mt+1} - E_t r^2_{Mt+1} \right) - \frac{\kappa}{2 \theta^3} R_f^2 \left( r^3_{Mt+1} - E_t r^3_{Mt+1} \right)
\]

\[\text{(D60)}\]

\[\text{I derive the pricing kernel when the order in (D36) is } Q = 3. \text{ The risk premium (D35) is}\]

\[
E_t (R_{kt+1} - R_f) = \frac{1}{\theta} R_f Cov_t (r_{Mt+1}, R_{kt+1}) - \frac{\rho}{\theta^2} R_f^2 Cov_t (r^2_{Mt+1}, R_{kt+1})
\]

\[\text{(D59)}\]

\[\text{where } k_{Mt+1} = E_{t+1} [r^4_{Mt+1}] \text{ is the kurtosis of the market return. I compare (D59) to (D50) and recover the pricing kernel}\]

\[
M_{t+1} = \frac{1}{R_f} - \frac{1}{\theta} r^2_{Mt+1} + \frac{\rho}{\theta^2} R_f \left( r^2_{Mt+1} - E_t r^2_{Mt+1} \right) - \frac{\kappa}{2 \theta^3} R_f^2 \left( r^3_{Mt+1} - E_t r^3_{Mt+1} \right)
\]

\[\text{(D60)}\]

I follow \( ? \) and consider in log form the SDF in the \( ? \) model

\[
\log \left( \frac{M_{t+1}}{M_t} \right) = \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) \log (R_{M_{t+1}}), \tag{E1}
\]

with \( \theta \equiv \frac{1 - \gamma}{1 - 1/\psi} \) where \( \gamma \) is the coefficient of relative risk aversion, \( \psi \) is the elasticity of intertemporal substitution, \( \beta \) is the subjective discount factor and \( g_{t+1} \) is the log consumption growth. \( ? \) specify the evolution of (log) consumption growth \( g_{t+1} \) as

\[
g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \quad x_{t+1} = \rho x_t + \varphi \sigma_t e_{t+1}, \tag{E2}
\]

\[
\sigma_{t+1}^2 = \sigma^2 + v_1 (\sigma_t^2 - \sigma^2) + \varphi \sigma_t \xi_{t+1}, \quad \xi_{t+1}, e_{t+1}, \eta_{t+1} \sim i.i.d. \mathcal{N}(0,1). \tag{E3}
\]

In the model, \( x_t \) is a persistent component of the expected consumption growth rate, and \( \sigma_t^2 \) is the conditional variance of consumption with unconditional mean \( \sigma^2 \). The time-series dynamics (E2)–(E3), combined with the SDF (E1), produces a log SDF that can be expressed as a function of market variance.

Because the expressions are derived using standard techniques, I intend to be brief. The return on a consumption claim can be approximated as

\[
\log R_{M_{t+1}} = \kappa_0 + \kappa_1 z_{t+1} + g_{t+1} - z_t, \tag{E4}
\]

with

\[
\kappa_0 = \log \left( 1 + e^z \right) - \frac{e^z}{(1 + e^z) \bar{z}}, \quad \kappa_1 = \frac{e^z}{(1 + e^z)},
\]

where \( z_t \) is the log price-consumption ratio. The approximate solution for the log price-consumption ratio is

\[
z_t = A_0 + A_1 x_t + A_2 \sigma_t^2. \tag{E5}
\]

To show that the log SDF is a function of the innovation in the market variance, I write the innovation in the log market portfolio return as

\[
\log (R_{M_{t+1}}) - E_t \log (R_{M_{t+1}}) = \kappa_1 A_1 \varphi \sigma_t e_{t+1} + \kappa_1 A_2 (\sigma_t \xi_{t+1}) + g_{t+1}. \tag{E6}
\]
I derive the conditional variance of the market portfolio return

\[ \sigma^2_{M,t} = (1 + \kappa_1^2 A_1^2 \varphi_e^2) \sigma_t^2 + \kappa_1^2 A_2^2 \sigma_z^2, \]  

(E7)

which implies

\[ \sigma^2_{M,t+1} - E_t (\sigma^2_{M,t+1}) = (1 + \kappa_1^2 A_1^2 \varphi_e^2) \sigma_z \xi_{t+1}. \]  

(E8)

The variance risk premium is

\[ E_t (\sigma^2_{M,t+1}) - E_t^Q (\sigma^2_{M,t+1}) = - \text{Cov}_t \left( \log \frac{M_{t+1}}{M_t}, \sigma^2_{M,t+1} \right) \]

\[ = \lambda_{m, \xi} \left( 1 + \kappa_1^2 A_1^2 \varphi_e^2 \right) \sigma_z^2, \]  

with \( \lambda_{m, \xi} = (1 - \theta) \kappa_1 A_2. \)  

(E10)

The log pricing kernel (E1) simplifies to

\[ \log \frac{M_{t+1}}{M_t} = \theta \log \beta + (\theta - 1) (\kappa_0 - z_t) + (\theta - 1) \kappa_1 z_{t+1} + \left( (\theta - 1) - \frac{\theta}{\psi} \right) g_{t+1}. \]  

(E11)

Under the time-series assumptions (E2)-(E3), I replace \( g_{t+1}, z_{t+1} \) and \( z_t \) in equation (E11) and use (E8) to simplify the log pricing kernel:

\[ \log \frac{M_{t+1}}{M_t} = \zeta_t + D_1 (g_{t+1} - E_t (g_{t+1})) + D_2 (x_{t+1} - E_t (x_{t+1})) + D_3 (\sigma^2_{M,t+1} - E_t (\sigma^2_{M,t+1})), \]  

(E12)

where \( D_1, D_2, \) and \( D_3 \) are

\[ D_1 = (\theta - 1) - \frac{\theta}{\psi}, D_2 = -(1 - \theta) \kappa_1 A_1, D_3 = -\frac{(1 - \theta) \kappa_1 A_2}{(1 + \kappa_1^2 A_1^2 \varphi_e^2)}, \]

with

\[ \zeta_t = \theta \log \beta + (\theta - 1) (\kappa_0 - z_t) + (\theta - 1) \kappa_1 A_0 + (\theta - 1) \kappa_1 A_1 \rho x_t \]

\[ + \left( (\theta - 1) - \frac{\theta}{\psi} \right) (\mu + x_t) + (\theta - 1) \kappa_1 A_2 \left( \sigma^2 + \nu \left( \sigma^2 - \sigma_e^2 \right) \right). \]  

(E13)

This was the final step.
Appendix F: The Pricing Kernel When the Utility Function Depends on the Market Return and Volatility Factors

I consider a representative agent who maximizes his expected utility over wealth. The parsimonious way to allow the representative agent’s utility function to be a function of the market volatility is to assume that the agent can only invest in the market return and the variance swap contract. A return variance swap has zero net market value at entry \((?)\). At maturity, the payoff to the long side of the swap is equal to the difference between the realized variance over the life of the contract and a constant called the variance swap rate

\[ V_{SMt+1} = \sigma_{Mt+1}^2 - E_t^Q \sigma_{Mt+1}^2, \quad (F1) \]

where \(\sigma_{Mt+1}^2\) denotes the proxy for the realized variance between \(t\) and \(t+1\). \(E_t^Q (\sigma_{Mt+1}^2)\) denotes the fixed variance swap rate that is determined at time \(t\). The representative agent chooses the portfolio weights \(\omega_M\) and \(\omega_v\) to maximize his expected utility

\[ \max_{\{\omega_M, \omega_v\}} E_t u [W_{t+1}], \quad (F2) \]

where

\[ W_{t+1} = R_f + \omega_M (R_{Mt+1} - R_f) + \omega_v (V_{SMt+1} - R_f). \quad (F3) \]

The first-order conditions are

\[ E_t \left( u' [W_{t+1}] (V_{SMt+1} - R_f) \right) = 0 \quad \text{and} \quad E_t \left( u' [W_{t+1}] (R_{Mt+1} - R_f) \right) = 0, \quad (F4) \]

and the pricing kernel is

\[ M_{t+1} = \frac{u' [W_{t+1}]}{E_t (u' [W_{t+1}])}. \quad (F5) \]

The first-order Taylor expansion series of the marginal utility \(u' (W_{t+1})\) around \((E_t R_{Mt+1}, E_t V_{SMt+1})\) produces a pricing kernel

\[ M_{t+1} = G_0 + G_1 (R_{Mt+1} - E_t R_{Mt+1}) + G_2 (\sigma_{Mt+1}^2 - E_t \sigma_{Mt+1}^2), \quad (F6) \]

with

\[ G_0 = \frac{u' [E_t W_{t+1}]}{E_t (u' [W_{t+1}])}, \quad G_1 = \omega_M \frac{u'' [E_t W_{t+1}]}{E_t (u' [W_{t+1}])}, \quad \text{and} \quad G_2 = \omega_v \frac{u'' [E_t W_{t+1}]}{E_t (u' [W_{t+1}])}. \quad (F7) \]
Since $u'' [E_t W_{t+1}] < 0$, the price of the market factor is positive if $\omega_M$ is positive, and the price of volatility risk is negative if $\omega_v$ is negative. Bondareko (2007) considers an investor who maximizes the expected value of the constant (CRRA) utility function where the investor’s wealth is defined by (F3) and shows (in Table 6, page 47) that $\omega_m < 0$ and $\omega_v < 0$ when the investor’s risk aversion takes the values 1, 2, 3, 5, 10, 20, and 50. Hence, $G_1 > 0$ and $G_2 > 0$.

Furthermore, in a one-period model, any high-order Taylor expansion series of the marginal utility $u' [W_{t+1}]$ in (F5) will not produce a pricing kernel function of the market volatility, market skewness, and market kurtosis as presented in Appendix D.
Table I: Preference Parameters and Implied Prices of Risk Using Industry Portfolio Returns (Robustness to Size, Book-to-Value, and Momentum Factors):

Table I presents results of GMM tests of the Euler equation condition, $EM_{t+1}R_{t+1} = 1$ using the pricing kernel derived in Proposition 1 when the investment horizon $T - t = 2$, augmented with $T$ size and book-to-market factors. I estimate the preference parameters by using the $T$ weighting matrix $ER_{t+1}$. Column (1) presents the mean of the pricing kernel and Columns (2) and (3) present the risk aversion and skewness preference, respectively. Column (4) presents the Hansen and Jagannathan distance measure with $p$-values for the test of model specification. Columns (5)–(7) present the annualized price of market, co-skewness, and market volatility risk, using the estimated preference parameters. The set of returns I use in my estimations are those of 30 industry-sorted portfolios augmented by the return on a one-month Treasury bill, covering the sample periods 01/1986–12/2000, 01/1990–12/2000, 01/1996–12/2006, 01/1990–12/2000, and 01/1986–12/2006. For the market portfolio, I use the value-weighted NYSE/AMEX/NASDAQ index, also known as the value-weighted index of the Center for Research in Security Prices (CRSP). As my proxy for the volatility of the market return, I use the Chicago Board Options Exchange (CBOE) VXO, the VIX implied volatilities, and the Realized Volatility RV, respectively. In Panel A, I present the results when I use the VXO. Panel B presents the results when I use the VIX. Panel C presents the results when I use the realized volatility RV.

### Panel A: Market Volatility: VXO

<table>
<thead>
<tr>
<th>Subperiod: 01/1986–12/2000</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\gamma}$</th>
<th>$\hat{\rho}$</th>
<th>HJ Dist</th>
<th>$\lambda_M$ (%)</th>
<th>$\lambda_{SKD}$ (%)</th>
<th>$\lambda_{VOL}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>0.996</td>
<td>4.211</td>
<td>1.095</td>
<td>0.116</td>
<td>10.082</td>
<td>-2.072</td>
<td>-0.424</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.032)</td>
<td>(0.002)</td>
<td>(0.162)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subperiod: 01/1996–12/2006</td>
<td>Coefficient</td>
<td>0.997</td>
<td>4.009</td>
<td>1.827</td>
<td>0.175</td>
<td>9.701</td>
<td>-3.185</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.169)</td>
<td>(0.02)</td>
<td>(0.002)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subperiod: 01/1996–12/2006</td>
<td>Coefficient</td>
<td>0.996</td>
<td>4.318</td>
<td>1.050</td>
<td>0.085</td>
<td>9.961</td>
<td>-1.976</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.023)</td>
<td>(0.001)</td>
<td>(0.081)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel B: Market Volatility: VIX

<table>
<thead>
<tr>
<th>Subperiod: 01/1990–12/2000</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\gamma}$</th>
<th>$\hat{\rho}$</th>
<th>HJ Dist</th>
<th>$\lambda_M$ (%)</th>
<th>$\lambda_{SKD}$ (%)</th>
<th>$\lambda_{VOL}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>0.996</td>
<td>3.403</td>
<td>1.741</td>
<td>0.199</td>
<td>6.931</td>
<td>-1.689</td>
<td>-0.896</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.268)</td>
<td>(0.270)</td>
<td>(0.037)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subperiod: 01/1996–12/2006</td>
<td>Coefficient</td>
<td>0.997</td>
<td>3.755</td>
<td>1.913</td>
<td>0.177</td>
<td>9.086</td>
<td>-2.926</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.206)</td>
<td>(0.264)</td>
<td>(0.003)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subperiod: 01/1990–12/2006</td>
<td>Coefficient</td>
<td>0.997</td>
<td>4.608</td>
<td>1.482</td>
<td>0.113</td>
<td>9.457</td>
<td>-2.667</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.061)</td>
<td>(0.027)</td>
<td>(0.006)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel C: Market Volatility: RV

<table>
<thead>
<tr>
<th>Subperiod: 01/1990–12/2000</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\gamma}$</th>
<th>$\hat{\rho}$</th>
<th>HJ Dist</th>
<th>$\lambda_M$ (%)</th>
<th>$\lambda_{SKD}$ (%)</th>
<th>$\lambda_{VOL}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>0.996</td>
<td>3.990</td>
<td>1.118</td>
<td>0.208</td>
<td>7.705</td>
<td>-1.376</td>
<td>-0.774</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.201)</td>
<td>(0.000)</td>
<td>(0.016)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subperiod: 01/1996–12/2006</td>
<td>Coefficient</td>
<td>0.997</td>
<td>3.636</td>
<td>1.138</td>
<td>0.172</td>
<td>8.174</td>
<td>-1.461</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.286)</td>
<td>(0.000)</td>
<td>(0.003)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subperiod: 01/1990–12/2006</td>
<td>Coefficient</td>
<td>0.997</td>
<td>3.623</td>
<td>1.215</td>
<td>0.097</td>
<td>7.027</td>
<td>-1.241</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.195)</td>
<td>(0.001)</td>
<td>(0.076)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table II: Descriptive Statistics on Dow 30 stocks

This table presents summary statistics for the monthly returns on Dow 30 Stocks. Maximum, minimum, mean, standard deviation (Std), skewness, and kurtosis are reported for each stock. The descriptive statistics are computed for the sample period from January 1990 to December 2006.

<table>
<thead>
<tr>
<th>Dow Jones Stocks</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Microsoft</td>
<td>-0.3435</td>
<td>0.4078</td>
<td>0.0251</td>
<td>0.1024</td>
<td>0.3978</td>
<td>4.6155</td>
</tr>
<tr>
<td>Honeywell</td>
<td>-0.3840</td>
<td>0.5105</td>
<td>0.0141</td>
<td>0.0901</td>
<td>-0.0367</td>
<td>9.3331</td>
</tr>
<tr>
<td>AT&amp;T Inc</td>
<td>-0.1876</td>
<td>0.2900</td>
<td>0.0097</td>
<td>0.0717</td>
<td>0.2064</td>
<td>4.1836</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>-0.1910</td>
<td>0.2228</td>
<td>0.0115</td>
<td>0.0660</td>
<td>-0.2103</td>
<td>4.0537</td>
</tr>
<tr>
<td>E.I. DuPont de Nemours</td>
<td>-0.1699</td>
<td>0.2174</td>
<td>0.0091</td>
<td>0.0673</td>
<td>0.1367</td>
<td>2.8528</td>
</tr>
<tr>
<td>Exxon Mobil</td>
<td>-0.1165</td>
<td>0.2322</td>
<td>0.0127</td>
<td>0.0465</td>
<td>0.6183</td>
<td>5.4267</td>
</tr>
<tr>
<td>General Electric</td>
<td>-0.1765</td>
<td>0.1924</td>
<td>0.0134</td>
<td>0.0626</td>
<td>0.1297</td>
<td>3.5668</td>
</tr>
<tr>
<td>General Motors</td>
<td>-0.2403</td>
<td>0.2766</td>
<td>0.0072</td>
<td>0.0951</td>
<td>0.1984</td>
<td>3.1706</td>
</tr>
<tr>
<td>International Business Machines</td>
<td>-0.2619</td>
<td>0.3538</td>
<td>0.0123</td>
<td>0.0890</td>
<td>0.3456</td>
<td>4.3209</td>
</tr>
<tr>
<td>Altria (was Philip Morris)</td>
<td>-0.2656</td>
<td>0.3427</td>
<td>0.0164</td>
<td>0.0830</td>
<td>-0.2708</td>
<td>5.0895</td>
</tr>
<tr>
<td>United Technologies</td>
<td>-0.3202</td>
<td>0.2461</td>
<td>0.0154</td>
<td>0.0711</td>
<td>-0.6543</td>
<td>6.1925</td>
</tr>
<tr>
<td>Procter and Gamble</td>
<td>-0.3570</td>
<td>0.2509</td>
<td>0.0135</td>
<td>0.0631</td>
<td>-0.7790</td>
<td>8.7933</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>-0.2146</td>
<td>0.4079</td>
<td>0.0157</td>
<td>0.0836</td>
<td>0.4065</td>
<td>4.7004</td>
</tr>
<tr>
<td>Boeing</td>
<td>-0.3457</td>
<td>0.1949</td>
<td>0.0120</td>
<td>0.0788</td>
<td>-0.5865</td>
<td>4.4729</td>
</tr>
<tr>
<td>Pfizer</td>
<td>-0.1707</td>
<td>0.2655</td>
<td>0.0151</td>
<td>0.0735</td>
<td>0.1404</td>
<td>2.9849</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>-0.1601</td>
<td>0.1881</td>
<td>0.0142</td>
<td>0.0629</td>
<td>0.0878</td>
<td>3.1813</td>
</tr>
<tr>
<td>3M Corporation</td>
<td>-0.1578</td>
<td>0.2580</td>
<td>0.0108</td>
<td>0.0581</td>
<td>0.4040</td>
<td>4.8393</td>
</tr>
<tr>
<td>Merck</td>
<td>-0.2577</td>
<td>0.2276</td>
<td>0.0114</td>
<td>0.0773</td>
<td>-0.0737</td>
<td>3.3652</td>
</tr>
<tr>
<td>Alcoa</td>
<td>-0.2387</td>
<td>0.5114</td>
<td>0.0115</td>
<td>0.0907</td>
<td>0.7795</td>
<td>6.9529</td>
</tr>
<tr>
<td>Walt Disney Co.</td>
<td>-0.2678</td>
<td>0.2415</td>
<td>0.0100</td>
<td>0.0775</td>
<td>-0.0686</td>
<td>3.9232</td>
</tr>
<tr>
<td>Hewlett-Packard</td>
<td>-0.3199</td>
<td>0.3539</td>
<td>0.0176</td>
<td>0.1099</td>
<td>0.0874</td>
<td>3.6069</td>
</tr>
<tr>
<td>McDonalds</td>
<td>-0.2567</td>
<td>0.1826</td>
<td>0.0114</td>
<td>0.0697</td>
<td>-0.2540</td>
<td>3.5148</td>
</tr>
<tr>
<td>JP Morgan Chase</td>
<td>-0.3468</td>
<td>0.3257</td>
<td>0.0161</td>
<td>0.1004</td>
<td>-0.1720</td>
<td>4.8161</td>
</tr>
<tr>
<td>Wal-Mart Stores</td>
<td>-0.2080</td>
<td>0.2643</td>
<td>0.0136</td>
<td>0.0730</td>
<td>0.1671</td>
<td>3.4579</td>
</tr>
<tr>
<td>Intel Corp</td>
<td>-0.4449</td>
<td>0.3382</td>
<td>0.0223</td>
<td>0.1213</td>
<td>-0.3005</td>
<td>3.6254</td>
</tr>
<tr>
<td>Verizon Communications</td>
<td>-0.2099</td>
<td>0.3901</td>
<td>0.0076</td>
<td>0.0721</td>
<td>0.8210</td>
<td>6.7551</td>
</tr>
<tr>
<td>Home Depot</td>
<td>-0.2059</td>
<td>0.3023</td>
<td>0.0193</td>
<td>0.0849</td>
<td>0.2564</td>
<td>3.5188</td>
</tr>
<tr>
<td>American Int’l Group</td>
<td>-0.2310</td>
<td>0.2387</td>
<td>0.0129</td>
<td>0.0665</td>
<td>0.0223</td>
<td>4.2054</td>
</tr>
<tr>
<td>CitiGroup</td>
<td>-0.3401</td>
<td>0.2608</td>
<td>0.0214</td>
<td>0.0889</td>
<td>-0.1032</td>
<td>4.3008</td>
</tr>
<tr>
<td>American Express IBM</td>
<td>-0.2933</td>
<td>0.2031</td>
<td>0.0139</td>
<td>0.0754</td>
<td>-0.8057</td>
<td>4.6244</td>
</tr>
</tbody>
</table>

(International Business Machines)
Table III: Preference Parameters and Implied Prices of Risk Using the Dow 30 Returns:

Table III presents results of GMM tests of the Euler equation condition, \( E_{t+1} R_{t+1} = 1 \) using the pricing kernel derived in Proposition 1 when the investment horizon \( T - t = 2 \) is augmented with size and book-to-market factors. I estimate the preference parameters by using the weighting matrix \( E_{t+1} R_{t+1}^\rho \). Column (1) presents the mean of the pricing kernel and Columns (2) and (3) present the risk aversion and skewness preference respectively. Column (4) presents the Hansen and Jagannathan distance measure with \( p \)-values for the test of model specification. Columns (5)–(7) present the annualized price of market, co-skewness, and market volatility risk, using the estimated preference parameters. The set of returns I use in my estimations are those of 30 Dow Jones returns augmented by the return on a one-month Treasury bill, covering the sample period 01/1990–12/2006. For the market portfolio, I use the value-weighted NYSE/AMEX/NASDAQ index, also known as the value-weighted index of the Center for Research in Security Prices (CRSP). As my proxy for the volatility of the market return, I use the Chicago Board Options Exchange (CBOE)’s VXO, the VIX implied volatilities, and the Realized Volatility RV, respectively. In Panel A, I present the results when I use the VXO, the VIX, and the realized volatility RV. Panel B presents the results when I control for the Fama and French and the momentum factors.

| Panel A | VXO | Coefficient 0.997 | 3.921 | 1.603 | 0.121 | 8.047 | –2.088 | –0.897 |
|         |     | \( p \)-value | (0.000) | (0.039) | (0.045) | (0.003) |         |         |
| VIX     | Coefficient 0.997 | 3.782 | 1.705 | 0.120 | 7.761 | –2.066 | –0.906 |
|         | \( p \)-value | (0.000) | (0.053) | (0.082) | (0.003) |         |         |
| RV      | Coefficient 0.997 | 5.750 | 1.003 | 0.101 | 11.154 | –2.582 | –0.148 |
|         | \( p \)-value | (0.000) | (0.008) | (0.000) | (0.319) |         |         |

Panel B

| VXO | Coefficient 0.997 | 4.771 | 1.394 | 0.115 | 9.791 | –2.689 | –0.869 |
|     | \( p \)-value | (0.000) | (0.046) | (0.008) | (0.004) |         |         |
| VIX | Coefficient 0.997 | 4.608 | 1.482 | 0.113 | 9.457 | –2.667 | –0.920 |
|     | \( p \)-value | (0.000) | (0.061) | (0.028) | (0.006) |         |         |
| RV  | Coefficient 0.9967 | 5.4074 | 1.0277 | 0.0913 | 10.4890 | –2.3396 | –1.2963 |
|     | \( p \)-value | (0.000) | (0.065) | (0.000) | (0.410) |         |         |
Table IV: Preference Parameters and Implied Prices of Risk When the Pricing Kernel Depends on Both Stochastic Volatility and Stochastic Skewness

Table IV presents results of GMM tests of the Euler equation condition, $EM_{t+1} \Delta R_{t+1} = 1$ using the pricing kernel derived in Proposition 2 when the investment horizon $T - t = 2$. I estimate the preference parameters by using the weighting matrix $ER_{t+1} \Delta R_{t+1}$. Column (1) presents the mean of the pricing kernel and Columns (2), (3), and (4) present the risk aversion, skewness preference, and kurtosis preference respectively. Column (5) presents the Hansen and Jagannathan distance measure with $p$-values for the test of model specification. Columns (6)–(9) present the annualized price of market, co-skewness, market co-kurtosis, market skewness risk, and market volatility risk, using the estimated preference parameters. In Panel A, the set of returns I use in my estimations are those of 30 industry returns augmented by the return on a one-month Treasury bill, covering the sample period 01/1990–12/2006. For the market portfolio, I use the value-weighted NYSE/AMEX/NASDAQ index, also known as the value-weighted index of the Center for Research in Security Prices (CRSP). As my proxy for the volatility of the market return, I use the Chicago Board Options Exchange (CBOE)’s VIX implied volatilities. As my proxy for the market skewness, I use the skewness measure implied from options prices. I use the formula to compute the skewness measure. In Panel B, I use the Dow 30 returns as the test asset.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$\frac{1}{\pi_r}$</th>
<th>$\frac{1}{\pi_p}$</th>
<th>$\rho$</th>
<th>$\kappa$</th>
<th>HJ Dist</th>
<th>$\lambda_M(%)$</th>
<th>$\lambda_{SKD}(%)$</th>
<th>$\lambda_{KUR}(%)$</th>
<th>$\lambda_{VOL}(%)$</th>
<th>$\lambda_{SM}(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: 30 Industry Returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient</td>
<td>0.997</td>
<td>2.958</td>
<td>1.264</td>
<td>1.913</td>
<td>0.1138</td>
<td>6.174</td>
<td>-3.781</td>
<td>2.152</td>
<td>-0.961</td>
<td>-2.130</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.107)</td>
<td>(0.010)</td>
<td>(0.061)</td>
<td>(0.077)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel B: Dow 30 returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient</td>
<td>0.997</td>
<td>3.390</td>
<td>1.197</td>
<td>1.848</td>
<td>0.109</td>
<td>6.687</td>
<td>-3.707</td>
<td>2.875</td>
<td>-1.099</td>
<td>-2.850</td>
</tr>
<tr>
<td>$p$-value</td>
<td>(0.000)</td>
<td>(0.010)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.102)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure I. Preference Parameters and Prices of Risk

Figure I depicts the risk aversion, skewness preferences, price of the volatility risk, and the price of co-skewness risk when I estimate the pricing kernel with constant preference parameters for different sample periods [1986 + j, 1986 + 10 + j] when \( j = 0, \ldots, 10 \). I report the preference parameters for different years \( j \). I estimate the preference parameters of the pricing kernel via GMM utilizing the Euler equation condition \( E_t M_{t+1} R_{t+1} = 1 \), where \( M_{t+1} \) represents the pricing kernel. I estimate the parameters by using the \( \Sigma \) weighting matrix. The sets of returns I use in my estimations are those of 30 industry-sorted portfolios covering the period January 1986 through December 2006, augmented by the return on a 30-day Treasury bill. I use the Chicago Board Options Exchange (CBOE)’s VOX as my proxy for the market volatility.
Figure II. Estimated Pricing Kernels

Figure II depicts point estimates of the pricing kernels estimated with constant preference parameters. The support for the graphs is the range of the return on the value-weighted index and the implied volatility difference. I estimate the preference parameters of the pricing kernel via GMM utilizing the Euler equation condition $E_t[M_{t+1} R_{t+1}] = 0$, where $M_{t+1}$ represents the pricing kernel. I estimate the parameters by using the $\omega$ weighting matrix. The sets of returns I use in my estimations are those of 30 industry-sorted portfolios covering the period January 1986 through December 2006, augmented by the return on a 30-day Treasury bill. I use the Chicago Board Options Exchange (CBOE)'s VXO as my proxy for the market volatility.
Figure III. **Projected Pricing Kernels**

Figure III depicts the projection of the estimated pricing kernel estimated with constant preference parameters (see Figure 2) on a polynomial function of the market return, $PM_{t+1} = \sum_{j=0}^{5} b_j r_{Mt+1}^j$. The support for the graphs is the observed range of the return on the value-weighted index.
Figure IV. Model Implied Projected Pricing Kernel and Observed Pricing Kernel: 1988–1995

Figure IV depicts the model implied projected pricing kernel and the observed pricing kernel. The support for the graphs is the observed range of the return on the S&P 500 index. Each year, we find the set of parameters \((1/R_f, \frac{1}{\mathbb{P}}, \rho)\) that generates a projected pricing kernel close to the observed pricing kernel in terms of the distance measure (23).
Figure V. Model Implied “projected” Absolute Risk Aversion and Observed Absolute Risk Aversion: 1988–1995

Figure V depicts the model implied “projected” absolute risk aversion and the observed absolute risk aversion. The support for the graphs is the observed range of the return on the S&P 500 index. Each year, we find the set of parameters \((1/R_f, \frac{1}{\rho}, \rho)\) that generates a “projected” absolute risk aversion close to the observed absolute risk aversion in terms of the distance measure (27).